MOMENTS OF ORDER STATISTICS FROM THE EQUICORRELATED MULTIVARIATE NORMAL DISTRIBUTION[1]

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1. Summary. Let $Z_1, Z_2, \cdots, Z_n$ be jointly normally distributed random variables with $EZ_i = 0$, $EZ_i^2 = 1$, $EZ_iZ_j = \rho$, $i \neq j$, $-1/(n-1) \leqq \rho \leqq 1$. Let the collection of random variables $\{Z_i\}$ be ordered so that $Z^{(1)} \geqq Z^{(2)} \geqq \cdots \geqq Z^{(n)}$. It is the purpose of this note to show how the moments and product moments of the $\{Z^{(i)}\}$ for any ρ can be obtained from the corresponding moments and product moments of the $\{Z^{(i)}\}$ for $\rho = 0$.

2. Results. Following Dunnett and Sobel [2] and Stuart [10], we construct the variables $\{Z_i\}$ from a collection of independent variables $\{X_i\}$. To be precise, let $X_1, X_2, \cdots, X_n$ be independently and normally distributed random variables with $EX_i = 0$, $EX_i^2 = 1$, $i = 1, 2, \cdots, n$. Let X_0 be another standardized normally distributed variable with $EX_0X_i = \lambda$, and let $Z_i = aX_0 + bX_i$, $i = 1, 2, \cdots, n$. Clearly, $EZ_i = 0$. The conditions that $EZ_i^2 = 1$, $EZ_iZ_j = \rho$, $i \neq j$, imply, respectively, the relations $a^2 + 2ab\lambda + b^2 = 1$, and $a^2 + 2ab\lambda = \rho$, so that $b = (1-\rho)^{\frac{1}{2}}$.

Let the collection $\{X_i\}$ be ordered so that $X^{(1)} \geqq X^{(2)} \geqq \cdots \geqq X^{(n)}$. Then it is clear that

$$Z^{(i)} = aX_0 + (1-\rho)^{\frac{1}{2}}X^{(i)}. \tag{1}$$

Let $\varphi_i(t \mid \rho, n)$ denote the characteristic function of $Z^{(i)}$ and let $\varphi_{i,j}(s, t \mid \rho, n)$ denote the joint characteristic function of $Z^{(i)}$ and $Z^{(j)}$, $i \neq j$.

The construction of the $\{Z_i\}$ depends on whether ρ is positive or negative.

Assume first that $\rho \geqq 0$. Then λ can be taken as zero (implying independence of X_0 and X_i, $i = 1, 2, \cdots, n$) and $a^2 = \rho$, so that

$$Z^{(i)} = (\rho)^{\frac{1}{2}}X_0 + (1-\rho)^{\frac{1}{2}}X^{(i)}, \qquad \rho \geqq 0,$$

and

$$\varphi_i(t \mid \rho, n) = e^{-\rho t^2/2}\varphi_i(t(1-\rho)^{\frac{1}{2}} \mid 0, n), \qquad \rho \geqq 0. \tag{2}$$

Now assume $\rho = -\alpha^2 < 0$. If $\lambda = -a/b$, then $EX_0Z_i = 0$, $i = 1, 2, \cdots, n$, and $a = \alpha$, so that $Z^{(i)} = \alpha X_0 + (1-\rho)^{\frac{1}{2}}X^{(i)}$. Since X_0 is independent of Z_i (through proper choice of λ) it follows that

$$\varphi_i(t \mid \rho, n)e^{-\alpha^2 t^2/2} = \varphi_i(t(1-\rho)^{\frac{1}{2}} \mid 0, n), \qquad \rho = -\alpha^2 < 0. \tag{3}$$

From (2) and (3) it follows that

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$$\varphi_i(t \mid \rho, n) = e^{-\rho t^2/2}\varphi_i(t(1-\rho)^{\frac{1}{2}} \mid 0, n), \qquad -1/(n-1) \leqq \rho \leqq 1. \tag{4}$$

In a similar fashion it is possible to show that

$$\varphi_{i,j}(s, t \mid \rho, n) = e^{-\rho(s+t)^2/2}\varphi_{i,j}(s(1-\rho)^{\frac{1}{2}}, t(1-\rho)^{\frac{1}{2}} \mid 0, n), \qquad -1/(n-1) \leqq \rho \leqq 1. \tag{5}$$

Equations (4) and (5) indicate the simple way in which the moments and product moments of order statistics from equicorrelated normal samples depend on those of order statistics from independent normal samples.

If $\kappa_{r,i}^*$ and $\kappa_{r,i}$ denote the rth cumulants of $Z^{(i)}$ and $X^{(i)}$, respectively, then from (4) it follows that

$$\begin{aligned}
\kappa_{1,i}^* &= (1-\rho)^{\frac{1}{2}}\kappa_{1,i} \\
\kappa_{2,i}^* &= \rho + (1-\rho)\kappa_{2,i} \\
\kappa_{r,i}^* &= (1-\rho)^{r/2}\kappa_{r,i}, \; r \geqq 3.
\end{aligned}$$

The moments of $Z^{(i)}$ about the origin are obtained from the cumulants (see Kendall [7] for formulae). The first four moments about the origin and about the mean are given below.

$$\begin{aligned}
EZ^{(i)} &= (1-\rho)^{\frac{1}{2}}EX^{(i)}, \\
E[Z^{(i)}]^2 &= \rho + (1-\rho)E[X^{(i)}]^2, \\
E[Z^{(i)}]^3 &= 3\rho(1-\rho)^{\frac{1}{2}}EX^{(i)} + (1-\rho)^{3/2}E[X^{(i)}]^3, \\
E[Z^{(i)}]^4 &= 3\rho^2 + 6\rho(1-\rho)E[X^{(i)}]^2 + (1-\rho)^2E[X^{(i)}]^4, \\
E[Z^{(i)} - EZ^{(i)}]^2 &= \rho + (1-\rho)E[X^{(i)} - EX^{(i)}]^2, \\
E[Z^{(i)} - EZ^{(i)}]^3 &= (1-\rho)^{3/2}E[X^{(i)} - EX^{(i)}]^3, \\
E[Z^{(i)} - EZ^{(i)}]^4 &= 3\rho^2 + 6\rho(1-\rho)E[X^{(i)} - EX^{(i)}]^2 + (1-\rho)^2E[X^{(i)} - EX^{(i)}]^4.
\end{aligned}$$

The product moments of $Z^{(i)}$ and $Z^{(j)}$ are similarly obtained from (5). In particular, $EZ^{(i)}Z^{(j)} = \rho + (1-\rho)EX^{(i)}X^{(j)}$.

3. Available tables. Several tables exist from which the moments of the $\{Z^{(i)}\}$ can be computed. Ruben [8] has tabulated the first ten moments of $X^{(1)} = \max_i X_i$ for $n = 1(1)50$. Teichroew [11] has tabulated $EX^{(i)}$ and $EX^{(i)}X^{(j)}$ for $n = 2(1)20$. And Harter [5] has tabulated $EX^{(i)}$ for $n = 2(1)100$ and other $n \leqq 400$ such that none of the prime factors of n exceeds seven. In addition, Jones [6] and Godwin [3] give some exact values of $EX^{(i)}$ and $EX^{(i)}X^{(j)}$ for $n \leqq 6$, and Bose and Gupta [1] give exact values of $E[X^{(i)}]^3$ and $E[X^{(i)}]^4$, for $n \leqq 5$ and $n \leqq 6$, respectively.

In order to provide a rough idea of the dependence of the moments of the

$\{Z^{(i)}\}$ on ρ, we have provided two tables. Table I gives the mean, standard deviation, and third and fourth central moments of $Z^{(1)}$, computed from Ruben's table, for $n = 2(1)5(5)50$ and for $\rho = -1/(n-1)$, 0, $\frac{1}{2}$, and 1. Table II gives the mean and standard deviation of $Z^{(n/2)}$, computed from Teichroew's table, for $n = 4(2)20$ and the above values of ρ.

TABLE I

Means and standard deviations and third and fourth central moments for the extreme order statistic from an n-dimensional Normal distribution with correlations equal to $-1/(n-1)$, 0, $\frac{1}{2}$ *and* 1

n	$\rho = -1/(n-1)$				$\rho = 0$			
	μ	σ	μ_3	μ_4	μ	σ	μ_3	μ_4
2	0.79788	0.60281	0.21801	0.51091	0.56419	0.82565	0.07708	1.42280
3	1.03648	0.58241	0.16387	0.43639	0.84628	0.74798	0.08920	0.97955
4	1.18862	0.56770	0.14022	0.38138	1.02938	0.70122	0.09121	0.76460
5	1.30023	0.55625	0.12660	0.35005	1.16296	0.66898	0.09059	0.64108
10	1.62199	0.52105	0.09701	0.26964	1.53875	0.58681	0.08283	0.39502
15	1.79684	0.50111	0.08516	0.23200	1.73591	0.54867	0.07679	0.30918
20	1.91599	0.48742	0.07825	0.20879	1.86748	0.52507	0.07246	0.26364
25	2.00584	0.47709	0.07355	0.19258	1.96531	0.50844	0.06918	0.23472
30	2.07768	0.46886	0.07006	0.18040	2.04276	0.49582	0.06659	0.21441
35	2.13736	0.46206	0.06733	0.17080	2.10661	0.48577	0.06446	0.19919
40	2.18830	0.45629	0.06510	0.16298	2.16078	0.47748	0.06268	0.18725
45	2.23267	0.45128	0.06324	0.15643	2.20772	0.47048	0.06115	0.17758
50	2.27191	0.44689	0.06166	0.15084	2.24907	0.46445	0.05982	0.16955

n	$\rho = \frac{1}{2}$				$\rho = 1$			
	μ	σ	μ_3	μ_4	μ	σ	μ_3	μ_4
2	0.39894	0.91698	0.02725	2.12823	0	1	0	3
3	0.59841	0.88303	0.03154	1.83409	0	1	0	3
4	0.72788	0.86363	0.03225	1.67872	0	1	0	3
5	0.82234	0.85074	0.03203	1.58157	0	1	0	3
10	1.08806	0.81986	0.02928	1.36527	0	1	0	3
15	1.22748	0.80655	0.02715	1.27886	0	1	0	3
20	1.32050	0.79865	0.02562	1.22946	0	1	0	3
25	1.38969	0.79326	0.02446	1.19645	0	1	0	3
30	1.44445	0.78925	0.02354	1.17236	0	1	0	3
35	1.48960	0.78612	0.02279	1.15375	0	1	0	3
40	1.52790	0.78358	0.02216	1.13880	0	1	0	3
45	1.56109	0.78146	0.02162	1.12643	0	1	0	3
50	1.59034	0.77965	0.02115	1.11596	0	1	0	3

TABLE II

Means and standard deviations for the central order statistic for an n-dimensional normal distribution with correlations equal to $-1/(n-1)$, 0, $\frac{1}{2}$ and 1

n	$\rho = -1/(n-1)$		$\rho = 0$		$\rho = \frac{1}{2}$		$\rho = 1$	
	μ	σ	μ	σ	μ	σ	μ	σ
4	0.34296	0.38376	0.29701	0.60038	0.21002	0.82476	0	1
6	0.22078	0.30896	0.20155	0.49620	0.14252	0.78937	0	1
8	0.16304	0.26659	0.15251	0.43265	0.10784	0.77045	0	1
10	0.12930	0.23817	0.12267	0.38866	0.08674	0.75863	0	1
12	0.10715	0.21735	0.10259	0.35586	0.07254	0.75055	0	1
14	0.09149	0.20122	0.08816	0.33019	0.06234	0.74466	0	1
16	0.07982	0.18824	0.07729	0.30939	0.05465	0.74018	0	1
18	0.07080	0.17750	0.06880	0.29208	0.04865	0.73665	0	1
20	0.06361	0.16842	0.06200	0.27739	0.04384	0.73381	0	1

Note that in both tables the odd moments decrease to zero and the even moments increase to values for the corresponding moments of the unit normal distribution as ρ increases. Convergence is to be expected since $Z^{(i)}$ has a unit normal distribution when $\rho = 1$.

4. Multinomial order statistics. We now apply the results of Section 2 to obtain approximate means and variances of order statistics from the multinomial distribution with equal cell probabilities. Let $Y_1, Y_2, \cdots, Y_n$ be jointly multinomially distributed in equally likely cells with $\sum Y_i = N$. Then $EY_i = N/n$, $\sigma^2(Y_i) = N(n-1)/n^2$, and $\rho(Y_i, Y_j) = -1/(n-1)$. Let $W_i = (Y_i - EY_i)/\sigma(Y_i)$ and let the $\{Y^{(i)}\}$ be the ordered sample as before. For large N, the joint distribution of the $\{W_i\}$ is approximated by an equicorrelated multivariate normal distribution (degenerate) with common correlation $\rho = -1/(n-1)$. Using the results of Section 2, it can be shown that

$$EY^{(i)} \cong (N/n) + (N/n)^{\frac{1}{2}}EX^{(i)}$$
(6)
$$\sigma^2(Y^{(i)}) \cong (N/n)[\sigma^2(X^{(i)}) - (1/n)].$$

Greenwood and Glasgow [4] give approximations to the mean and variance of max (Y_1, Y_2) for the special case $n = 3$. Specializing their results by setting all cell probabilities equal and using the relation max (Y_1, Y_2) + max (Y_2, Y_3) + max $(Y_1, Y_3) = 2Y^{(1)} + Y^{(2)}$, one has, assuming all cell probabilities equal,

$$EY^{(1)} = \tfrac{1}{2}\,[3E \max (Y_1, Y_2) - (N/3)]$$
$$\cong (N/3) + \tfrac{1}{2}(3N/\pi)^{\frac{1}{2}}.$$

This is in agreement with (6) since, for $n = 3$, Jones [6] gives $EX^{(1)} = (\frac{3}{2})\pi^{-\frac{1}{2}}$.

TABLE III

Comparison of exact and approximate means and variances of $Y^{(1)}$

n	N	$EY^{(1)}$		$\sigma^2(Y^{(1)})$	
		approx.	exact	approx.	exact
3	10	4.878	4.927	0.754	0.834
	20	8.852	8.887	1.508	1.688
	30	12.676	12.717	2.261	2.462
	50	20.122	20.162	3.769	4.037
4	10	4.128	4.198	0.604	0.800
	20	7.302	7.383	1.209	1.451
	30	10.319	10.399	1.813	2.123
	50	16.139	16.222	3.021	3.417
5	10	3.645	3.756	0.495	0.692
	20	6.326	6.440	0.990	1.285
	30	8.849	8.965	1.485	1.846
	50	13.678	13.798	2.475	2.939
10	10	2.539	2.749	0.244	0.518
	20	4.176	4.410	0.489	0.846
	30	5.665	5.907	0.733	1.156
	50	8.441	8.690	1.222	1.759

In Table III, some approximate means and variances of $Y^{(1)}$ as given by (6) are compared with exact values obtained from a table of the distribution of $Y^{(1)}$ prepared by Steck [9].

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