ASYMPTOTIC RELATIVE EFFICIENCY OF MOOD'S AND MASSEY'S TWO SAMPLE TESTS AGAINST SOME PARAMETRIC ALTERNATIVES[1]

By I. M. Chakravarti[2], F. C. Leone and J. D. Alanen

University of North Carolina and Case Institute of Technology

1. Introduction and summary. In [2], the exact power of Mood's and Massey's two-sample tests for discriminating between two populations was derived. Two types of alternatives were considered—change in the location of an exponential distribution and change in the location and scale of a rectangular distribution. The asymptotic relative efficiency of Mood's test based on the median against an alternative of change in location of a normal distribution was shown ([1], [8]) to be $2/\pi$. A limited comparison of powers of the tests based on the median, on the first quartile and the median and on the likelihood-ratio against the exponential alternative is given in [3].

In this paper, the asymptotic relative efficiencies of Mood's test based on the median and Massey's test based on the first quartile and the median are shown to be zero, when these tests are compared against the likelihood-ratio test appropriate for detecting a shift in location of an exponential distribution. Massey's test is found to be about three times as efficient as Mood's test, for exponential distribution. But so also is the test based on the first quartile alone. If the order of the fractile is lowered, the efficiency of the test based on it is increased. Similar comparisons are also made for the normal distribution.

2. Mood's and Massey's test statistics and their limiting distributions. Let $X_1, X_2, \cdots, X_{n_1}$ and $Y_1, Y_2, \cdots, Y_{n_2}$ be independently distributed with continuous cumulative distribution functions (c.d.f.'s) $F(x)$ and $G(y)$ respectively. Let $n = n_1 + n_2 = 4r + 1$ where r is an integer. Let $Z_{(1)} < \cdots \cdots < Z_{(n)}$ be the ordered combined sample and let $Z_1 = Z_{(r+1)}$ and $Z_2 = Z_{(2r+1)}$ be the first quartile and the median respectively of the combined sample. Let U_1 and U denote the number of observations in the first sample, that are less than Z_1 and Z_2 respectively. Then $U_2 = U - U_1$ is the number of observations in the first sample that are greater than or equal to Z_1 but less than Z_2.

Mood's one-sided test based on U for the hypothesis

$$\mathcal{H}_o : F(x) = G(x) \tag{2.1}$$

rejects $\mathcal{H}_o$ if $U \geqq u_o$ where $\Pr(U \geqq u_o \mid \mathcal{H}_o) \leqq \alpha$ and $0 < \alpha < 1$ is a preassigned constant.

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[2] Present address: Laboratoire de Statistique, Rue Alcide Jentzer 22, Geneva, Switzerland.

The expression for the null distribution $\varphi_o(u)$ of U

$$\Pr\{U = u\} = \varphi_0(u) = \binom{2r}{u}\binom{2r+1}{n_1 - u}\bigg/\binom{n}{n_1}, \tag{2.2}$$

as well as the non-null distribution of U and Z_2, was first derived by Mood [8]. The limiting form of the joint distribution of U and Z_2 obtained by him is bivariate normal.

Let $H(x) = (n_1/n)F(x) + (n_2/n)G(x)$ be the mixture of the cdf's $F(x)$ and $G(x)$. Also, let c_1 and c_2 denote the first quartile and the median of $H(x)$, that is,

$$H(c_1) = \tfrac{1}{4} \quad \text{and} \quad H(c_2) = \tfrac{1}{2}. \tag{2.3}$$

The derivatives $F'(x)$ and $G'(x)$ are assumed not to vanish at c_1 and c_2. Then the asymptotic distribution of $V = n_1^{\frac{1}{2}}[(U/n_1) - F(c_2)]$ is normal. The expressions for the asymptotic mean and the asymptotic variance of U are

$$\begin{aligned} &E(U) \sim \mu_0 = n_1 F(c_2) \\ &\sigma^2(U) \sim \sigma_o^2 = \left[\frac{n_2}{n_1 n_2 F(c_2)(1 - F(c_2))} + \frac{n_1}{n_1 n_2 G(c_2)(1 - G(c_2))}\right]^{-1}. \end{aligned} \tag{2.4}$$

Massey's extension [7] of Mood's test, based on U_1 and U_2 alone, is considered next. Let M denote the statistic

$$M = \frac{n^2}{n_1 n_2}\left[\frac{U_1^2}{r} + \frac{U_2^2}{r} + \frac{(n_1 - U_1 - U_2)^2}{2r+1} - \frac{n_1^2}{n}\right]. \tag{2.5}$$

Then the test consists in rejecting $\mathcal{H}_o$ if $M \geqq m_o$, where $\Pr(M \geqq m_o \mid \mathcal{H}_o) \leqq \alpha$. The joint distribution of U_1 and U_2 under $\mathcal{H}_o$ was shown [7] to be

$$\varphi_o(u_1, u_2) = \binom{r}{u_1}\binom{r}{u_2}\binom{2r+1}{n_1 - u_1 - u_2}\bigg/\binom{n}{n_1}. \tag{2.6}$$

Hence, the distribution of M under $\mathcal{H}_o$ can be approximated by the Chi-square distribution with two degrees of freedom.

The non-null distribution of U_1, U_2, Z_1 and Z_2 was derived in [2]. Let $P_{ij}(u_1, u_2, z_1, z_2)$ denote the joint probability density of U_1, U_2, Z_1 and Z_2 when Z_1 belongs to the ith sample and Z_2 to the jth sample, $i, j = 1, 2$. Then the expression for P_{11} when F and G are any two continuous cdf's (given here only as an example) is

$$\begin{aligned} P_{11} = {} & \frac{n_1!}{u_1!(u_2 - 1)!(n_1 - u_1 - u_2 - 1)!}[F(z_1)]^{u_1}[F(z_2) - F(z_1)]^{u_2-1} \\ & \cdot[1 - F(z_2)]^{n_1-u_1-u_2-1}\frac{n_2!}{(r - u_1)!(r - u_2)!(n_2 - 2r + u_1 + u_2)!} \\ & \cdot[G(z_1)]^{r-u_1}[G(z_2) - G(z_1)]^{r-u_2}[1 - G(z_2)]^{n_2-2r+u_1+u_2}\frac{dF(z_1)}{dz_1}\frac{dF(z_2)}{dz_2}. \end{aligned} \tag{2.7}$$

The probability density of the joint distribution of U_1, U_2, Z_1 and Z_2 is

$$P(u_1, u_2, z_1, z_2) = P_{11} + P_{12} + P_{21} + P_{22}. \tag{2.8}$$

The joint distribution $\varphi_{\mathcal{H}}(u_1, u_2)$ of U_1 and U_2 is obtained by integrating $P(u_1, u_2, z_1, z_2)$ over the appropriate range of Z_1 and Z_2. Defining

$$V_1 = n_1^{\frac{1}{2}}[(U_1/n_1) - F(c_1)] \quad \text{and} \quad V_2 = n_1^{\frac{1}{2}}[(U_2/n_1) - F(c_2) + F(c_1)]$$

and proceeding on the same lines and under similar regularity conditions as Mood [8], the asymptotic distribution of V_1 and V_2 is found to be bivariate normal. The expressions for the asymptotic means, variances and covariance of U_1 and U_2 are

$$E(U_1) \sim \mu_1 = n_1 F(c_1), \quad E(U_2) \sim \mu_2 = n_1(F(c_2) - F(c_1))$$

$$\sigma^2(U_1) \sim \sigma_{11} = \left[\frac{n_2}{n_1 n_2 F(c_1)[1 - F(c_1)]} + \frac{n_1}{n_1 n_2 G(c_1)[1 - G(c_1)]}\right]^{-1}$$

$$\sigma^2(U_2) \sim \sigma_{22} = \left[\frac{n_2}{n_1 n_2[F(c_2) - F(c_1)][1 - F(c_2) + F(c_1)]} + \frac{n_1}{n_1 n_2[G(c_2) - G(c_1)][1 - G(c_2) + G(c_1)]}\right]^{-1} \tag{2.9}$$

$$\text{Covar}(U_1, U_2) \sim \sigma_{12} = \tfrac{1}{2}(\sigma_o^2 - \sigma_{11} - \sigma_{22})$$

where σ_o^2 has the expression of (2.4) and c_1, c_2 are as defined in (2.3). Under the null hypothesis $\mathcal{H}_o$, the expressions for σ_o^2, σ_{11}, σ_{22} and σ_{12} become

$$\sigma_o^2 = n_1 n_2/4n, \qquad \sigma_{11} = 3n_1 n_2/16n, \qquad \sigma_{22} = 3n_1 n_2/16n, \qquad \sigma_{12} = -n_1 n_2/16n. \tag{2.10}$$

3. Two-sample likelihood-ıatio test for sh'ft in location of exponential distribution. Suppose $X_1, X_2, \cdots, X_{n_1}$ form a random sampl from the exponential distribution $F(x) = 1 - e^{-(x-\theta_1)}$, $x \geqq \theta_1$, and $Y_1, Y_2, \cdots, Y_{n_2}$, a random sample from the distribution $G(y) = 1 - e^{-(y-\theta_2)}$, $y \geqq \theta_2$.

Consider the hypotheses

(a) $\mathcal{H}_o : \theta_1 = \theta_2$ against $\mathcal{H}_1 : \theta_1 \neq \theta_2$,

(b) $\mathcal{H}_o : \theta_1 = \theta_2$ against $\mathcal{H}_1 : \theta_1 < \theta_2$.

Let X denote the minimum of the sample $X_1, X_2, \cdots, X_{n_1}$ and Y the minimum of the sample $Y_1, Y_2, \cdots, Y_{n_2}$ and $Z = \min(X, Y)$. Then the likelihood ratio statistic for testing the hypothesis (a) is

$$\xi = n_1 X + n_2 Y - nZ, \tag{3.1}$$

which can be also written as $\xi = n_2(Y - X)$ if $Y > X$, $= n_1(X - Y)$ if $X > Y$. The procedure for testing $\mathcal{H}_o$ in (a) is

$$\text{reject } \mathcal{H}_o \quad \text{if} \quad \xi > \xi_0$$

$$\text{accept } \mathcal{H}_o \quad \text{if} \quad \xi \leqq \xi_0,$$

where ξ_o is determined so that

$$\Pr\{\xi \leqq \xi_o \mid \mathcal{H}_o\} = 1 - \alpha, \tag{3.2}$$

or, alternatively, $\Pr\{-(\xi_o/n_1) \leqq Y - X \leqq (\xi_o/n_2)\} = 1 - \alpha$.

For testing the hypothesis $\mathcal{H}_o$ against one-sided alternatives, as stated in (b), the likelihood ratio test is based on the statistic W^* which is defined as $W^* = Y - X$ if $Y > X$, $= 0$ otherwise. The procedure for testing $\mathcal{H}_o$ in (b) is

$$\begin{aligned} &\text{reject } \mathcal{H}_o \quad \text{if} \quad W^* > w_0^*, \\ &\text{accept } \mathcal{H}_o \quad \text{if} \quad W^* \leqq w_0^*, \end{aligned} \tag{3.3}$$

where w_0^* is determined so that $\Pr\{W^* \leqq w_0^* \mid \mathcal{H}_o\} = 1 - \alpha$. The power function $\mathcal{C}_n(\theta)$ of the test of size α, based on W^*, is given by

$$\begin{aligned} \mathcal{C}_n(\theta) &= \alpha e^{n_2\theta} && \text{if} \quad e^{n_2\theta} \leqq (n_1/n\alpha) \\ &= 1 - (n_2/n)e^{-n_1\theta}(n_1/n\alpha)^{n_1/n_2} && \text{if} \quad e^{n_2\theta} \geqq (n_1/n\alpha). \end{aligned} \tag{3.4}$$

4. Asymptotic relative efficiency of Mood's and Massey's tests.

Theorems 1 and 2 quoted below, on asymptotic relative efficiency and efficiency index, are well known and in their present form are due to Hoeffding and Rosenblatt [5].

Let $\beta_n(\theta)$ denote the power function of a test of the hypothesis $\mathcal{H}_o : \theta = \theta_o$ against the alternative $\mathcal{H}_1 : \theta \neq \theta_0$.

THEOREM 1. *Suppose that*

(a) $\beta_n(\theta_0) \leqq \alpha$, $\lim_{n\to\infty}\beta_n(\theta_0) = \alpha$;

(b) *for each* n, $\beta_n(\theta)$ *is non-decreasing in* θ *for* $\theta \geqq \theta_0$ *and continuous at* $\theta = \theta_0$;

(c) $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$;

(d) *there is a positive* r *such that for any* $d \geqq 0$, *the limit* $\lim_{n\to\infty}\beta_n(\theta_0 + dn^{-r}) = H(d)$ *exists*;

(e) $H(d)$ *is continuous and increasing for all* $d \geqq 0$ *and* $\lim_{d\to\infty}H(d) = 1$. *Then*

(I) *the equation* $H(d) = 1 - \beta$, *has a unique positive root* D;

(II) *for any* $\delta > 0$, *we have* $\beta_n(\theta_0 + \delta) \geqq 1 - \beta$ *for some* n;

(III) *if* $N(\delta)$ *is the least* n, *such that* $\beta_n(\theta_0 + \delta) \geqq 1 - \beta$, *then asymptotically as* $\delta \to 0$;

$$N(\delta) \sim (D/\delta)^{1/r}. \tag{4.1}$$

THEOREM 2. *Suppose that for a test based on the statistic* t_n *from a random sample of size* n, *which rejects the hypotheses if* t_n *exceeds a constant,*

(a) $\beta_n(\theta_0) \leqq \alpha$, $\lim_{n\to\infty}\beta_n(\theta_0) = \alpha$;

(b) *for each* n, $\beta_n(\theta)$ *is non-decreasing in* θ *for* $\theta \geqq \theta_0$ *and continuous at* $\theta = \theta_0$;

(c) *there exist a positive* r *and functions* $\mu(\theta)$ *and* $\sigma(\theta)$ *such that for any real* x *and any* $d \geqq 0$, *the probability*

$$\lim_{n\to\infty} P_{\theta_n}\left\{n^r \frac{t_n - \mu(\theta_n)}{\sigma(\theta_n)} \leqq x\right\} = \Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy,$$

where $\theta_n = \theta_0 + dn^{-r}$;

(d) $\mu(\theta)$ *has a derivative* $\mu'(\theta_0)$ *at* θ_0 *and* $\mu'(\theta_0) > 0$;

(e) $\sigma(\theta)$ *is continuous and positive at* $\theta = \theta_0$.

Then

$$\lim_{n\to\infty}\beta_n(\theta_0 + dn^{-r}) = \Phi\{d[\mu'(\theta_0)/\sigma(\theta_0)] - \lambda_\alpha\}$$

when $\Phi(-\lambda_\alpha) = \alpha$. *The efficiency index* $N(\delta)$ *of the test based on* t_n, *has the expression*

$$N(\delta) \sim \left[\frac{\lambda_\alpha + \lambda_\beta}{\delta}\frac{\sigma(\theta_0)}{\mu'(\theta_0)}\right]^{1/r}. \tag{4.2}$$

Let us define the hypothesis $\mathcal{H}$: $G(y) = F(y - \theta)$. Let $n_1 = s_1n/(s_1 + s_2)$ and $n_2 = s_2n/(s_1 + s_2)$ where s_1 and s_2 are two positive numbers.

For Mood's test

$$\lim_{n_1\to\infty} P_{\theta_{n_1}}\left[\frac{n_1^{\frac{1}{2}}(U/n_1 - F(c_2))}{(n_2/4n)^{\frac{1}{2}}} \leqq x\right] = \Phi(x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt,$$

where $\theta_{n_1} = dn_1^{-\frac{1}{2}}$ and $\theta_0 = 0$. To show that condition (b) of Theorem 2 is satisfied for the (one-sided) tests based on U (or U_1), one might proceed as follows. Assume $G(y) = F(y - \theta)$. Let $X_{n_1+i} = Y_i - \theta$ for $i = 1, 2, \cdots, n_2$ and $\underset{\sim}{X} = (X_1, \cdots, X_{n_1}, \cdots, X_{n_1+n_2})$. Then U can be expressed as a function $G(\theta, \underset{\sim}{X})$ for θ and $\underset{\sim}{X}$. It is easy to see that $G(\theta, \underset{\sim}{X})$ is non-decreasing in θ for $\underset{\sim}{X}$ fixed. The distribution of $\underset{\sim}{X}$ does not involve θ. Hence the probability of $U \geqq u_o$, that is, Pr $\{G(\theta, \underset{\sim}{X}) \geqq u_o\}$, is a non-decreasing function of θ. This proof was suggested by W. Hoeffding. It is easy to verify that the other conditions of Theorem 2 are satisfied for this test. Hence the limiting power function of the test is given by

$$\lim_{n_1\to\infty}\beta_{n_1}(dn_1^{-\frac{1}{2}}) = \Phi\{-\lambda_\alpha + d[dF(c_2)/d\theta](4n/n_2)^{\frac{1}{2}}\} = A(d), \tag{4.3}$$

where $dF(c_2)/d\theta$ is evaluated at $\theta = 0$ and $\Phi(-\lambda_\alpha) = \alpha$.

For the test based on M, it is easy to verify that the conditions (a), (c), (d) and (e) are satisfied. Numerical calculations showed that condition (b) is not satisfied. To compute the efficiency of this test relative to the other (one-sided) tests, we appeal to a weaker version of the definition of relative asymptotic efficiency $e_{A^*,A}$ of Hodges and Lehmann [4]. They require that the limit $e_{A^*,A}$ be independent of α and β and of two sequences $\{\theta_n\}$ and $\{h(n)\}$ which appear in the definition [4]. In our case, as pointed out by W. Hoeffding, under conditions (a), (c), (d) and (e) of Theorem 1, the limit $e_{A^*,A}$ exists for the particular sequences $\{\theta_n\}$ and $\{h(n)\}$ implicit in condition (d) and is equal to the limit of the ratio of the corresponding $N(\delta)$ values as δ tends to 0. All comparisons of efficiency relative to the test M are thus subject to this definition.

Similarly for the test based on M, for the sequence of alternative hypotheses $\{\theta_{n_1} = dn_1^{-\frac{1}{2}}\}$, it can be shown that the limiting power function will be given by

$$B(d) = \int_{m_0}^{\infty} f(\chi_2^2, \Delta^2)\,d\chi_2^2. \tag{4.4}$$

In the above expression m_0 is the critical value of M at a preassigned significance level α. $f(\chi_2^2, \Delta^2)$ is the density of a noncentral chi-square with two degrees of freedom. The non-centrality parameter Δ^2 is given by

$$(4.5)\quad \Delta^2 = \frac{4n}{n_2} d^2 \left[\left(\frac{dF(c_1)}{d\theta}\right)^2 + \left(\frac{dF(c_2)}{d\theta} - \frac{dF(c_1)}{d\theta}\right)^2 + \frac{1}{2}\left(\frac{dF(c_2)}{d\theta}\right)^2\right],$$

where the expressions within brackets are evaluated at $\theta = 0$.

4.1. *Exponential distribution.* The alternative considered here is

$$\mathcal{H}_1 : F(x) = 1 - e^{-x}\, x \geqq 0, \qquad G(y) = 1 - e^{-(y-\theta)}\, x \geqq \theta, \theta > 0.$$

The efficiency index for the test based on U, computed using (4.2) and (4.3), is

$$(4.6)\qquad N_1(\delta) \sim \left[\frac{\lambda_\alpha + \lambda_\beta}{\delta}\left(\frac{s_1 + s_2}{s_2}\right)^{\frac{1}{2}}\right]^2 = \bar{N}_1(\delta) \text{ (say)}.$$

Similarly, the index for the test based on U_1, that is, the number of observations less than the first quartile, is

$$(4.7)\qquad N_2(\delta) \sim \left[\frac{\lambda_\alpha + \lambda_\beta}{\delta}\left(\frac{s_1 + s_2}{3s_2}\right)^{\frac{1}{2}}\right]^2 = \bar{N}_2(\delta) \text{ (say)}.$$

For the test based on U_1 and U_2, the expression for the power function $B(d)$ defined in (4.4) depends on θ through the non-centrality parameter Δ^2 alone. The expression for Δ^2 for the exponential distribution is seen to be

$$(4.8)\qquad \Delta^2 = [3s_2/(s_1 + s_2)]\, d^2,$$

where $\theta_{n_1} = dn_1^{-\frac{1}{2}}$ and $\theta_0 = 0$.

If the null-distribution of M is approximated by the central χ^2 distribution with two degrees of freedom and α is the significance level, then $m_0 = -2 \log_e \alpha$. It is also known [6] that $\int_{m_0}^{\infty} f(\chi_2^2, \Delta^2)\, d\chi_2^2$ can be expressed as

$$(4.9)\qquad \int_{m_0}^{\infty} f(\chi_2^2, \Delta^2)\, d\chi_2^2 = \Pr\{R - S \leqq 0\},$$

where R and S are independently distributed as Poisson variates with parameters $\frac{1}{2}m_0 = -\log_e \alpha$ and $\frac{1}{2}\Delta^2$ respectively. To find the efficiency index for Massey's test, one needs to solve the equation

$$(4.10)\qquad H(d) = \int_{m_0}^{\infty} f(\chi_2^2, \Delta^2)\, d\chi_2^2 = 1 - \beta$$

for a preassigned power $1 - \beta$.

Using (4.9) and a normal approximation to the Poisson, a first approximation to the solution of (4.10) might be obtained. For several sets of values of (α, β) correct solutions of (4.10) were computed. These are shown in Table 1.

If Δ^2 is a solution of the equation (4.10) then the solution in terms of d^2 is $D^2 = [(s_1 + s_2)/3s_2]\Delta^2$. Hence the efficiency index $N_3(\delta)$ of Massey's test based on the first quartile and the median, is

$$N_3(\delta) \sim [(s_1 + s_2)/3s_2](\Delta/\delta)^2 = \bar{N}_3(\delta), \tag{4.11}$$

where Δ^2 is a solution of (4.10) for preassigned α and β.

For the likelihood-ratio test based on W^* described in Section 3, let $\theta_{n_1} = \theta_0 + dn_1^{-1}$ for some $d > 0$. Under $\mathcal{H}_o$, $\theta_0 = 0$. Hence substituting the value of θ_{n_1} in $\mathcal{C}_n(\theta)$ defined in (3.4), one gets

$$\begin{aligned} C_n(\theta_{n_1}) &= \alpha \exp(s_2 d/s_1), && \text{if } \exp(s_2 d/s_1) \leqq s_1/\alpha(s_1 + s_2) \\ &= 1 - [s_2/(s_1 + s_2)] \exp(-d)(s_1/\alpha(s_1 + s_2))^{s_1/s_2}, && \text{otherwise.} \end{aligned} \tag{4.12}$$

Hence, the above defines the limiting power function $H(d)$ as $n \to \infty$. This function is continuous in d and $\lim_{d\to\infty} H(d) = 1$. Hence the conditions of Theorem 1 are satisfied. For this test then, if D^* denotes the solution of $H(D^*) = 1 - \beta$ the efficiency index $N^*(\delta)$ is

$$N^*(\delta) \sim (D^*/\delta) = \bar{N}^*(\delta) \tag{4.13}$$

since $r = 1$.

Then it is easily seen that the asymptotic relative efficiency of the tests based on the median, the first quartile and both the first quartile and the median, computed as $\bar{N}^*(\delta)/\bar{N}_i(\delta)$, $i = 1, 2, 3$, tends to zero as δ tends to zero.

On the other hand, the asymptotic relative efficiency of the test based on the first quartile, computed relative to the median test is

$$\epsilon(U_1, U) = \bar{N}_1(\delta)/\bar{N}_2(\delta) = 3. \tag{4.14}$$

The asymptotic relative efficiency $\epsilon(M, U)$ of Massey's test based on the first quartile and the median, relative to the median test is

$$\epsilon(M, U) = \bar{N}_1(\delta)/\bar{N}_3(\delta) = 3[(\lambda_\alpha + \lambda_\beta)/\Delta]^2. \tag{4.15}$$

For several sets of values of (α, β), $\epsilon(M, U)$ is tabulated in Table 1. Here a Massey's test of size 2α is compared against a median test of size α.

TABLE 1

Asymptotic relative efficiency $\epsilon(M, U)$ of Massey's test (M) relative to Mood's test (U) against shift in location of exponential distribution

2α	β	Δ^2	$\epsilon(M, U)$
.01	.01	27.4142	2.630
.01	.025	23.6613	2.608
.01	.05	20.6498	2.588
.05	.01	21.3958	2.576
.05	.025	18.0788	2.550
.05	.05	15.4432	2.524
.05	.10	12.6539	2.491
.10	.10	10.4579	2.457

As seen from the table, the test based on M which is a quadratic form in U_1 and U_2 is asymptotically about twice as efficient as the one based on $U = U_1 + U_2$ but is less efficient than the test based on U_1 alone. This is expected because the test based on M rejects values of U_1 and U_2 which are either too large or too small, whereas the test based on either U_1 or U rejects only large values as significant and the alternative considered to compute the efficiency index is one-sided.

4.2. *Normal distribution.* The alternative considered here is

$$\mathcal{H}_1 : F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt, \qquad G(y) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}(t-\theta)^2}\, dt, \qquad \theta > 0.$$

It is known ([1], [8]) that the asymptotic efficiency of the median test relative to the normal deviate test, which is the most powerful, is $2/\pi$. To evaluate the efficiency of the test based on the first quartile relative to the median test, the approximate solutions for equation (2.3) in the neighborhood of $\theta = 0$, are required. These are

$$c_1 = n_2\theta/(n_1 + n_2) - .6745, \qquad c_2 = n_2\theta/(n_1 + n_2) \tag{4.16}$$

and also the value of $(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}t^2}$ at c_1 and c_2 for $\theta = 0$ are .31778 and .39894. Using (2.10), (4.2) and (4.3), the value of $\epsilon(U_1, U)$ for normal alternatives, works out to be

$$\epsilon(U_1, U) = \bar{N}_1(\delta)/\bar{N}_2(\delta) = (4/3)(.31778/.39894)^2 = .846. \tag{4.17}$$

To evaluate the efficiency of the Massey test relative to the median test, it seems reasonable to compare a median test of size α against Massey's test of size 2α.

From (4.5), the expression for Δ^2, for normal alternatives reduces to

$$\Delta^2 = [4s_2/(s_1 + s_2)]\, d^2(.187148) \tag{4.18}$$

where $\theta_{n_1} = dn_1^{-\frac{1}{2}}$. For $2\alpha = .05$ and $\beta = .01$, the solution to (4.10) is $\Delta^2 = 21.3958$. Thus

$$\bar{N}_3(\delta) = \frac{s_1 + s_2}{4s_2} \frac{1}{\delta^2} \frac{21.3958}{.187148}$$

and the corresponding

$$\bar{N}_1(\delta) = \frac{(\lambda_\alpha + \lambda_\beta)^2}{\delta^2} \frac{s_1 + s_2}{4s_2} \frac{1}{(.39894)^2} = \frac{s_1 + s_2}{4s_2} \frac{1}{\delta^2} \frac{18.3698}{(.39894)^2}.$$

Hence, for these values of α, β, the value of

$$\epsilon(M, U) = \bar{N}_1(\delta)/\bar{N}_2(\delta) = 1.010. \tag{4.19}$$

Computations for other values of α, β show that the value of $\epsilon(M, U)$ is very close to 1. The two tests, then, asymptotically, are equivalent for normal alternatives.

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