A CHARACTERIZATION OF THE WISHART DISTRIBUTION

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1. Summary and introduction. It is known that if X and Y are independent random variables having a Gamma distribution with parameters (θ, n) and (θ, m), i.e., with density function

$$p(x, \theta, n) = \frac{\theta^{n/2} x^{n/2-1} e^{-(\frac{1}{2})\theta x}}{2^{n/2}\Gamma(n/2)}, \qquad 0 < x, \theta; 1 \leqq n,$$

then $X + Y$ and $X/(X + Y)$, or equivalently X/Y, are statistically independent. Lukacs [1] proved that this independence property characterizes the Gamma distribution, namely, if X and Y are two nondegenerate positive random variables, and if $X + Y$ is independent of $X/(X + Y)$, or equivalently of X/Y, then X and Y have a Gamma distribution with the same scale parameter.

In the present paper we present an extension to the case where U and V are symmetric positive definite matrices having a Wishart distribution. A number of difficulties are encountered in the generalization. First, there is no natural extension of a ratio, and we consider $Z = W^{-1}UW'^{-1}$, where the "square root" $W = (U + V)^{\frac{1}{2}}$ is any factorization $WW' = (U + V)$. In the matrix case Z is not a function of $V^{-\frac{1}{2}}UV^{-\frac{1}{2}}$ as was true in one dimension, and indeed if U and V are independent random matrices having a Wishart distribution, $U + V$ and $V^{-\frac{1}{2}}UV^{-\frac{1}{2}}$ need not be statistically independent, depending on which square root is chosen. This aspect will be treated in another paper.

In the univariate case it is relatively straightforward to generate differential equations by differentiating under the expectation sign, but this is no longer true since the elements of $(U + V)^{\frac{1}{2}}$ do not bear a simple relation to the elements of $(U + V)$, and it is this point which leads to the difficulties in the proof. The characterization theorem is stated in Section 2. In Section 3 the differential equation is set up, and is solved in Section 4.

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2. Characterization of the Wishart distribution. We write $X \sim \mathcal{W}(\Lambda, p, n)$, $(n > p - 1, \Lambda{:}\, p \times p, \Lambda > 0)$, to mean that X is a $p \times p$ symmetric matrix with density function

$$(1) \qquad p(X) = c\,|\Lambda|^{n/2}\,|X|^{(n-p-1)/2} e^{-(\frac{1}{2})\operatorname{tr}\Lambda X}, \qquad X > 0,$$

where

$$c^{-1} = 2^{pn/2}\pi^{p(p-1)/4}\prod_{1}^{p}\Gamma[\tfrac{1}{2}(n - i + 1)].$$

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The characteristic function (c.f.) is

$$Ee^{i\operatorname{tr} AX} = |\Lambda|^{n/2}\, |\Lambda - 2iA|^{-n/2}, \qquad \text{for } A \text{ symmetric.}$$

If $n < p$, n integral, then this expression is the c.f. of a Wishart d.f., but for any fractional $n < p - 1$, it is not the c.f. of a d.f. (See Appendix.)

The following theorems characterize the Wishart distribution.

THEOREM 1. *If $U \sim \mathcal{W}(\Lambda, p, n)$, $V \sim \mathcal{W}(\Lambda, p, m)$ are independently distributed, then $U + V \equiv WW'$ and $Z = W^{-1}UW'^{-1}$ are statistically independent. Furthermore, the distribution of Z is invariant under the transformation $Z \to \Gamma Z \Gamma'$, where Γ is orthogonal.*

The proof of this theorem is straightforward, and will be omitted.

THEOREM 2. *If U and V are $p \times p$ positive definite matrices which are independently distributed, and* (i) *$U + V = WW'$ is statistically independent of $Z = W^{-1}VW'^{-1}$,* (ii) *the distribution of Z is invariant under the transformation $Z \to \Gamma Z \Gamma'$, where Γ is orthogonal, then U and V have a Wishart distribution with the same scale matrix.*

The proof is given in two parts: (a) derivation of the differential equation, (b) its solution.

3. The differential equation. From the independence hypothesis we have

$$\begin{aligned} Ee^{\operatorname{tr}(AW + BWW' + CZ)} &= Ee^{\operatorname{tr}(AW + BWW')}\, E(e^{\operatorname{tr} CZ}) \\ &\equiv f(A, B)\, g(C), \end{aligned} \tag{2}$$

where A, B, and C are $p \times p$ matrices, and $\mathcal{R}(B + B')$ is negative definite.

The motivation for using the particular form of equation (2) is twofold: (i) the conditions on B are such as to permit differentiation, (ii) the inclusion of A permits the derivation of certain relations, since differentiating twice with respect to the elements of A and summing is equivalent to differentiating once with respect to the elements of B.

We adopt the following notation:

$$f^{ij} = \frac{\partial f}{\partial a_{ij}},\ f_{ij} = \frac{\partial f}{\partial b_{ij}},\ g_{ij} = \frac{\partial g}{\partial c_{ij}}.$$

The indices using Greek letters will generally be those which are summed.

Since we can differentiate under the expectation sign, there is a relation between second derivatives with respect to A and first derivatives with respect to B, namely,

$$\sum_\alpha \frac{\partial^2}{\partial a_{i\alpha}\partial a_{j\alpha}} \frac{\partial^q f}{\partial t_1 \cdots \partial t_q} = \frac{\partial}{\partial b_{ij}} \frac{\partial^q f}{\partial t_1 \cdots \partial t_q}, \tag{3}$$

where $t_1, \cdots, t_q$ are any (not necessarily distinct) arguments of f.

We now obtain a consequence of the assumption of orthogonal invariance.

LEMMA 1. *If*

$$g(C) = E \exp \operatorname{tr} CZ = E \exp \operatorname{tr} C\Gamma' Z \Gamma = g(\Gamma C \Gamma'), \tag{4}$$

for all orthogonal matrices Γ, *then*

$$g_{ij}(0) \equiv \frac{\partial g(C)}{\partial c_{ij}}\bigg|_{C=0} = c_1\delta_{ij}\,, \tag{5}$$

$$g_{ij,kl}(0) = c_2\delta_{ij}\delta_{kl} + c_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{6}$$

where c_1, c_2, c_3 *are constants and* δ_{ij} *is the Kronecker delta.*

Proof. We first note that by considering permutation matrices Γ, that

$$g_{ii} = g_{jj}\,,\quad g_{ij,kl} = g_{ij,lk} = g_{kl,ij} = \text{etc.}$$

By choosing $\gamma_{ii} = 1$, $\gamma_{jj} = -1$, $g_{ij} = -g_{ij}$, which yields (5).

By considering reflections,

$$g_{ii,ij} = g_{ii,jk} = g_{ij,ik} = g_{ij,kl} = 0.$$

For example, $\gamma_{ii} = \gamma_{jj} = 1$, $\gamma_{kk} = -1$ yields $g_{ij,ik} = -g_{ij,ik}$. From (4) we obtain

$$g_{ij,kl} = \sum_{\alpha,\beta,\gamma,\delta} \gamma_{i\alpha}\gamma_{j\beta}\gamma_{k\gamma}\gamma_{l\delta}\, g_{\alpha\beta,\gamma\delta}\,. \tag{7}$$

We now set up the table of values of $g_{ij,kl}$ by the nature of the indices:

x	g_x
ii,ii	d_1
ii,jj	d_2
ij,ij	d_3

Set $\Gamma = \exp \epsilon\, \Xi$, where Ξ is skew-symmetric. Let $\xi_{ij} = -\xi_{ji}$ be the only non-zero elements of Ξ, then the first order terms of ϵ in (7) for $g_{ii,ij}$ yields

$$\begin{aligned} 0 &= \Sigma\, \gamma_{i\alpha}\gamma_{i\beta}\gamma_{i\gamma}\gamma_{j\delta}\, g_{\alpha\beta,\gamma\delta} \\ &= g_{ii,ii}\xi_{ji} + g_{ii,jj}\xi_{ij} + g_{ij,ij}\xi_{ij} + g_{ji,ij}\xi_{ij}\,, \end{aligned}$$

so that $d_1 = d_2 + 2d_3$, which leads to (6). ||[2]

From (2) we set up the basic differential equations. Differentiating (2) successively with respect to $a_{i\lambda}$, $c_{\lambda\mu}$, $a_{j\mu}$, and summing over λ, μ we obtain (noting that $U = WZW'$)

$$E u_{ij} e^{\operatorname{tr}(AW + BWW' + CZ)} = \sum_{\lambda,\mu} f^{i\lambda,j\mu} g_{\lambda\mu}\,, \tag{8}$$

and differentiating (8) successively with respect to $a_{k\nu}$, $c_{\nu\sigma}$, $a_{l\sigma}$, and summing, we obtain

$$E u_{ij} u_{kl} e^{\operatorname{tr}(AW + BWW' + CZ)} = \sum_{\lambda,\mu,\nu,\sigma} f^{i\lambda,j\mu,k\nu,l\sigma} g_{\lambda\mu,\nu\sigma}\,. \tag{9}$$

Setting $C = 0$, $A = 0$, using Lemma 1 and (3), (8) and (9) become

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[2] The symbol || denotes end of proof.

$$Eu_{ij}e^{\operatorname{tr}B(U+V)} = c_1 \sum_{\lambda,\mu} f^{i\lambda,j\mu}\delta_{\lambda\mu} = c_1 \sum_{\lambda} f^{i\lambda,j\lambda} = c_1 f_{ij}, \tag{10}$$

$$\begin{aligned} Eu_{ij}u_{kl}e^{\operatorname{tr}B(U+V)} &= \sum_{\lambda,\mu,\nu,\sigma} f^{i\lambda,j\mu,k\nu,l\sigma}[c_2\delta_{\lambda\mu}\delta_{\nu\sigma} + c_3(\delta_{\lambda\nu}\delta_{\mu\sigma} + \delta_{\lambda\sigma}\delta_{\mu\nu})] \\ &= \sum_{\lambda,\mu} [c_2 f^{i\lambda,j\lambda,k\mu,l\mu} + c_3(f^{i\lambda,j\mu,k\lambda,l\mu} + f^{i\lambda,j\mu,k\mu,l\lambda})] \\ &= c_2 f_{ij,kl} + c_3(f_{ik,jl} + f_{il,kj}). \end{aligned} \tag{11}$$

If we define

$$\varphi(B) = Ee^{\operatorname{tr}BU},\ \psi(B) = Ee^{\operatorname{tr}BV} \tag{12}$$

then (10) and (11) can be written as

$$\varphi_{ij}\psi = c_1(\varphi\psi)_{ij}, \tag{13}$$

$$\varphi_{ij,kl}\psi = c_2(\varphi\psi)_{ij,kl} + c_3[(\varphi\psi)_{il,kj} + (\varphi\psi)_{ik,jl}]. \tag{14}$$

Equation (13) implies that

$$\varphi = (\varphi\psi)^{c_1}. \tag{15}$$

For B sufficiently close to zero, we can write

$$\varphi\psi = e^{\chi},$$

and using (15), we obtain for (14):

$$\begin{aligned} c_1\chi_{ij,kl} + c_1^2\chi_{ij}\chi_{kl} &= c_2(\chi_{ij,kl} + \chi_{ij}\chi_{kl}) \\ &\quad + c_3(\chi_{ik,jl} + \chi_{ik}\chi_{jl} + \chi_{il,jk} + \chi_{il}\chi_{jk}). \end{aligned} \tag{16}$$

For $i = j = k = l$, (16) reduces to

$$(c_2 + 2c_3 - c_1)\chi_{ii,ii} + (c_2 + 2c_3 - c_1^2)\chi_{ii}^2 = 0, \tag{17}$$

the solution of which is

$$\chi = -c \log [s(B) + b_{ii}\, t(B)], \tag{18}$$

where $s(B)$ and $t(B)$ do not depend on the element b_{ii}, and c is a function of c_1, c_2, c_3, namely

$$c = -(c_2 + 2c_3 - c_1)/(c_2 + 2c_3 - c_1^2). \tag{19}$$

Write $\chi = -c \log H$, then

$$\begin{aligned} \chi_{ij} &= -cH_{ij}/H, \\ \chi_{ij,kl} &= cH_{ij}H_{kl}/H^2 - cH_{ij,kl}/H, \end{aligned} \tag{20}$$

so that (16) can be written as

$$(21)\qquad \begin{aligned}&(c_2 - c_1)[H_{ij}H_{kl} - HH_{ij,kl}] + c(c_2 - c_1^2)H_{ij}H_{kl}\\ &\qquad + c_3[(1 + c)H_{ik}H_{jl} - HH_{ik,jl} + (1 + c)H_{il}H_{jk} - HH_{il,jk}] = 0.\end{aligned}$$

We now prove the following lemma.

LEMMA 2. *If* (21) *holds, then when* B *is symmetric,*

$$(22)\qquad 2HH_{ij,kl} = \begin{vmatrix} H_{ij} & H_{il} \\ H_{jk} & H_{kl} \end{vmatrix} + \begin{vmatrix} H_{ij} & H_{ik} \\ H_{jl} & H_{kl} \end{vmatrix}.$$

PROOF. Writing (21) for indices (i, j, k, l), (i, k, j, l), and (i, l, j, k), we obtain the matrix equation

$$d\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} H_{ij}H_{kl} \\ H_{ik}H_{jl} \\ H_{il}H_{jk} \end{pmatrix} = \begin{pmatrix} c_2 - c_1 & c_3 & c_3 \\ c_3 & c_2 - c_1 & c_3 \\ c_3 & c_3 & c_2 - c_1 \end{pmatrix}\begin{pmatrix} HH_{ij,kl} \\ HH_{ik,jl} \\ HH_{il,jk} \end{pmatrix},$$

where $d = c_3(1 + c)$. Inversion and simplification leads to the equation

$$(23)\qquad -c_4HH_{ij,kl} = \begin{vmatrix} H_{ij} & H_{il} \\ H_{jk} & H_{kl} \end{vmatrix} + \begin{vmatrix} H_{ij} & H_{ik} \\ H_{jl} & H_{kl} \end{vmatrix},$$

$c_4 = c_3(1 + c)/(c_2 - c_1 - c_3)$.

Furthermore, from Lemma 1 and the definition of H, it follows that the third derivatives of H with respect to (i, j) are 0. Therefore, for B symmetric,

$$H = \lambda + \mu b_{ii} + \gamma b_{jj} + \sigma b_{ij} + \tau(b_{ii}b_{jj} - b_{ij}^2),$$

where the Greek letters are functions of the elements of B except b_{ii}, b_{ij}, b_{jj}. Thus $H_{ii} = \mu + \tau b_{jj}$, $H_{jj} = \gamma + \tau b_{ii}$,

$$H_{ij} = \tfrac{1}{2}\sigma - \tau b_{ij},\quad H_{ij,ij} = -\tfrac{1}{2}H_{ii,jj} = -\tfrac{1}{2}\tau.$$

(Because of the assumption of symmetry, certain differentiations require the insertion of a factor of $\frac{1}{2}$.) Thus in (23) for $H_{ii,jj}$ we obtain

$$-\tfrac{1}{2}\tau Hc_4 = \begin{vmatrix} H_{ii} & H_{ij} \\ H_{ij} & H_{jj} \end{vmatrix} = \begin{vmatrix} \mu + \tau b_{jj} & \frac{1}{2}\sigma - \tau b_{ij} \\ \frac{1}{2}\sigma - \tau b_{ij} & \gamma + \tau b_{ii} \end{vmatrix},$$

and equating of coefficients of b_{ii}, b_{jj}, or b_{ij} yields $c_4 = -2$. ||

We now use Lemma 2 as a basis for induction.

LEMMA 3. *If* B *is symmetric, then*

$$(24)\qquad 2^n n!\, H^{n-1}H_{i_1j_1,\cdots,i_nj_n} = \Sigma'\rho(\alpha)\rho(\beta)H_{\alpha_1\beta_1}\cdots H_{\alpha_n\beta_n},$$

where Σ' *is over all assignments of* i*'s and* j*'s to* α*'s and* β*'s with the restriction that for each pair* (i_k, j_k) *one index is an* α *and one a* β*, and where* $\rho(\alpha)$ *is the sign of the permutation of the indices.*

REMARK. The above notation is not precise. However, we feel that the present simpler terminology will be clearer to the reader than a more cumbersome precise form. Verification of the reduction of $n = 2$ will serve to clarify the notation.

PROOF. By Lemma 2, the result holds for $n = 2$, and we assume it holds for n. Multiply (24) by 2, differentiate with respect to $b_{i_{n+1}j_{n+1}}$, then multiply by H. The LHS becomes

$$2^{n+1}n!\{(n-1)H^{n-1}H_{i_{n+1}j_{n+1}}H_{i_1j_1,\cdots,i_nj_n} + H^nH_{i_1j_1,\cdots,i_{n+1}j_{n+1}}\}. \tag{25}$$

The RHS becomes, using (23),

$$\begin{aligned}\Sigma'\rho(\alpha)\rho(\beta)\sum_{h=1}^{n}\left(\prod_{k\neq h}^{n}H_{\alpha_k\beta_k}\right)2H_{\alpha_h\beta_h,i_{n+1}j_{n+1}}H &= \Sigma'\rho(\alpha)\rho(\beta)\sum_{h=1}^{n}\left(\prod_{k\neq h}^{n}H_{\alpha_k\beta_k}\right)\\ &\cdot\{2H_{\alpha_h\beta_h}H_{i_{n+1}j_{n+1}} - H_{\alpha_hi_{n+1}}H_{\beta_hj_{n+1}} - H_{\alpha_hj_{n+1}}H_{\beta_hi_{n+1}}\}.\end{aligned} \tag{26}$$

By equations (25) and (26), and using the induction hypothesis, we obtain

$$\begin{aligned}&2^{n+1}n!H^nH_{i_1j_1,\cdots,i_{n+1}j_{n+1}}\\ &= \Sigma'\rho(\alpha)\rho(\beta)\left\{\sum_h\left(\prod_{k\neq h}H_{\alpha_k\beta_k}\right)\right.\\ &\qquad\cdot[2H_{\alpha_h\beta_h}H_{i_{h+1}j_{n+1}} - H_{\alpha_hi_{n+1}}H_{\beta_hj_{n+1}} - H_{\alpha_hj_{n+1}}H_{\beta_hi_{n+1}}]\\ &\qquad\left. - 2(n-1)\left(\prod_{k=1}^{n}H_{\alpha_k\beta_k}\right)H_{i_{n+1}j_{n+1}}\right\}\\ &= \Sigma'\rho(\alpha)\rho(\beta)\left\{2\left(\prod_{k=1}^{n}H_{\alpha_k\beta_k}\right)H_{i_{n+1}j_{n+1}}\right.\\ &\qquad\left. - \sum_h\prod_{k\neq h}H_{\alpha_k\beta_k}(H_{\alpha_hi_{n+1}}H_{\beta_hj_{n+1}} + H_{\alpha_hj_{n+1}}H_{\beta_hi_{n+1}})\right\}\\ &= \frac{1}{n+1}\Sigma'\rho(\alpha^*)\rho(\beta^*)H_{\alpha_1^*\beta_1^*}\cdots H_{\alpha_{n+1}^*\beta_{n+1}^*},\end{aligned} \tag{27}$$

which is (24) for $(n + 1)$, and completes the induction.

To see the last step, the evaluation can be made by examining two cases as $\{\alpha_h^*\beta_h^*\} = \{i_{n+1}j_{n+1}\}$ for some h, or not. In the former $(\alpha_h^*\beta_h^*) = (i_{n+1}j_{n+1})$ or $(j_{n+1}i_{n+1})$. Define

$$\alpha_t = \alpha_t^* \text{ if } t < h,$$

$$\alpha_t = \alpha_{t+1}^* \text{ if } t \geqq h,$$

and β_t similarly. Then $\rho(\alpha^*)\rho(\beta^*) = \rho(\alpha)\rho(\beta)$ and

$$H_{\alpha_1^*\beta_1^*}\cdots H_{\alpha_{n+1}^*\beta_{n+1}^*} = H_{\alpha_1\beta_1}\cdots H_{\alpha_n\beta_n}H_{i_{n+1}j_{n+1}},$$

and there are $2(n + 1)$ such terms.

If $\alpha_h^* = i_{n+1}$ and $\beta_m^* = j_{n+1}(h < m)$, define

$$\alpha_t = \alpha_t^* \quad \text{for} \quad t < h, \qquad \beta_t = \beta_t^* \quad \text{for} \quad t < h,$$
$$\alpha_t = \alpha_{t+1}^* \quad \text{for} \quad t \geqq h, \qquad \beta_t = \beta_{t+1}^\alpha \quad \text{for} \quad t \geqq h, t \neq m-1,$$
$$\beta_t = \beta_h^* \quad \text{for} \quad t = m-1,$$

then $\rho(\alpha^*)\rho(\beta^*) = -\rho(\alpha)\rho(\beta)$, and

$$H_{\alpha_1^*\beta_1^*} \cdots H_{\alpha_{n+1}^*\beta_{n+1}^*} = H_{\alpha_1\beta_1} \cdots H_{\alpha_{m-2}\beta_{m-2}} H_{\alpha_m\beta_m} \cdots H_{\alpha_n\beta_n} H_{\alpha_{n-1}j_{n+1}} H_{\beta_{m-1}i_{n+1}} .$$

By considering $h > m$ we obtain $(n + 1)$ such terms. From the other cases, i.e., $\alpha_h^* = j_{n+1}$, $\beta_h^* = i_{n+1}$ for $h < m$ and $h > m$, we obtain $(n + 1)$ terms

$$H_{\alpha_1^*\beta_1^*} \cdots H_{\alpha_{n+1}^*\beta_{n+1}^*} = H_{\alpha_1\beta_1} \cdots H_{\alpha_{m-2}\beta_{m-2}} H_{\alpha_m\beta_m} \cdots H_{\alpha_n\beta_n} H_{\alpha_{m-1}i_{n+1}} H_{\beta_{m-1}j_{n+1}} . \|$$

4. Solution of the differential equation. From Lemma 3 we obtained a differential equation in H. If we can prove that, for symmetric matrices B, $H = |\Lambda - 2B|/|\Lambda|$, the proof will be complete, since our original characteristic functions $\varphi(B)$, $\psi(B)$ defined in (12) are related to H by $\varphi = H^{-cc_1}$, $\psi = H^{-c(1-c_1)}$.

In the following we adopt the notation: if $M = (m_{ij}):p \times p$, then $M_k = (m_{ij})$: $1 \leqq i, j \leqq k$, and $m^{(k)}$ is a vector consisting of the elements not in M_k arranged in any order.

We now prove by induction that for symmetric matrices, B,

$$H = \theta_k(b^{(k)})|B_k + D^k(b^{(k)})|, \tag{28}$$

where $\theta_k(b^{(k)})$ is a scalar function of $b^{(k)}$, and $D^k(b^{(k)})$ is a symmetric matrix function of $b^{(k)}$.

Before proving (28), we point out how this will give us the desired result. Suppose (28) holds, then for $k = p$, $H(B) = \alpha|B + D|$. Recall that $\chi = -c \log H$ and $H(0) = 1$, so that $\alpha = |D|^{-1}$ and $H(B) = |B + D|/|D|$. We assert that D is real and is negative definite. To see this we can diagonalize D with diagonal elements d_i and choose B to be diagonal with diagonal elements b_i. Hence, from $\theta\psi = H^{-c}$,

$$\theta\psi = [\prod d_i/(b_i + d_i)]^c.$$

But $\theta\psi \leqq 1$ when $b_i < 0$, and hence $d_i < 0$. Thus we can write $D = -\frac{1}{2}\Lambda$, where Λ is positive definite, in which case $H = |\Lambda - 2B|/|\Lambda|$.

We now prove (28). Since $H_{ii,ii} = 0$ from (22), (28) holds for $k = 1$. From the development of Lemma 2, we also have that

$$H = \frac{1}{\tau}\begin{vmatrix} b_{jj} + (\mu/\tau) & b_{ij} - \frac{1}{2}(\sigma/\tau) \\ b_{ij} - \frac{1}{2}(\sigma/\tau) & b_{ii} + (\nu/\tau) \end{vmatrix},$$

so that (28) holds for $k = 2$. Assume that (28) holds for general n, i.e.,

$$H = \theta_n(b^{(n)})|B_n + D^n(b^{(n)})|. \tag{29}$$

From the derivatives of order $(n + 2)$ with respect to the elements of B_{n+1} in (24), we see that H is a polynomial in the elements of B_{n+1} with the elements of $b^{(n+1)}$ as coefficients, i.e., $H = P_{n+1}(B_{n+1}, b^{(n+1)})$. The terms of degree $(n+1)$ in P_{n+1} are, by (24), a multiple of $|B_{n+1}|$, so that

$$H = \alpha_{n+1}(b^{(n+1)})|B_{n+1}| + \cdots . \tag{30}$$

The coefficient of $b_{11} \cdots b_{nn}$ in H is, by (29), $\theta_n(b^{(n)})$, and by (30),

$$\alpha_{n+1}(b^{(n+1)})b_{n+1,n+1} + \beta_{n+1}(b^{(n+1)}).$$

Consequently,

$$\theta_n(b^{(n)}) = \alpha_{n+1}(b^{(n+1)})b_{n+1,n+1} + \beta_{n+1}(b^{(n+1)}).$$

The coefficient of $b_{33} \cdots b_{nn}$ in H is, by (29),

$$\theta_n(b^{(n)}) \begin{vmatrix} b_{11} + d_{11}(b^{(n)}) & b_{12} + d_{12}(b^{(n)}) \\ b_{12} + d_{12}(b^{(n)}) & b_{22} + d_{22}(b^{(n)}) \end{vmatrix}.$$

By (30), $\theta_n(b^{(n)})d_{11}$, $\theta_n(b^{(n)})d_{22}$, and $\theta_n(b^{(n)})d_{12}$ are polynomials of degree $\leqq 2$ in $b_{1,n+1}$, $b_{2,n+1}$, and $(b_{1,n+1}b_{2,n+1})$, respectively. Hence

$$d_{11} = \frac{Q_{11}(b_{1,n+1})}{b_{n+1,n+1} + \beta/\alpha} + \gamma_{11},$$

$$d_{22} = \frac{Q_{22}(b_{2,n+1})}{b_{n+1,n+1} + \beta/\alpha} + \gamma_{22},$$

$$d_{12} = \frac{Q_{12}(b_{1,n+1}, b_{2,n+1})}{b_{n+1,n+1} + \beta/\alpha} + \gamma_{12},$$

where Q_{12} is linear in each variable separately. Thus $Q_{11}Q_{22} - Q_{12}^2 = 0$, so that

$$Q_{11} = k_1(b_{1,n+1} + \delta_1)^2, \qquad Q_{22} = k_2(b_{2,n+1} + \delta_2)^2,$$

$$Q_{12} = k_3(b_{1,n+1} + \delta_1)(b_{2,n+1} + \delta_2).$$

By examining the coefficient of $|B_{n+1}|$ in (30), we obtain $k_1 = k_2 = k_3 = -1$.

Set $\theta_{n+1}(b^{(n+1)}) = \alpha$, $d_{n+1,n+1}^{n+1}(b^{(n+1)}) = \beta/\alpha$, $d_{ij}^{n+1}(b^{(n+1)}) = \gamma_{ij}$, $d_{i,n+1}^{n+1}(b^{(n+1)}) = \delta_i$, $i, j \leqq n$. This is as above with $(i, j) = (1, 2)$. Then according to the above definition, we have, with $b_{n+1} = (b_{1,n+1}, \cdots, b_{n,n+1})$,

$$\begin{aligned} &\theta_{n+1}(b^{(n+1)})\ |B_{n+1} + D^{n+1}(b^{(n+1)})| \\ &\quad = \alpha \begin{vmatrix} B_n + \Gamma & (b_{n+1} + \delta)' \\ b_{n+1} + \delta & b_{n+1,n+1} + \beta/\alpha \end{vmatrix} \\ &\quad = \alpha\left(b_{n+1,n+1} + \frac{\beta}{\alpha}\right)\left|B_n + \Gamma - \frac{(b_{n+1} + \delta)'(b_{n+1} + \delta)}{b_{n+1,n+1} + \beta/\alpha}\right| \\ &\quad = \theta_n(b^{(n)})\ |B_n + D^n(b^{(n)})| = H. \ \| \end{aligned}$$

5. Appendix. We now consider the expression $|I_p - 2T|^{-n/2}$ for fractional $n < p - 1$, and show that it is not the moment generating function (m.g.f.) of a d.f.

Let q be the largest integer less than n. Now let $X = (x_{ij})$ be a $p \times (q + 1)$ random matrix whose elements are independently distributed as follows: each x_{ij} is $N(0, 1)$ if $i > j$, $x_{ij} \equiv 0$ if $i < j$, and x_{ii} $(i = 1, \cdots, q + 1)$ has a χ_{n-i+1} distribution. Then X has a non-singular distribution, and if $XX' \equiv Z$, and the right-hand $(p - q - 1) \times (p - q - 1)$ corner of T is equal to 0, then $|I_p - 2T|^{-n/2} = E \text{ tr } TZ$ is the moment generating function of Z, except for the same right-hand corner. Furthermore, if W has m.g.f. $|I_p - 2T|^{-n/2}$, then W except for the lower right-hand corner can be decomposed uniquely into XX', where X is as above. Thus $W = XX' + U$. But $Eu_{ii} = n - q - 1 < 0$ for $i > q + 1$. Hence W is not positive semi-definite with positive probability.

Consequently, there is a matrix B, not positive semi-definite, every neighborhood of which has positive probability. Let $kn > p - 1$, and let $W_1, \cdots, W_k$ be independent random matrices with m.g.f. $|I_p - 2T|^{-n/2}$. There is a neighborhood N of B such that if $C_i \varepsilon N$, $i = 1, \cdots, k$, C_i is not positive semi-definite. Therefore ΣW_i is not positive semi-definite with positive probability, and hence is not Wishart.

REFERENCE

[1] Lukacs, Eugene (1955). A characterization of the gamma distribution. *Ann. Math. Statist.* **26** 319–324.