ON THE EFFICIENCY OF OPTIMAL NONPARAMETRIC PROCEDURES IN THE TWO SAMPLE CASE[1]

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1. Summary. A series of papers has been published recently dealing with the efficiency of nonparametric procedures in testing statistical hypotheses. A frequently discussed problem is that of the efficiency of nonparametric procedures compared to some parametric methods in the two sample case, when the hypothesis tested asserts no shift versus the alternative that two samples are drawn from two populations with distributions differing only by the location parameter.

Hodges and Lehmann in [5] compared the Wilcoxon test with the t-test for this case. Chernoff and Savage [3] have proved that the Fisher-Yates test compared to the t-test has Pitman's efficiency exceeding one, with equality sign achieved only if the underlying distribution is normal. In [3] it has also been shown that under mild regularity restrictions the optimal nonparametric procedure as compared to the best parametric procedure (in the sense of the likelihood ratio test) has Pitman's efficiency equal to unity assuming that the underlying distribution is known. Also in [3] the authors implicitly stated the following question: "Suppose we construct two tests for the two sample problem, one parametric and one nonparametric for some fixed distribution believed to occur in investigated populations. How does Pitman's efficiency behave if the true distribution departs from the assumed one?" The present investigation deals with this particular problem.

It turns out that among a class of distributions satisfying some regularity conditions, the normal is the only one possessing the property proved in [3].

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2. Assumptions, definitions and notation. Let $X_1X_2 \cdots X_n\ Y_1Y_2 \cdots Y_m$ be independent random variables such that $\Pr\{X_k \leqq z\} = K\{z + (1 - \lambda)\Delta\}$ for $k = 1, 2, \cdots, n$ and $\Pr\{Y_j \leqq z\} = K(z - \lambda\Delta)$ for $j = 1, 2, \cdots, m$, with $K(\cdot)$ being a Lebesgue absolutely continuous distribution function, $\lambda = n/(n + m)$ and $\Delta \geqq 0$ an unknown parameter.

For the purpose of constructing test procedures for the hypothesis $\mathrm{H}: \Delta = 0$

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against the alternative $H:\Delta > 0$ we shall assume that $K(z) = F(z)$, for some completely specified F satisfying regularity restrictions listed below. For construction of nonparametric tests this will be assumed only under the alternative, leaving K arbitrary under the hypothesis, and for construction of parametric tests it will be assumed that $K(z) = F(z)$ for any $\Delta \geqq 0$.

2.1. *Regularity assumptions.* Concerning the common distribution function F we shall make the following regularity assumptions:

Assumption 1. $F(z)$ is twice continuously differentiable for all $z \varepsilon (-\infty; +\infty)$.

Assumption 2. $f(z) = F'(z) > 0$ for all $z \varepsilon (-\infty; +\infty)$ and

$$\int_{-\infty}^{+\infty} [f'(z)/f(z)]^2 f(z+\xi)\, dz \to \int_{-\infty}^{+\infty} [f'(z)/f(z)]^2 f(z)\, dz < \infty \text{ as } \xi \to 0.$$

(This implies also $\int_{-\infty}^{+\infty} f'(z)dz = 0$.)

Assumption 3. $g(z) = -f'(z)/f(z)$ is monotone increasing.

Assumption 4. $F(z)$ is such that either

case (a) $g(-\infty) = c_1$; $g(+\infty) = c_2$ with both c_1 and c_2 finite, or

case (b) $g(-\infty) = -\infty$; $g(+\infty) = +\infty$.

In *case* (b) it will be assumed in addition that $g(z)$ is twice continuously differentiable and $0 < g'(z) < +\infty$ for all $z \varepsilon (-\infty; +\infty)$.

Assumption 5. Let $F^*(z) = F[h(z)]$, where $h(z) = g^{-1}(z)$ (inverse function) and let F^{*-1} be the inverse function of F^*. It will be assumed that:

$$|d^{(i)}F^{*-1}(u)/du^i| \leqq C[u(1-u)]^{-i-\frac{1}{2}+\delta} \qquad i = 0, 1, 2, \tag{2.1.1}$$

for all $u \varepsilon (0; 1)$ for some C and some $\delta > 0$.

Remark 1. If U is a random variable with c.d.f. $F(u)$, density $f(u)$ and the logarithmic derivative of the density $-g(u)$, then $F^*(z)$ is the c.d.f. of $Z = g(U)$.

Remark 2. Using directly the definitions involved we find that Condition (2.1.1) is equivalent to the following:

$$|g(z)| \leqq C\{F(z)\ [1 - F(z)]\}^{\delta-1/2}$$

$$|g'(z)| \leqq C\{F(z)\ [1 - F(z)]\}^{\delta-3/2} f(z)$$

$$|g''(z) + g(z)g'(z)| \leqq C\{F(z)[1 - F(z)]\}^{\delta-5/2} f^2(z).$$

2.2. *Discussion of assumptions.* Most of the assumptions imposed on F are needed in order to make the problem and the asymptotic approach meaningful. That concerns in particular Assumptions 1 and 2 needed for construction of parametric tests and Assumption 5 needed for the asymptotic normality of the nonparametric test statistic. (In [3] p. 974 Chernoff and Savage express the belief that the asymptotic normality holds without this assumption, however this has not yet been proved.)

Additional requirements imposed on F in this paper are therefore Assumptions 3 and 4. Assumption 3 in case (a) (of Assumption 4) is not essential, however, it simplifies the notation and allows a relatively concise formulation of Assumption 5, which otherwise would have to be more involved. In some cases (as with F being a Cauchy distribution) the asymptotic normality of the rank test statistic can be verified directly and Assumption 3 disregarded. Under Assumption 4 (b) Assumption 3 is simply a consequence of 4 (b) and hence does not impose additional restrictions. The restrictions on the derivative of g in 4 (b) allow the use of variational technique in the proof of the theorem. Although it seems to be likely that the result of the theorem remains valid if one of the c's, defined in Assumption 4, is finite and the other infinite, the method of the proof does not work in that case and this is the reason for considering cases (a) and (b) only.

3. Optimal tests. The locally most powerful rank test for testing the hypothesis $H:\Delta = 0$ versus $\bar{H}:\Delta > 0$ has the critical function ([6] p. 237)

$$\Phi_{T_N^*}(X; Y) = \begin{cases} 1 & T_N^* \geqq c_\alpha \\ 0 & T_N^* < c_\alpha \end{cases} \tag{3.1}$$

where

$$T_N^* = \sum_{j=1}^{N} E\{g[V^{(j)}]\} Z_j \tag{3.2}$$

with $V^{(j)}$ being jth order statistic in the joint sample of X's and Y's and $Z_j = 0$ or 1 according to whether the jth order statistic is an observation of one of Y's or X's. The expectation $E\{\cdot\}$ is computed under the hypothesis $\Delta = 0$, and $N = n + m$.

The asymptotically optimal parametric test in the sense of [7] has the critical function

$$\Phi_{T_N}(X; Y) = \begin{cases} 1 & T_N \geqq c_\alpha \\ 0 & T_N < c_\alpha \end{cases} \tag{3.3}$$

where

$$T_N = -(1-\lambda)\sum_{i=1}^{n} g(X_i) + \lambda \sum_{j=1}^{m} g(Y_j) \tag{3.4}$$

and λ defined in Section 2 is assumed to satisfy the relation $0 < \lambda < 1$. (One can show that this test is equivalent to a large sample likelihood ratio test).

Assume now that N increases. For reasons of simplicity we assume that λ is constant and hence N runs through a subset of integers defined by λ. All conclusions remain valid if we allow N to take on all integer values in such a way that corresponding $\lambda_N \to \lambda \neq 0$ or 1.

Pitman's efficiency can be defined in several equivalent versions. We shall quote here the following:

If for a statistic T_N there are functions $\alpha_N(\Delta)$ and $\beta_N(\Delta)$ such that for Δ in a vicinity of zero $\mathcal{L}[(T_N - \alpha_N(\Delta))/\beta_N(\Delta)] \to N(0; 1)$ and $\beta_N(\Delta_N)/\beta_N(0) \to 1$, then

$$E_T = \lim_{N\to\infty}\left[\frac{\alpha_N(\Delta_N) - \alpha_N(0)}{\Delta_N N^{1/2}\beta_N(0)}\right]^2$$

with $\Delta_N = 0\ (N^{-\frac{1}{2}})$ is called the efficacy of the T procedure, provided this limit exists.

The comparison of two procedures T^* and T reduces to comparison of two sequences of sample sizes N_k^* and N_k with the property, that at the same significance level α, values N_k^* and N_k give the same power of $T_{N_k^*}$ and T_{N_k} correspondingly for some alternative value Δ close to zero. (This involves a slight difficulty in achieving exactly the same value of the power, which can be removed by randomization. The question is well known and we shall not discuss it here.)

The limit

$$e(F;F) = \lim_{N_k, N_k^* \to \infty} N_k/N_k^*$$

is called Pitman's efficiency of T^* with respect to T.

It is known [3], that $e(F;F) = E_{T^*} / E_T$ provided the proportions of X's in joint samples tend to a common limit. This requirement was assumed to be satisfied, in our case, in the remark following Formula 3.4. Now suppose that the true distribution of X's and Y's is not F but some Ψ with the density ψ. The question to be investigated is how our tests constructed for F will perform under Ψ. On Ψ we must put mild restrictions to assure normal limits of T_N and T_N^*.

ASSUMPTION 6. We shall assume throughout this paper that Ψ is such that the integral $\int_{-\infty}^{+\infty} g^2(x)\ \psi\ (x)\ dx$ is finite and T_N^*, properly standardized, has under Ψ a normal limit.

By κ_F we shall denote the class of distributions satisfying Assumption 6 for some fixed F.

We define e_F in the following way: $e_F = \inf_{\Psi \varepsilon \kappa_F} e(F;\Psi)$ where $e(F;\Psi)$ is Pitman's efficiency of the LMPR test to the asymptotically optimal test in the sense of [7], both derived for F, with $e(F;\Psi)$ computed under Ψ.

4. Main theorem. We shall prove the following:

THEOREM. *Under Assumptions* 1–6 *the relation*

$$e_F = 1 \tag{4.1}$$

holds if and only if F is $N(\mu;\sigma)$ for some $\mu \,\varepsilon\, (-\infty; +\infty)$ and $\sigma \,\varepsilon\, (0; +\infty)$.

The sufficiency part has been proved in [3] and it remains to prove the necessity. We shall proceed in the following way.

First we shall compute Pitman's efficiency $e(F;\Psi)$. Next it will be shown that if F satisfies the Assumption 4(a) then $e_F = 0$. Hence if F satisfies Assumptions 1–5 and $e_F = 1$, then F necessarily satisfies 4(b). To conclude the proof we will show by variational methods that Assumption 4(b) together with Formula 4.1 necessarily imply normality of F.

4.1. *Derivation of Pitman's efficiency.* Let us denote by J the function

$$J(z) = g[F^{-1}(z)] \tag{4.1.1}$$

where $F^{-1}(\cdot)$ is the inverse function of the c.d.f. $F(\cdot)$. From [3] we find

$$\alpha_N^*(\Delta) = N\lambda \int_{-\infty}^{+\infty} J[\lambda\Psi(y) + (1-\lambda)\Psi(y-\Delta)]\psi(y)\ dy$$

and hence

$$\alpha_N^{*\prime}(0) = -N\lambda(1-\lambda)\left.\frac{d}{d\Delta}\right|_{\Delta=0}$$

$$(4.1.2)\qquad -\frac{1}{1-\lambda}\int_{-\infty}^{+\infty} J[\lambda\Psi(y) + (1-\lambda)\Psi(y-\Delta)]\psi(y)\,dy$$

$$= -N\lambda(1-\lambda)I_{1\Psi}\,.$$

This can be written $\alpha_N^{*\prime}(0) = -N\lambda(1-\lambda)\int_{-\infty}^{+\infty} J'[\Psi(y)]\psi^2(y)dy$ provided the differentiation can be carried out under the integral. Also from [3] (p. 978) we find

$$\beta_N^{*2}(0) \cong N\lambda(1-\lambda)\left\{\int_0^1 J^2(u)\,du - \left[\int_0^1 J(u)\,du\right]^2\right\}$$

where

$$\int_0^1 J^2(u)\,du = \int_0^1 g^2[F^{-1}(u)]\,du = \int_{-\infty}^{+\infty} g^2(x)f(x)\,dx$$

and

$$\int_0^1 J(u)\,du = -\int_{-\infty}^{+\infty} f'(x)\,dx = 0.$$

Hence we obtain

$$(4.1.3)\qquad \beta_N^{*2}(0) \cong N(1-\lambda)\lambda\,\mathrm{Var}_F\,g = N\lambda\,(1-\lambda)I_2\,,$$

where I_2 is also known as $\inf_{F_0}$, that is the information of the c.d.f. $F(z-\Delta)$ at $\Delta = 0$.

(The difference in constants in (4.1.2) and (4.1.3) compared to [3] occurs because of slightly different definition of T_N^*.) Now let us find α' and β corresponding to T. We have

$$\alpha_N(\Delta) = -E\left[(1-\lambda)\sum_{i=1}^{n} g(X_i) - \lambda\sum_{j=1}^{m} g(Y_j)\right]$$

and

$$Eg(X) = \int_{-\infty}^{+\infty} g(x)\psi[x + (1-\lambda)\Delta]\,dx$$

$$Eg(Y) = \int_{-\infty}^{+\infty} g(y)\psi[y - \lambda\Delta]\,dy.$$

Hence

$$(4.1.4)\qquad \alpha_N'(0) = N\lambda(1-\lambda)\left.\frac{d}{d\Delta}\right|_{\Delta=0}\int_{-\infty}^{+\infty} g(x+\Delta)\psi(x)\,dx = N\lambda(1-\lambda)I_{3\Psi}$$

and finally

$$\beta_N^2(0) = N\lambda(1-\lambda) \operatorname{Var}_\Psi g = N\lambda(1-\lambda)I_{4\Psi}. \tag{4.1.5}$$

Hence we find

$$e_N(F:\Psi) \cong \left[\frac{N\lambda(1-\lambda)I_{1\Psi}}{N\lambda(1-\lambda)I_{3\Psi}}\right]^2 \frac{N\lambda(1-\lambda)I_{4\Psi}}{N\lambda(1-\lambda)I_2}$$

and

$$e(F;\Psi) = [I_{1\Psi}/I_{3\Psi}]^2\,[I_{4\Psi}/I_2]. \tag{4.1.6}$$

We shall remark here, that there is no restriction of generality in assuming that:

$$I_2 = 1. \tag{4.1.7}$$

This follows from the fact that $e(F;\Psi)$ remains invariant under multiplication of all observations by a constant.

4.2. *A special case.* Let us consider the following:

LEMMA 1. *If $k(z)$ is a continuous bounded function on $(-\infty; +\infty)$ and there exists a finite number M such that k is monotone on $(M; +\infty)$ and $R(a_1; a_2)$ is a rectangular distribution on the interval $(a_1; a_2)$ then* $\lim_{a_2\to+\infty} \operatorname{Var}[k(Z) \mid R(a_1; a_2)] = 0.$ for any a_1.

Proof of this lemma is immediate and will be omitted. Using Lemma 1 we can prove a special case of the main problem contained in Lemma 2.

LEMMA 2. *Under Assumptions* 1, 2, 3, 4 (a), 5, 6, $e_F = 0$.

PROOF. Let $\Psi = R(a_1; a_2)$. Inserting this in (4.1.6), after elementary computations, we obtain

$$e(F;R) = \left[\frac{c_2 - c_1}{g(a_2) - g(a_1)}\right]^2 \frac{\operatorname{Var}_R g}{\operatorname{Var}_F g}.$$

Hence by Assumption 3 and Lemma 1

$$e_F = \lim_{a_2\to\infty} e(F;R) = \left[\frac{c_2 - c_1}{c_2 - g(a_1)}\right]^2 \lim_{a_2\to\infty} \frac{\operatorname{Var}_R g}{\operatorname{Var}_F g} = 0$$

4.3. *Proof of the theorem.* From Lemma 2 it follows that the relation $e_F = 1$ implies case (b) of Assumption 4. It remains to show that this implies the normality of F.

Let $\kappa_{1F} \subset \kappa_F$ be the class of distributions Ψ such that the identities

$$I_{1\Psi} = \int_{-\infty}^{+\infty} J'[\Psi(x)\psi^2(x)\, dx$$

$$I_{3\Psi} = \int_{-\infty}^{+\infty} g'(y)\psi(y)\, dy$$

hold. Then $\inf_{\Psi\varepsilon\kappa_F} e(F;\Psi) \leqq \inf_{\Psi\varepsilon\kappa_{1F}} e(F;\Psi)$ and it suffices to show that $\inf_{\Psi\varepsilon\kappa_{1F}} e(F;\Psi) < 1$ unless F is normal. Using (4.1.7) we can see that the inequality

$$[I_{1\Psi}/I_{3\Psi}]^2\, I_{4\Psi} \geqq 1 \tag{4.3.1}$$

is equivalent to the inequality

$$I_{1\Psi}\,(I_{4\Psi})^{\frac{1}{2}} - I_{3\Psi} \geqq 0. \tag{4.3.2}$$

Let $\Psi^*(z) = \Psi[h(z)]$ and ψ^* be the density of Ψ^*. Let us denote

$$\sigma = (I_{4\Psi})^{\frac{1}{2}} \tag{4.3.3}$$

and define $x(J)$ by

$$\Psi^*[x(J)] = F^*(J). \tag{4.3.4}$$

By substitution $y = h(x)$; $dy = h'(x)\,dx$ in $I_{1\Psi}$ we obtain

$$I_{1\Psi} = \int_{-\infty}^{+\infty} J'[\Psi(h(x))]\psi^2[h(x)]h'(x)\,dx = \int_{-\infty}^{+\infty} J'[\Psi^*(x)]\,\frac{\psi^{*2}(x)}{h'(x)}\,dx. \tag{4.3.5}$$

From (4.3.4) it follows that

$$\begin{aligned} \psi^*(x)\,dx &= f^*(J)\,dJ \\ \psi^*(x) &= f^*(J)/x'. \end{aligned} \tag{4.3.6}$$

On the other hand by (4.1.1) we have $J[F^*(u)] = u$. Hence $F^*[J(z)] = z$. Differentiating both sides of this identity with respect to z we obtain

$$J'(z) = \{1/f^*[J(z)]\}.$$

Now if $z = \Psi^*(x)$, then using (4.3.4) we find

$$J'[\Psi^*(x)] = [1/f^*(J)]. \tag{4.3.7}$$

Hence by (4.3.5), (4.3.6) and (4.3.7), we obtain for $I_{1\Psi}$

$$I_{1\Psi} = \int_{-\infty}^{+\infty} \frac{f^*(J)\,dJ}{h'(x)x'}. \tag{4.3.8}$$

Now let us consider $I_{3\Psi}$

$$I_{3\Psi} = \int_{-\infty}^{+\infty} g'(y)\psi(y)\,dy.$$

Substituting $g(y) = x$; $g'(y)\,dy = dx$; $y = h(x)$ we have

$$I_{3\Psi} = \int_{-\infty}^{+\infty} \psi[h(x)\,dx] = \int_{-\infty}^{+\infty} [\psi^*(x)/h'(x)]\,dx$$

and hence

$$I_{3\Psi} = \int_{-\infty}^{+\infty} \frac{f^*(J)}{h'(x)}\,dJ. \tag{4.3.9}$$

Combining (4.3.8) and (4.3.9) we can write (4.3.2) in the form

$$\int_{-\infty}^{+\infty} \frac{f^*(J)}{h'(x)x'}\,(\sigma - x')\,dJ \geqq 0 \tag{4.3.10}$$

where σ in terms of $x(J)$ is defined by formulae (4.3.11), obtained from relations (4.1.5), (4.3.3) and (4.3.4).

$$(4.3.11) \quad \int_{-\infty}^{+\infty} x(J)f^*(J)\, dJ = \mu_1 \, ; \quad \int_{-\infty}^{+\infty} x^2(J)f^*(J)\, dJ = \mu_2 \, ; \quad \mu_2 - \mu_1^2 = \sigma^2.$$

From definition (4.3.4) it follows that $x(J)$ is a nondecreasing finite function, mapping the whole real line onto itself. Hence $x(J)$ satisfies side conditions

$$(4.3.12) \qquad x(-\infty) = -\infty; \qquad x(+\infty) = +\infty.$$

Let us consider the functional on the left hand side of the inequality (4.3.10). The minimization of this functional for each class of functions $x(J)$, satisfying side conditions (4.3.11) and (4.3.12) for fixed μ_1 and σ, is equivalent to the minimization of

$$(4.3.13) \qquad \int_{-\infty}^{+\infty} \left[\frac{\sigma - x'}{x'h'(x)} - a(\sigma; \mu_1)x - b(\sigma; \mu_1)x^2\right] f^*(J)\, dJ,$$

where $a(\sigma; \mu_1)$ and $b(\sigma; \mu_1)$ are Lagrange multipliers.

A necessary condition for a function x to minimize (4.3.13) is that x satisfies Euler's differential equation $(\partial G/\partial x) - (d/dJ)\,(\partial G/\partial x') = 0$ where G is the integrand of (4.3.13).

For our particular G we obtain

$$(4.3.14) \quad \begin{aligned} &-\frac{h''(x)}{h'(x)^2}\left(\frac{2\sigma}{x'} - 1\right) + \frac{\sigma}{(x')^2}\frac{h''(J)}{h'(J)h'(x)} \\ &\qquad - \sigma J\, \frac{h'(J)}{h'(x)(x')^2} - a - 2bx - \frac{2x''\sigma}{h'(x)(x')^3} = 0. \end{aligned}$$

We can observe that the function: $x_0(J) = \sigma J + \mu_1$.

(1) Satisfies side conditions (4.3.11) and (4.3.12).

(2) Gives the left-hand side of (4.3.10) equal to zero and hence $e(F; \Psi_0) = 1$ (for Ψ_0 corresponding to x_0 by (4.3.4)).

(3) Does not satisfy (4.3.14) for all σ and μ_1 unless F is normal. The first two conclusions are immediate. Conclusion 3 follows from the next Lemma 3. Before we state the lemma let us first rewrite (4.3.14). Substituting x_0 for x in (4.3.14) we obtain

$$(4.3.15) \quad \begin{aligned} &h'(J)\left[-J + \frac{h''(J)}{h'(J)^2}\right] = \sigma h'(\sigma J + \mu_1) \\ &\qquad \cdot \left[2b(\sigma; \mu_1)(\sigma J + \mu_1) + \frac{h''(\sigma J + \mu_1)}{h'(\sigma J + \mu_1)^2} + a(\sigma; \mu_1)\right]. \end{aligned}$$

LEMMA 3. *A unique solution for F (induced by h by definitions of Section 2.1) of the functional-differential equation* (4.3.15), *with a and b being arbitrary functions, such that F satisfies Assumptions* 1, 2, 3, 4(b), 5 *of Section* 2.1, *is a normal distribution.*

Proof. We shall introduce a transformation $\tau = \sigma J + \mu_1$; $\delta = \frac{1}{\sigma}$; $\xi = -\frac{\mu_1}{\sigma}$. We can easily observe that the range of τ coincides with that of J $(-\infty; +\infty)$, δ varies in $(0; \infty)$ the same as σ and ξ in $(-\infty; +\infty)$ the same as μ_1. Hence the identity (4.3.15) implies the identity

$$-\delta h'(\delta\tau + \xi)\left[\delta\tau - \frac{h''(\delta\tau + \xi)}{h'(\delta\tau + \xi)^2} + \xi\right] \underset{\tau;\delta;\xi}{\equiv} h'(\tau)\left[2b\left(\frac{1}{\delta}; -\frac{\xi}{\delta}\right)\tau + \frac{h''(\tau)}{h'(\tau)^2} + a\left(\frac{1}{\delta}; -\frac{\xi}{\delta}\right)\right].$$

This can be rewritten in the form

$$\begin{aligned} &\left\{\delta h'(\delta\tau + \xi)\left[2b(\delta; \xi)(\delta\tau + \xi) + \frac{h''(\delta\tau + \xi)}{h'(\delta\tau + \xi)^2} + a(\delta; \xi)\right]\right. \\ &\left. + h'(\tau)\left[\tau - \frac{h''(\tau)}{h'(\tau)^2}\right]\right\} - \delta h'(\delta\tau + \xi)[2b(\delta; \xi)(\delta\tau + \xi) \\ &+ a(\delta; \xi) + \xi + \delta\tau] - h'(\tau)\left[2b\left(\frac{1}{\delta}; -\frac{\xi}{\delta}\right)\tau + \tau + a\left(\frac{1}{\delta}; -\frac{\xi}{\delta}\right)\right] = 0. \end{aligned} \tag{4.3.16}$$

By the identity (4.3.15) the first bracket is identically equal to zero and hence (4.3.16) implies

$$\begin{aligned} &- h'(\delta\tau + \xi)\{[2b(\delta; \xi) + 1]\delta^2\tau + [2b(\delta; \xi) + 1]\delta\xi \\ &+ a(\delta; \xi)\delta\} = h'(\tau)\{[2b(1/\delta; -\xi/\delta) + 1]\tau + a(1/\delta; -\xi/\delta)\} \end{aligned} \tag{4.3.17}$$

If we consider ξ and δ as arbitrary but fixed and vary τ, then both sides of 4.3.17 vanish identically only if

$$\begin{aligned} 2b(\delta; \xi) &= -1; \qquad & 2b(1/\delta; -\xi/\delta) &= -1 \\ a(\delta; \xi) &= 0 & a(1/\delta; -\xi/\delta) &= 0. \end{aligned} \tag{4.3.18}$$

Now let us suppose that for some $\delta \neq 1$ and for some ξ 4.3.18 is not satisfied. Then for those τ for which both sides are different from zero we can write:

$$\frac{h'(\tau)}{h'(\sigma\tau + \xi)} = -\frac{[2b(\delta;\xi) + 1]\delta^2\tau + [2b(\delta;\xi) + 1]\delta\xi + a(\delta;\xi)\delta}{[2b(1/\delta; -\xi/\delta) + 1]\tau + a(1/\delta; -\xi/\delta)}. \tag{4.3.19}$$

Since the left-hand side is positive by assumption and the right-hand side, considered as a function of τ, by fixed ξ and δ, is the ratio of two linear functions, we can conclude that the right-hand side is positive for all $\tau \varepsilon (-\infty\ +\infty)$ only if it is a constant. Since δ was assumed to be different from one, taking $\tau = -\xi/(\delta - 1)$ we find that the left-hand side is equal to one (by Assumption 4(b)) and we can conclude that

$$[h'(\tau)/h'(\delta\tau + \xi)] \underset{\tau}{\equiv} 1. \tag{4.3.20}$$

On the other hand for fixed δ and τ the left-hand side of (4.3.19) is a continuous finite function of ξ (by Assumption 4(b)). Hence the numerator and the denominator of (4.3.19) have the same set of zeros in ξ. This implies that the relation (4.3.20) considered as a function of ξ holds, provided that for some $\delta \neq 1$ some ξ and τ, (4.3.19) is defined. Under this assumption it follows that $h'(\tau)$ is necessarily constant.

On the other hand, if (4.3.19) is not defined for any $\delta \neq 1$ and any ξ, we have $2b(\delta;\xi) = -1; a(\delta;\xi) = 0$ for all $\delta \neq 1$ and all ξ. If that is the case, then denoting the left-hand side of (4.3.15) by φ, we have $\varphi(\tau) = \delta\varphi(\delta\tau + \xi)$ for any $\delta \neq 1$ all τ and all ξ. This is possible only if $\varphi(\tau) = 0$. Since $h' > 0$ (by Assumption 4(b)) this is possible only if h satisfies the following differential equation

$$[h''(\tau)/h'(\tau)^2] = \tau.$$

Solution of this differential equation leads to

$$-h(\tau) = \begin{cases} 2(2c)^{-\frac{1}{2}} \text{ arc tg}[\tau/(2c)^{\frac{1}{2}}] + \mu & c > 0 \\ -2/\tau + \mu & c = 0 \\ (-2c)^{-\frac{1}{2}} \log |[\tau - (-2c)^{\frac{1}{2}}]/[\tau + (-2c)^{\frac{1}{2}}]| + \mu & c < 0. \end{cases}$$

By Assumption 4(b) h is a monotone continuous function, mapping $(-\infty; +\infty)$ onto itself. Neither of these solutions satisfies this condition. Hence the case $\varphi(\tau) = 0$ can be excluded.

We have shown here that $h' =$ const. and hence $h(\tau) = \sigma\tau + \mu$ which gives $g(\tau) = h^{-1}(\tau) = (\tau - \mu/\sigma)$ and $g(\tau) = -f'(\tau)/f(\tau)$. Hence f is necessarily a normal density which concludes the proof of Lemma 3 and of the theorem.

5. Examples and concluding remarks.

EXAMPLE 1. Logistic distribution.

$$F(x) = 1/[1 + e^{-(x-\mu)/\sigma}].$$

Assumptions 1, 2, 3, 4(a), 5 can be verified. The LMPR test is the Wilcoxon test. The lower bound of its efficiency compared to the optimal parametric test in the sense of [7] is (by Lemma 2) $e_F = 0$.

EXAMPLE 2. Cauchy distribution. Let

$$F(x) = \frac{1}{\pi}\frac{1}{\sigma}\int_{-\infty}^{x} \frac{dy}{1 + [(y-\mu)/\sigma]^2}.$$

Here g is not monotone. However asymptotic normality of T_N^* can be verified directly by means of Corollary 1 of [3]. Hence we find $e_F = 0$.

EXAMPLE 3. Mixture of two normal distributions.

$$F(x) = \int_{-\infty}^{x} [\eta\varphi_1(z) + (1-\eta)\varphi_2(z)]\, dz \qquad 0 < \eta < 1,$$

with

$$\varphi_1(z) = \varphi(z - \mu)$$

$$\varphi_2(z) = \varphi(z - \xi)$$
$$\varphi(z) = (2\pi)^{-\frac{1}{2}} \exp(-z^2/2).$$

It can be verified that Assumptions 1, 2, 3, 4(b), 5 are satisfied if $|\xi - \mu| < 2$. In this case we can only conclude that, $e_F < 1$.

EXAMPLE 4. As another example we can consider the distribution

$$F(x) = c \int_{-\infty}^{x} e^{-\cosh(y-\mu)/\sigma}\, dy.$$

As in Example 3 we find that it satisfies assumptions 1, 2, 3, 4(b), 5 and $e_F < 1$.

It could be interesting to find explicitly e_F in case (b) of Assumption 4.

Another kind of problem of interest could be to find e_F for restricted classes of distributions Ψ. For one such class when $\Psi(x) = F[(x - \mu/\sigma]$ the answer is an immediate consequence of formula (4.1.6) and is contained in the following:

LEMMA 4. *For* $\Psi(x) = F[(x - \mu)/\sigma]$ *we have*

$$e(F; \Psi) \geqq 1 \tag{5.1}$$

and $e_F = 1$. *If either* $\mu \neq 0$ or $\sigma \neq 1$ *then the equality in* (5.1) *is achieved only if* F *is normal and* $e(F; \Psi) > 1$ *for any other* F.

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