ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Eastern Regional Meeting, Cambridge, Massachusetts, May 6–7, 1963. Additional abstracts appeared in the March, 1963 issue.*)

6. **Asymptotic Joint Distribution of Quantiles From a Bivariate Distribution.**

J. B. Bartoo, W. L. Harkness and O. P. Srivastava, Pennsylvania State University.

Let $(X_i, Y_i), i = 1, 2, \cdots, n$ be a random sample of size n from an absolutely continuous d.f. $F(x, y)$, with joint p.d.f. $f(x, y)$. Let $Z_1^{(n)} \leqq Z_2^{(n)} \leqq \cdots \leqq Z_n^{(n)}$ and $W_1^{(n)} \leqq W_2^{(n)} \leqq \cdots \leqq W_n^{(n)}$ be the ordered values of $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ respectively. Further, let ξ_α, η_β be the unique real numbers satisfying $F_1(\xi_\alpha) = \alpha$, $F_2(\eta_\beta) = \beta$, where $F_1(x)$ and $F_2(y)$ are the marginal distribution functions of X and Y (with marginal p.d.f.'s $f_1(x)$ and $f_2(y)$) and $f_1(\xi_\alpha) > 0$, $f_2(\eta_\beta) > 0$. Also let r_n and s_n be two sequences of positive integers such that $\lim_{n\to\infty} r_n/n = \alpha$, $\lim_{n\to\infty} s_n/n = \beta$, $0 < \alpha < 1$, $0 < \beta < 1$, so that $Z_{r_n}^{(n)}$ and $W_{s_n}^{(n)}$ are sample quantiles of order α and β.

If

$$\Sigma = \begin{pmatrix} \alpha(1-\alpha) & \alpha_\beta - q_1 \\ \alpha_\beta - q_1 & \beta(1-\beta) \end{pmatrix},$$

where $q_1 = F(\xi_\alpha, \eta_\beta)$ is non-singular, then the limiting joint distribution of the r.v.'s $n(Z_{r_n}^{(n)} - \xi_\alpha) f_1(\xi_\alpha)$, $n(W_{s_n}^{(n)} - \eta_\beta) f_2(\eta_\beta)$ is a bivariate normal distribution with mean vector 0 and variance covariance matrix Σ. We note that $Z_{r_n}^{(n)}$, $W_{s_n}^{(n)}$ are asymptotically independent if and only if $F(\xi_\alpha, \eta_\beta) = q_1 = \alpha\beta = F_1(\xi_\alpha) F_2(\eta_\beta)$. This result has obvious generalizations.

7. **On the Pessimum Interference With Random Signals** (Preliminary report).

Nelson M. Blachman, Sylvania Electronic Defense Laboratories, Mountain View, California.

For a communication channel accepting as input one real number per unit time of variance $\leqq P$, adding to it independent normal noise of variance N and interference of variance $\leqq J$ which may depend on the signal (sequence of $n \to \infty$ numbers representing one of $M \to \infty$ messages) being sent and the $M - 1$ alternative signals, the channel capacity can be approached by selecting the M signals randomly from among the vectors of length $(nP)^{\frac{1}{2}}$ for sufficiently large N (*Trans. Information Theory* IT-8 (1962), 48–55 and S53–S57). The proof depends on establishing that the worst interference with such random signals is either (1) half the difference between the transmitted signal and a nearest alternative signal or (2) that negative fraction of the transmitted signal plus additional noise which minimizes the effective signal-to-noise ratio. To show no other sort of interference can do worse, we suppose J too small for (1), and (2) unlikely to cause errors. Consequently, the caps which the faces of the Dirichlet region of the transmitted signal cut from the "noise sphere" of radius $(nN)^{\frac{1}{2}}$ surrounding the sum of the transmitted signal plus interference have radii $< \frac{1}{2}\pi - \epsilon$, and no more than a bounded number of them cover any point of the noise sphere (*Ann. Math. Statist.* **32** (1961) 916). Hence (N. M. Blachman and L. Few, "Multiple Packing of Caps on a Sphere", *Mathematika* **10** (1963)), only a vanishing fraction of the surface of the noise sphere lies outside the Dirichlet region, and the error probability $\to 0$.

8. An Analysis for Modified Triangle Tests in Sensory Difference Experimentation. Ralph A. Bradley and Tom J. Harmon, Florida State University.

In a triangle sensory difference test two like samples and an odd sample are presented for sensory selection of the odd sample. It may be assumed that stimulus responses x_1, x_2 and y exist on a subjective continuum and are normal, independent, have common variance σ^2 and means, 0, 0, μ respectively. In this way Bradley (*Biometrics* **14** (1958) 566, **19** (1963) in press) and Ura (*Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs.* **7** (1960) 107–119) obtained p_Δ, the probability of correct selection, in terms of μ/σ and certain relationships with other sensory tests. The determination of correct selection was based on the configurations and spacings of x_1, x_2, y.

In a modified triangle test, the respondent also scores the intensity of the supposed difference. If the selection is correct, he approximates to $W = |y - \frac{1}{2}(x_1 + x_2)|$ and if incorrect to $Q = |x_1 - \frac{1}{2}(y + x_2)|$ or $|x_2 - \frac{1}{2}(y + x_1)|$. The conditional probability density function for W is $f(W \mid \Delta) = 4(3\pi\sigma^2)^{-\frac{1}{2}} \cdot I(2^{\frac{1}{2}}W/3\sigma) \cosh (2\mu W/3\sigma^2) [\exp\{-(W^2 + \mu^2)/3\sigma^2\}]/p_\Delta$ where $I(A)$ is the integral of the standard normal p.d.f. from 0 to A. Similarly, $f(Q \mid \Delta') = 4(3\pi\sigma^2)^{-\frac{1}{2}}[I\{(2^{\frac{1}{2}}Q/3\sigma) - (\mu/2^{\frac{1}{2}}\sigma)\} + I\{(2^{\frac{1}{2}}Q/3\sigma) + (\mu/2^{\frac{1}{2}}\sigma)\}] \cosh (\mu Q/3\sigma^2) \cdot [\exp\{-(Q^2 + \frac{1}{4}\mu^2)/3\sigma^2\}]/p_{\Delta'}$ where $p_{\Delta'} = 1 - p_\Delta$ and Δ' indicates incorrect selection. The likelihood function for N triangle tests is developed, estimates of μ and σ from observed values of W and Q are obtained by iterative means, and the large-sample, likelihood-ratio test criteria is applied. Applications to experimental data have been made with apparent success.

9. On the Estimation of Mixing Distributions. D. C. Boes, Bell Telephone Laboratories, Inc., Holmdel, New Jersey.

Let $\mathfrak{H} = \{H_\theta(x):H_\theta(x) = \sum_{i=1}^{k+1} \theta_i F_i\ (x),\ \theta_i > 0,\ i = 1, \cdots, k + 1,\ \sum_{i=1}^{k+1} \theta_i = 1\}$ be the family of finite mixtures of any fixed set of $k + 1$ (distinct) distribution functions (c. d. f.'s) $F_1, \cdots, F_{k+1}$. Estimation of the parameter $\theta = (\theta_1, \cdots, \theta_k)$ $(\theta_{k+1} = 1 - \sum_{i=1}^{k} \theta_i)$ for identifiable (Teicher, "Identifiability of Mixtures," *Ann. Math. Statist.* **32** (1961) 244–248) families $\mathfrak{H}$ is considered. Estimation of the mixing ratio ($k = 1$ in $\mathfrak{H}$) is discussed at length. Identifiability is then evident. Necessary and sufficient conditions (NSC) for the uniform attainment of the Cramér-Rao lower bound are derived. The class of θ-efficient estimators is found; also, the minimax unbiased estimator is a member of this class and it is characterized. NSC for the identifiability of finite mixtures ($\mathfrak{H}$) are given. It is proved that under mild regularity conditions a NSC for the existence of an estimator uniformly attaining the minimal ellipsoid of concentration is that the density be of Darmois-Koopman form. This result is utilized to give NSC for the existence of an estimator which uniformly attains the minimal ellipsoid of concentration when the density is a finite mixture. The θ-efficient family of estimators is derived; also, those estimators within the θ-efficient family which are CANE (consistent asymptotically normal efficient) are characterized.

10. On Certain Bounds Useful in the Theory of Factorial Experiments and Error-Correcting Codes. R. C. Bose and J. N. Srivastava, University of North Carolina.

Let the function $m_t(r, s)$ denote the maximum number of points that one could choose in an $(r - 1)$-dimensional projective space $PG(r - 1, s)$ so that no t of the points are dependent. This function is of great interest in the theory of factorial experiments and also of error-correcting codes. In this paper the following bound has been obtained for this function for the case $t = 3$: $N_r \geqq m + \binom{m}{2}(s-1)^2/f(m)$, where $f(m) = (s-1) + \frac{1}{2}(m-2)(s-2)$ where N_r is the total number of points in $PG(r-1, s)$. This bound is better than the previous known ones for s even $(s > 2)$ and also for some odd values of s, including $s = 3, 5$.

11. **The Present Value of a Renewal Process** (Preliminary report). Giorgio Dall'Aglio, University of North Carolina. (Introduced by G. E. Nicholson, Jr.)

Given a renewal process with inter-arrivals times X_i equally and independently distributed with d.f. $F(x)$, the "present value" of the renewal process is defined by $C = \sum_1^\infty \exp(-\rho T_i)$, where $T_0 = 0$, $T_i = X_1 + X_2 + \cdots + X_i$, and ρ is the force of interest. Thus C is the sum of the costs of renewals (supposed all equal to 1), considered at time T_0. An explicit expression for the characteristic function of C is found when X_1 is negatively exponentially distributed. From this expression, it is easily seen that the distribution of C tends to the normal distribution as ρ tends to zero. The asymptotic normality of C can be investigated also for other classes of distributions. We have thus: *Theorem 1*. If all the moments of X_1 exist, then C is asymptotically normally distributed as ρ tends to zero. The existence of all the moments of X_1 does not appear to be necessary for the asymptotic normal distribution of C. However, the following can be established: *Theorem 2*. If $EX_1^{2r+\delta} < \infty$, $EX_1^{2r+\delta'} = \infty$, for some positive integer r, and δ, δ' with $0 < \delta < \delta' < 1$, then EC^{4r} tends to infinity as ρ tends to zero.

12. **Optimum Allocation of Measurements** (Preliminary report). M. H. DeGroot, Carnegie Institute of Technology. (Invited)

The following example, initially suggested by J. Marschak, is a prototype of the problems studied. Consider a population of coins. Associated with each coin is a probability Z of heads and it is desired to make some inference about the distribution of Z in the population. One can select a random sample of k coins from the population and toss the ith coin n_i times $(i = 1, \cdots, k)$. Given that the total number n of tosses is fixed, the problem is to find an optimum choice of k and $n_1, \cdots, n_k$. An optimum allocation will in general depend on the family of possible distributions of Z being considered as well as on the type of inference to be made. However, for some families of distributions it can be shown that one allocation, say D^*, is optimum regardless of the type of inference to be made. This occurs when the allocation D^* is sufficient, in the sense of Blackwell (*Ann. Math. Statist.* **24** (1953) 265–272), for any other allocation. In the general allocation problem, Z is an arbitrary random variable with an unknown distribution and the (known) conditional distribution of the observations given $Z = z$ need not be binomial.

13. **Asymptotic Efficiencies of the Moment Estimators for the Parameters of the Weibull Laws** (Preliminary report). Satya D. Dubey, Procter & Gamble Company, Cincinnati, Ohio. (By title)

Let the probability density function of a random variable X be represented by $f_X(x) = m_0\theta^{-1}(x-G)^{m_0-1}\exp[-\theta^{-1}(x-G)^{m_0}]$, $x > G$, $\theta > 0$ and m_0 (> 0) known, and $f_X(x) = 0$ otherwise. The asymptotic properties of the moment estimators for G and θ are investigated. The expression for the covariance matrix of these estimators, valid for large samples, is derived. In a previous paper entitled, "On Some Statistical Inferences for Weibull Laws" this author has considered the maximum likelihood estimators for the parameters of Weibull laws and derived the large sample covariance matrix of such estimators. It is valid when the shape parameter is larger than two (*Ann. Math. Statist.* **33** (1962) 1504–1505). Taking the determinant of a covariance matrix as a measure of the generalized variance, the joint asymptotic efficiency of the moment estimators for G and θ with respect to their maximum likelihood estimators is obtained. Some special cases are discussed. The asymptotic efficiencies of moment estimators depend only on the shape parameter, m_0. For some values of m_0, the moment estimators seem to be very highly efficient. The asymptotic properties of the moment estimators allow us to consider testing of hypotheses and construction of confidence regions for meaningful parametric functions.

14. Bayes Solutions Involving a Dummy Parameter. FRIEDRICH GEBHARDT, University of Connecticut.

Let (y, z) be a point of the sample space (R^1, Z) with non-atomic measure μ and let $f_i(y - a, z)$ be the probability density under hypothesis H_i, $i = 1, \cdots, n$, where a is an unknown parameter. For any terminal decision d, the loss function L_i under hypothesis H_i may depend on the outcome (y, z) of the experiment and on the unknown parameter a in the form $L_i = L_i(y - a, z; d)$. A decision function $d_\phi(z)$ (not depending on y) is called a (generalized) Bayes solution to the a priori probability $(\phi_1, \cdots, \phi_n)$, if

$$\sum \phi_i \int L_i(y, z; d_\phi(z))f_i(y, z)\, d\mu$$

$$\leq \liminf_{t=1,2,\cdots} \frac{1}{2t}\int_{-t}^{+t} \sum \phi_i \left[\int L_i(y - a, z; d(y, z))\cdot f_i(y - a, z)\, d\mu\right] da$$

for any decision function $d(y, z)$ which fulfills certain measurability conditions. Under some assumptions regarding the compactness of the decision space and the integrability of L_i and $y \cdot L_i$, it can be shown that Bayes solutions exist.

15. Exact Moments and Percentage Points of Order Statistics From the Logistic Distribution and Applications to Estimation of Parameters of the Distribution. SHANTI S. GUPTA and BHUPENDRA K. SHAH, Purdue University.

Let $X_{(k)}$ denote the kth order statistic in a random sample of size n from the logistic distribution $L(0, 1)$ with the c. d. f. $F(x) = 1/[1 + \exp)(3^{-\frac{1}{2}}\pi x)]$. General expressions are derived for the moments of $X_{(k)}$ in terms of Bernoulli and Stirling numbers. Exact moments are given for all order statistics for $n = 1(1)10$ and numerical values for the first four cumulants are computed. Percentage points of $X_{(k)}$ are tabulated for $n = 1(1)10$ for all k and for $n = 10(1)25$, for some order statistics. The use of the percentage points for interval estimators of the location parameter is discussed. Efficiency of estimators based on single order statistics is studied. Application to life testing problems are discussed. An expression for the generating function of the product moment is obtained. Unbiased nearly best linear estimates based on order statistics are obtained. This paper extends some work of Plackett (*Ann. Math. Statist.* (1958) 131–142) and Birnbaum (*Ann. Math. Statist.* (1958) 1285 abstract) and has several new results.

16. Estimation of the Three Weibull Parameters, Using an Overlooked Consequence of a Well-Known Property of the Weibull Distribution. LEON H. HERBACH, New York University.

In much routine maintenance in life testing work, actual *times* of failures and replacement are not known. However, data are often kept indicating the *number* of failures and replacements made in some time interval. It is well known that if the time to failure, T has a Weibull distribution, i.e., if $P(T > t) = \exp \{-[(t - \alpha)/\eta]^\beta\}$, then the reduced variable $U = (T - \alpha)^\beta$ has an exponential distribution with mean, $\delta = \eta^\beta$. It is equally well known that if times between failures occur independently according to an exponential distribution, the number of failures in a fixed time interval has a Poisson distribution. These two facts enable one to estimate the parameters, α, β and η from the type of data often kept routinely.

17. On an Unbiased Ratio Estimator in Sampling With Replacement With Unequal and Varying Selection Probabilities. J. C. Koop, North Carolina State College.

In sampling with replacement with selection probabilities varying from draw to draw an unbiased estimator of the population total $T = \sum_{i=1}^{n} x_i$ is given by $T'(s_v) = [\Delta^v 0^n/N^n P(s_v)] \cdot (N/v) \sum_{i\varepsilon s_v} x_i$, where s_v is the sample of v distinct units derived from a sample of n and $P(s_v)$ is the total probability obtaining s_v. The optimum value of $P(s_v)$, i.e., $(\Delta^v 0^n/N^n) \cdot (N/v) \sum_{i\varepsilon s_v} x_i/T$, leads to the ideal result $V[T'(s_v)] = 0$. Then if there is a characteristic y_i closely related to x_i $(i = 1, 2, \cdots, N)$, and we choose $P'(s_v) = (\Delta^v 0^n/N^n)\ (N/v) \cdot \sum_{i\varepsilon s_v} y_i/\sum_{i=1}^{N} y_i$, then a simple unbiased estimator, $T''(s_v) = (\sum_{i\varepsilon s_v} x_i/\sum_{i\varepsilon s_v} y_i) \sum_{i=1}^{N} y_i$, will be obtained with a variance deviating from zero to the extent of departure from proportionality of the y's. A sampling procedure (where $P'(s_v)$ is exactly as expressed in its formula) and the estimate of the variance of $T''(s_v)$, are both available.

18. Some Estimators in Sampling With or Without Replacement With Unequal Probabilities (Preliminary report). J. C. Koop, North Carolina State College. (By title)

On the basis of a sample s of n drawn one by one from a finite universe U of N elements without replacement, an unbiased estimator $T'_{(s_n)} = \sum_{i\varepsilon s_n} P(s_n \mid i)\, x_i/P(s_n)$ of the population total $T = \sum_{i=1}^{N} x_i$, where $P(s_n)$ is the total probability of obtaining s, and $P(s_n \mid i)$ is the total probability of obtaining s without element i, given that it was removed from U at the first draw, has zero variance, when x_i is exactly proportional to $p(i_1)$ (> 0), the probability of selecting element i $(i = 1, 2, \cdots N)$ at the first draw. This is so because $\sum_{i\varepsilon s_n} p(i_1)P(s_n \mid i) = P(s_n)$. In sampling with replacement with unequal probabilities varying at each draw, the analogue of T' possesses a similar property, but whereas there the key condition for zero variance hinges only on the selection probabilities at the *first draw*, here, it hinges only on the selection probabilities at the *last draw*. The practical implications of these ideal results in sample survey work are obvious. Unbiased estimates of variance are available. Murthy's (1957) and Basu's (1958) estimators (Sankhyā, **18** (3 and 4), 382, and **20** (3 and 4), 294, respectively) are special cases of T' and its analogue.

19. On the Distribution of Sum of Identically Distributed Correlated Gamma Variables (Preliminary report.) Samuel Kotz and John W. Adams, University of North Carolina.

A distribution of a sum of identically distributed Gamma-variables correlated according to an "exponential" autocorrelation law $\rho_{ij} = \rho^{|i-j|}$ $(i, j = 1, \cdots, n)$ where ρ_{ij} is the correlation coefficient between the ith and jth random variables and $0 < \rho < 1$ is a given number is derived. This is performed by determining the characteristic roots of the appropriate variance-covariance matrix using a special method and by applying Robbin's and Pitman's result on mixtures of distributions for the case of Gamma-variables. An "approximate" distribution of the sum of these variables under the assumption that the sum itself is a Gamma variable is given. A comparison between the "exact" and the "approximate" distributions for certain values of the correlation coefficient, the number of variables in the sum and the values of parameters of the initial distributions are presented.

20. On Multitreatment Rank-Order Tests for Paired-Comparisons. KRISHEN L. MEHRA, Michigan State University. (By title)

For testing the hypothesis of no-difference among k treatments on the basis of $\binom{k}{2}$ independent samples of paired observations viz (X_{il}, X_{jl}) $l = 1, \cdots N_{ij}$ for the pair (i, j), consider the following family of rank-order statistics: $L_N(\zeta_N, \zeta) = \sum_{i=1}^k \{\sum_{j\neq i} (V_N^{(i,j)}/N_{ij}^{\frac{1}{2}})\}^2 \div [\int_0^1 \zeta^2(u)\, du]k$, where $V_N^{(i,j)} = \sum_{l=1}^{N_{ij}} \zeta_N(R_{N,l}^{(i,j)}/N+1) \cdot \text{sign } Z_l^{(i,j)}$, $R_{N,l}^{(i,j)}$ is the rank of $|Z_l^{(i,j)}| = |X_{il} - X_{jl}|$ in a combined ranking of the $N = \sum_{i=1}^k \sum_{j>i} N_{ij}$ absolute $|Z_l^{(i,j)}|$ $(l = 1, \cdots N_{ij}, 1 \leqq i < j \leqq k)$ and $\{\zeta_N(u)\}$ a sequence of step functions: $\zeta_N(u) = \zeta_N(\alpha/N+1)$ for $\alpha/N \leqq u < (\alpha+1)/N$, $\alpha = 0, \cdots N-1$ such that $\lim_{N\to\infty} \int_0^1 \{\zeta_N(u) - \zeta(u)\}^2\, du = 0$ (see also Puri, *Ann. Math. Statist.* **33** (1962) 827). The relative asymptotic efficiencies of these statistics are obtained for certain "contiguous" alternatives by extending the results of Hajek (*Ann. Math. Statist.* **33** (1962) 1124–1167). One observes, however, that the statistic $L_N^*(\zeta_N, \zeta)$, constructed similarly but with each $V_N^{(i,j)}$ now based on separate rankings for each pair (i, j), is equally Pitman-efficient as $L_N(\zeta_N, \zeta)$. The question (unresolved under Pitman criterion) of preference between the two procedures—"joint" or "separate" rankings—is investigated by comparing the local powers of L_N and L^* for large k and finite sample sizes. The results suggest that for testing against shift in location, the statistic $L_N(\zeta_N, \zeta)$ is preferable to its counterpart $L_N^*(\zeta_N, \zeta)$ except for alternatives for which Durbin's statistic is relatively Pitman-efficient than $L_N(\zeta_N, \zeta)$. Similar conclusions also hold for Lehmann's distribution-free alternatives.

21. Note on a Coin Tossing Game. SRI GOPAL MOHANTY, State University of New York.

Given two coins, 1 and 2, with probabilities p_1 and p_2 of obtaining heads in a single trial $(p_1 + p_2 > 1)$, a game is played with the following rules: (1) Toss coins 1 and 2 alternately and (2) stop making further trials when for the first time the total number of heads exceeds the total number of tails by exactly a $(a \geqq 1)$. It has been shown that the game $\mathcal{H}_a$, which starts with coin 1, and the game $\mathcal{H}_a'$, which starts with coin 2, are complete.

22. A Characterization of the Exponential-Type Distribution. G. P. PATIL, McGill University.

A real-valued random variable X is said to have the exponential-type distribution if its frequency function is given by $f(x; w) = a(x)e^{wx}/g(w)$ where the symbols follow usual restrictions. We show that the exponential-type distribution is characterized by a recursion relation, between the cumulants given by $K_{r+1} = K_r'$ where K_j is the jth order cumulant of a distribution and the prime denotes differentiation with respect to the parameter.

23. Unbiased Ratio and Regression Estimators in Multi-Stage Sampling. J. N. K. RAO, Iowa State University.

In recent years considerable attention has been given in the literature to the construction of unbiased ratio and regression estimators, since the classical ratio and regression estimators are biased. Mickey (*J. Amer. Statist. Assoc.*, 1959) has given a method of constructing a broad class of unbiased ratio and regression estimators in simple random sampling which includes the well-known Hartley-Ross unbiased ratio estimator. However, since in practice multi-stage sampling is often employed, it seems necessary to extend Mickey's method to multi-stage designs. In this paper we extend Mickey's method to two-stage sampling and construct two different classes of "combined" unbiased ratio and regression estimators. The estimators in class 1 depend on the sum of the population totals X_i of the n primaries selected in the sample, where X_i is the ith primary population total of the supplementary variate "x" (this sum may not be known in practice). The estimators in class 2 depend only on the overall population mean $\bar{X}$ of the variate "x", so that the estimators in class 2 may become more useful in practice. A simple method of variance estimation, similar to that of Goodman and Hartley (*J. Amer. Statist. Assoc.*, 1958) for the unbiased ratio estimator in simple random sampling, is proposed.

24. **The Robustness of ANOVA for a Class of 2-Associate PBIB Designs.** P. V. Rao, University of Georgia.

Let U = [Treatment (eliminating blocks) SS]/[Treatment (eliminating blocks) SS + Error SS]. Under the hypothesis that the treatment effects are all equal and certain randomization conditions are satisfied, the permutation moments of U were obtained by Welch (1937) for Randomized Blocks design and by Mitra (1960) for Balanced Incomplete Block design. In this paper, the first two permutation moments of U are obtained for 2-associate PBIB designs with $\lambda_1 = 0$ and $\lambda_2 = 1$ under the following assumptions: (i) the treatment effects are all equal, (ii) the blocks of the design are assigned to the blocks of the experimental material completely at random, and (iii) within each block, the treatments are assigned to plots at random. The permutation theory mean of U under these assumptions is the same as its mean with ANOVA assumptions, but the variances differ in the two situations. The permutation theory variance of U for a 2-associate PBIB design is equal to $\{2(v-1)[(v-1)(r-1)(k-1)+m]/bk(k-1)\}[1-(b-1)^{-1}V]$, where v, b, k, m are the parameters of the design and $V^{\frac{1}{2}}$ is the coefficient of variation of the block variances. This result is used to assess the robustness of ANOVA for this design.

25. **Bayes Sequential Procedures for Some Binomial Problems** (Preliminary report). Sudhindra Narayan Ray, University of North Carolina.

Consider a sequential decision problem for deciding whether or not the binomial parameter p exceeds $\frac{1}{2}$ when a loss, proportional to $a(p) = |p - \frac{1}{2}|$, is incurred only if a wrong decision is made. Moriguti and Robbins (*Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs.* (1962)) studied the optimum "boundary" (sampling rule) and its asymptotic behavior when the cost is a constant per observation. Their work is here extended to the case of absolute deviation cost, $a(p)$, per observation. The proposed power series representation of the asymptotic boundary is verified by the derivation of upper and lower bounds by a method due to Bather (*Proc. Cambridge Philos. Soc.* (1962)).

All of the above results are extended to a two-population problem in which one wishes to decide which of the two binomial parameters is the larger; the role of $|p - \frac{1}{2}|$ above is played by $|p_1 - p_2|$, and sampling is done in pairs with no distinction made between the two kinds of "ties". Finally, a modified loss function, proposed by F. J. Anscombe (unpublished) for use in clinical trials, in which the proportionality "constant" is reduced as the sampling progresses, is introduced in each of the above problems. Here, the exact boundaries are computable from simple recursion formulas. The asymptotic behavior is also studied.

26. **An Asymptotically Optimal Sequential Design for Comparing Several Experimental Categories With a Control** (Preliminary report). CHARLES DEWITT ROBERTS, University of North Carolina.

Let $X^{(j)}$ be the random variable resulting from a measurement with the jth category. We denote the probability density of $X^{(j)}$ by $g(x, \tau_j)$. We say $\theta = 0$ when $\tau_1 = \tau_2 = \cdots = \tau_k = \tau_0$, and say $\theta = j$ when $\tau_1 = \cdots = \tau_{j-1} = \tau_{j+1} = \cdots = \tau_k = \tau_0$ and $\tau_j = \tau_0 + \Delta$ where $\Delta > 0, j = 1, 2, \cdots, k$. For deciding on the true value of θ three sequential procedures are examined with specification of how the procedures are carried out in practice. One procedure is that of sampling in k (or less)-tuples of one observation on each category beginning with a k-tuple. With this procedure after each observation either we decide a category is better than the control and stop further sampling or we continue after (possibly) eliminating categories that appear no better than the control. The second procedure selects at random an order to examining the categories, and then one-by-one we decide if a category is better than the control. The third procedure selects after each single observation a category on which to sample next. With a definite loss function and a cost $C > 0$ per observation the three sequential procedures and fixed sample size procedures are compared in a certain asymptotic sense as $C \to 0$. In particular, it is shown that the third procedure is optimal and that the other two procedures are not optimal in this asymptotic sense.

27. **Bounds on the Probability and Coverage of a Multivariate Tolerance Region** (Preliminary report). ERNEST M. SCHEUER, The RAND Corporation, Santa Monica, California.

Let $\mathbf{x} = (x_1, \cdots, x_p)$ be a random vector with unknown continuous density f. Based on a sample $\mathbf{x}_j = (x_{1j}, \cdots, x_{pj})$, $j = 1, \cdots, n$ a tolerance region R is constructed where $R = \{\mathbf{x}: x_i(r_i) < x_i < x_i(s_i), i = 1, \cdots, p\}$, $x_i(1) \leqq x_i(2) \leqq \cdots \leqq x_i(n), i = 1, \cdots, p$, and $1 \leqq r_i < s_i \leqq n$, $i = 1, \cdots, p$. For fixed n and γ $(0 < \gamma < 1)$ bounds are given on $\alpha = P\{$at least $100\gamma\%$ of the population of the $\mathbf{x}$'s lie in $R\}$ and on $E\{\int \cdots \int_R f(\mathbf{x})\, d\mathbf{x}\}$ by use of a Bonferroni inequality and of results of Wald (these *Annals* **14** (1943) 45–55). A relation of these results to confidence regions for the multivariate median is discussed.

28. **On a Construction of a Class of Resolvable BIBD.** ESTHER SEIDEN, Michigan State University, (By title)

It is shown that one can construct resolvable BIBD for the parameters $(v, b, r, k, \lambda) = (2^{2n-1} - 2^{n-1}, 2^{2n}-1, 2^{n-1}, 2^n + 1, 1)$ for n, a positive integer greater than or equal to 2. The method of construction is geometrical.

29. **On the Non-Existence of Balanced Incomplete Block Designs BIBD, With Parameters (46, 69, 9, 6, 1) and (51, 85, 10, 6, 1).** ESTHER SEIDEN, Michigan State University. (By title)

The work of many authors provided either proofs of non-existence or solutions of all BIBD for $r \leqq 10$ except in two cases (46, 69, 9, 6, 1) and (51, 85, 10, 6, 1). This gap is now bridged. It is shown that these two designs do not exist.

30. **On Random Variables Which Have the Same Distribution as Their Reciprocals.** V. SESHADRI, McGill University.

Let x be a random variable with p.d.f. (probability density function) $f(x)$. The p.d.f. of $1/x$ is known to be $(1/x^2) f(1/x)$. If x and $1/x$ have the same distribution, the p.d.f. will satisfy the functional equation $f(x) = (1/x^2) f(1/x)$. The solution of the functional equation leads to the existence of certain Mellin Transforms. By looking up a table of Mellin Transforms and their inverse Transforms, many well-known and commonly used probability distributions are obtained. Mention is also made of a more general functional equation which is obtained by considering a two parameter family of p.d.f's. The equation then is $f(x; a, b) = (1/x^2) f(1/x; b, a)$. It is shown that the p.d.f. of the ratio of two independent samples from the same distribution satisfy the functional equations. Examples of random variables (non-negative) satisfying the two functional equations are given and some of the interesting properties of the distributions are examined.

31. **On the Probability of Large Deviations of Families of Sample Means** (Preliminary report). J. SETHURAMAN, University of North Carolina.

Let $\xi_1(\omega)$, $\xi_2(\omega)$, $\cdots$ be a sequence of independently and identically distributed random variables from (Ω, S, P) into (X, B) where X is a complete separable metric space and B is the σ-field of Borel sets. The distribution of ξ_1 is μ and the sample distribution of $\xi_1(\omega)$, $\cdots$ $\xi_n(\omega)$ is $\mu(n, \omega)$. *Theorem 1.* For any equicontinuous class, $\mathfrak{F}$, of continuous nonconstant functions bounded a continuous function with a finite moment generating function, as $n \to \infty$, $(1/n) \log P \{\omega: \sup_{f \varepsilon \mathfrak{F}} | \int f d\mu(n, \omega) - \int f d\mu | \geqq \epsilon\} \to \log \rho_{\mathfrak{F}}(\epsilon)$ where $0 < \rho_{\mathfrak{F}}(\epsilon) < 1$. *Theorem 2.* For any compact (under the u.c.c. topology) class, $\mathfrak{F}$, of continuous functions from X into R^k with non-atomic induced distributions, as $n \to \infty$, $(1/n) \log P\{\omega: \sup_{A \varepsilon \mathfrak{A}} | \mu(n, \omega)[A] - \mu[A] | \geqq \epsilon\} \to \log \rho^*(\epsilon)$ where $\mathfrak{A} = \{A\}$, $A = \{x: f_1(x) \leqq a_1, \cdots f_k(x) \leqq a_k\}$ for some $f(x) = (f_1(x), \cdots f_k(x)) \varepsilon \mathfrak{F}$ and $-\infty < a_1, \cdots a_k < \infty$, and $0 < \rho^*(\epsilon) < 1$. These results are allied to the results of Sanov [*Mat. Sb.* **84** (1957) 11–44] and possibly, could be deduced from them. However, direct and simple proofs of these results are presented here.

32. **Significance Probability Bounds for Rank Orderings.** PAUL SWITZER, Harvard University. (Introduced by Jerome H. Klotz)

Let $X_1, \cdots, X_m$ be a sample of m from the p.d.f. $f(x)$ and let $Y_1, \cdots, Y_n$ be an independent sample of n from the p.d.f. $g(y)$, such that $g(u)/f(u)$ is a strictly increasing function of u. Let the ordered ranks of the X's in the pooled sample be denoted by the vector $c = (c_1, \cdots, c_m)$. If $c' = (c_1', \cdots, c_m')$ is any vector in the range of c, we will say that $c' < c$ provided $c_j' \leqq c_j$ all j, and $c_j' < c_j$ some j. I. R. Savage has shown that if $c' < c$ then $P(c') > P(c)$. Let $C(a, b)$ denote the number of combinations of a things b at a time. The minimum number of possible vectors, c', which are such that $P(c') > P(c)$ is given by the expression $L = C(c_m, m) - \sum_{j=1}^{m-1} C(c_m - c_j, m - j + 1)f_j$, where $f_j = C(c_j, j - 1) - \sum_{k=1}^{j-1} f_{j-k} C(c_j - c_{j-k}, k)$, $f_1 = 1$. Since all $C(m + n, m)$ vectors in the range of c are equally likely under the hypothesis $f = g$, a lower bound on the significance probability for an admissible test is then just $P = L/C(m + n, m)$. An upper bound is similarly obtained by using the vector of ordered ranks of the Y's in the expression for L and P. An analogous result is obtained for the one-sample case.

33. **Identifiability of Finite Mixtures.** HENRY TEICHER, Purdue University.

The class of *all* mixtures of an m-parameter additively closed family of distributions is identifiable for $m = 1$ but not, in general, for $m > 1$ ("Identifiability of Mixtures," *Ann. Math. Statist.* **32** (1961) 244–248). Here, two theorems on identifiability of the class of all *finite* mixtures of a family of distributions are proved, one of these yielding the identifiability of the class of all finite mixtures of normal (or Type III) distributions. An alternative proposition leads to a necessary and sufficient condition for a class of finite mixtures of binomial distributions to be identifiable.

(Abstract of papers to be presented at the Central Regional Meeting, Madison, Wisconsin, June 14–15, 1963. An additional abstract appeared in the March, 1963 issue, and others will appear in the September, 1963 issue.)

2. A Two-Stage Sampling Procedure for Estimating the Common Mean of Several Normal Populations With Unknown Variances. KHURSHEED ALAM, University of Minnesota.

Richter (1960) considered the problem of estimating the common mean of two normal populations using a two-stage sampling procedure with a fixed number of total observations, say, n. The variances of the two populations are assumed unknown. In the first-stage a sample of equal size, say, m, is taken from each population and the sample variances are computed. The population corresponding to the smallest sample variance is selected for sampling in the second-stage. Using squared error divided by the smallest population variance as a loss function and a suitable estimator for the mean, Richter showed that $m = \theta'(n^{2/3})$ is an asymptotic minimax solution for m. This paper generalizes the above result for $k > 2$ populations. It is also shown in this paper that the selected corresponding sampling rule for k populations is minimax in a class of invariant procedures. The corresponding estimator of the mean is compared with an optimal invariant estimator.

3. Stochastic Processes and Genotypic Frequencies Under Mixed Selfing and Random Mating. R. W. ALLARD, G. A. BAKER, JR., G. A. BAKER and J. CHRISTY, University of California, Davis; University of California, Los Alamos; University of California, Davis; University of California, Los Alamos. (By title.)

A theoretical stochastic model is developed from which it is possible to compute simply mean genotypic frequencies and variances expected in any generation. The parameters in the model specify hypotheses concerning the more important directed and non-directed processes which affect populations mating under mixed selfing and random outcrossing. Results based on this model agree closely with those obtained by using an electronic data processing machine to impose stochastic variation on deterministic population models. Application of these results to the interpretation of observations made on an experimental population are discussed. It turns out that data of this sort are very insensitive to the fraction of random outcrossing. This study was partially supported by the United States Atomic Energy Commission and the National Science Foundation under Research Grant NSF 6-4954.

4. Analysis of Genetic Change in Finite Populations Composed of Mixture of Pure Lines. G. A. BAKER, JR., R. W. ALLARD, G. A. BAKER and J. CHRISTY, University of California, Los Alamos; University of California, Davis; University of California, Los Alamos. (By title.)

A mathematically manageable log-normal approximation was developed of a multinominal genetic model which takes into account those directed and random processes which control genotypic frequencies in populations consisting of mixtures of pure lines. Results based on this model agreed closely with those obtained by numerical simulations performed on an electronic data processing computer. On the basis of this model we give formulas and figures which show the dispersion to be expected for values of the relevant parameters as determined in extensive field data on populations of largely self-pollinated plants (lima beans, barley, etc.). In general, for the magnitude of random fluctuations in the viabilities considered, it is true that, when the population size is of the order of a few hundred times the number of pure lines, the sampling errors are small compared to the intrinsic errors due to the fluctuations in the viabilities. A method of analysis of experimental data based on the log-normal approximation is developed and applied. In the example, the observed dominance of one variety of barley over another could be ascribed with confidence to a larger relative viability. This study was partially supported by the United States Atomic Energy Commission and the National Science Foundation under Research Grant NSF 6-4954.

5. **A New Table of Percentage Points of the Chi-Square Distribution.** H. LEON HARTER, Wright-Patterson Air Force Base, Ohio.

For certain applications a table of percentage points of the chi-square distribution is required which is more extensive and more accurate than any previously published. Having encountered a need for such a table, the author set out to compute a six-significant-figure table, accurate to within a unit in the last digit, of percentage points corresponding to cumulative probabilities $P = .0001, .0005, .001, .005, .01, .025, .05, .1\ (.1)\ .9, .95, .975, .99, .995, .999, .9995, .9999$ for $\nu = 1\ (1)\ 100$ degrees of freedom. The bulk of the table was computed by making use of the relation between the chi-square distribution and the incomplete Gamma-function ratio and interpolating inversely in a newly computed eight-decimal-place table of the latter. The accuracy of the results obtained by this method is not satisfactory for certain cases, especially those in which both P and ν are small. It was possible to obtain the required accuracy when both P and ν are small by interpolating inversely in an auxiliary table and/or by an iterative method based on an expansion in series. The only remaining tabular values which may be in error by more than a unit (but not more than 3 units) in the sixth significant digit are those for $P = .9999$ and $\nu \leqq 52$. The paper includes the table itself and a description in some detail of the method of computation.

6. **On the Independence of Certain Wishart Variables.** ROBERT V. HOGG, University of Iowa.

Let each of the mutually independent rows of the $(n \times p)$ matrix $\mathbf{x}$ have a p-variate normal distribution with the unknown positive definite covariance matrix $\mathbf{K}$. In each of most of the tests of hypotheses which concern the means of these multivariate normal distributions, the likelihood ratio, raised to an appropriate power, is equal to the ratio of two determinants, say $U = |\mathbf{x}'\mathbf{A}\mathbf{x}|/|\mathbf{x}'\mathbf{B}\mathbf{x}|$, where $\mathbf{A}$ and $\mathbf{B}$ are real symmetric matrices with ranks greater than or equal to p. In addition, we frequently know, or it can be easily shown, that both $\mathbf{x}'\mathbf{A}\mathbf{x}$ and $\mathbf{x}'\mathbf{B}\mathbf{x}$ have Wishart distributions. In this paper, we prove that the fact that the likelihood ratio is less than or equal to one implies that $\mathbf{x}'\mathbf{A}\mathbf{x}$ and $\mathbf{x}'(\mathbf{B} - \mathbf{A})\mathbf{x}$ are independent and that the latter form also has a Wishart distribution. This proof is based on a chi-square decomposition theorem of Hogg and Craig which can be extended to Wishart variables. Another example of the use of this decomposition theorem is given. In addition, the independence of Wishart variables and linear forms is considered.

7. **On the Busy Period of Single Server, Many Queue, Service Systems** (Preliminary report). PETER D. WELCH, IBM Thomas J. Watson Research Center, Yorktown Heights, New York.

Customers arrive from r sources and wait in r separate queues which are attended by a single server. The arrival process from each source is Poisson, and these r input processes are mutually independent. The service times have an arbitrary distribution which is a function of the source. Results are obtained which yield the joint distribution of the length of the busy period and the number of each type of customer served during it. These results hold for a wide class of service disciplines. The only requirements are that the server is busy whenever there are customers present and that the total time a customer is in contact with the server is his service time. In particular, they hold for the two priority service disciplines: head-of-line and pre-emptive resume.

(Abstracts of papers to be presented at the Western Regional Meeting, Eugene, Oregon, June 20–21, 1963. Additional abstracts will appear in the September, 1963 issue.)

1. A Review of the Literature on a Class of Coverage Problems. WILLIAM C. GUENTHER and PAUL J. TERRAGNO, University of Wyoming and Westat Research Analysts, Denver, Colorado. (By title)

In recent years a large number of publications have appeared on probability problems arising from ballistic applications. Many of these are concerned with topics which are often referred to as coverage problems. Some of the results are found only in obscure sources and are difficult to obtain, a fact which has led to considerable duplication of effort and waste of time. Coverage problems are defined as the evaluation of a special type of probability which depends upon a damage function, a distribution of aiming errors, and a target distribution. Section 1 presents the case in which the damage function is of the zero-one type and the target assumes a position with probability 1. In Section 2 the damage function is zero-one but the distribution of the target does not concentrate all its probability at one point. Some special results with an exponential damage function are given in Section 3. Reference is made to 58 papers and reports.

2. Confounding in the $3(2^{4-2})$ Designs. PETER W. M. JOHN, University of California, Davis.

The $3(2^{4-2})$ design is the three quarter replicate of the 2^4 factorial obtained by omitting one quarter. If the missing quarter is defined by $\mathrm{I} = A = BCD = ABCD$ or $\mathrm{I} = AD = ABC = BCD$ the design is of resolution V (main effects and 2 f.i. clear). It is shown that either design may be divided into two blocks, one of 8 runs and one of 4 runs, by using a high interaction from the defining contrasts as a blocking variable, and still be of resolution V. This $3(2^{4-2})$ design also provides designs of resolution III for 10 factors in 2 blocks of 6 runs, 9 in 3 blocks of 4 or 8 in 4 blocks of 3. The resolution IV design for six factors defined by $\mathrm{I} = AB = AC = BC$ splits into six blocks of two. The latter design is a special case of this general result; the design of resolution IV for $3(2^{n-3})$ factors in $3(2^{n-2})$ runs is divisible into $3(2^{n-3})$ blocks of size two.

3. A Note on a Multiple Minima in Least Squares. ROGER H. MOORE, Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

For the purposes of least squares estimation, it is common to relate the observations by a model which often takes the form $y_i = f(x_i; \alpha) + e_i$. The uniqueness of the least squares estimate of the parameter vector α is often simply taken for granted. This note is concerned with the characterization of statistical models which lead to more than one least squares solution. Recognition of this phenomenon is important in certain problems, notably those in which the sampling behavior of the parameter estimates is examined by Monte Carlo methods.

4. Estimation of the Interval Containing the Change in Mean Value. Donald E. Robison, Space Technology Laboratories, Inc., Redondo Beach, California.

Let X and Y be independent normal random variables with unknown means μ and ν and common unknown variance σ^2. Suppose an independent sample on X is taken at distinct ordered times $(s_1, s_2, \cdots)$ and that at some time, s_N, the mean switches from μ to ν and independent sampling on Y begins at distinct ordered times $(t_1, t_2, \cdots)$. In various problems it is usually assumed that N is known, so that $(x_1, \cdots, x_N, y_1, \cdots, y_M)$ is the correct division of the sample into X's and Y's. Probability statements subsequently made are conditional upon this correct classification.

By contrast, we assume N unknown, although requiring the total sample size $N + M$ to be known. The classification of the time ordered sample $(x_1, x_2, \cdots, y_M)$ into X's and Y's is the problem of estimating the change in mean value. This problem is solved by maximum likelihood estimation. Probability statements conditioned by the true known value of N are given for both correct and incorrect classification. These probabilities depend on the quantity $(\mu - \nu)/\sigma$.

(Abstracts of papers to be presented at the European Regional Meeting, Copenhagen, Denmark, July 8–10, 1963. Additional abstracts will appear in the September, 1963 issue.)

1. The Influence of Smoothing on the Correlation Between Two Synchronous Series. C. Levert, Royal Meteorological Institute, Netherlands.

In climatological studies one often is inclined to consider the existence of an "event" in a time-series (e.g. a slow change in the general mean level, possibly indicating a climatological change) as the more probable, the more various synchronous meteorological time-series, referring to neighbouring stations, show analogous tendencies. Moreover, the larger the similarity (resemblance, correlation) between these series, the more probable the event is not a random result. Since most series are very irregular, a smoothing process is applied, so that such a resemblance can be seen much easier in a visual way. The author has made some statistical computations on this subject. One starts with two synchronous stationary time-series, either mutually correlated or non correlated, these series possessing either identical or not identical autocorrelations; randomness is considered as a special case of autocorrelation. Next a moving-average smoothing process is carried out in each or only in one of the two series over equal or unequal numbers of terms and the expression for the new correlation coefficient between the new series is derived. Some conclusions which are based on this expression are formulated which may be important in investigations of synchronous meteorological and also time-series in other fields. Among others it once again turned out to be better to base a well justified statistical conclusion as to the similarity between synchronous series only on the initial series themselves and to smooth these series only for the benefit of a quick tentative orientation.

2. Selection of Categories for Some Permutation Results Based on Grouped Data (Preliminary report). John E. Walsh, System Development Corporation, Santa Monica, California.

Conversion of data to categorical form by grouping has computational advantages. Also, categorical data procedures are applicable for data from arbitrary univariate or multivariate populations. The major disadvantage is insensitivity of categorical data procedures.

When a specified alternative is of principal interest, selection of categories to emphasize this alternative can offset this disadvantage. Knowledge of null probability concentration is helpful in making suitable selections. Sometimes, for permutation situations, the data can be used to estimate the null probability concentration and re-used, with no conditional effects, for the procedure based on the resulting categories; e.g., consider the two-sample problem (possibly multivariate). All observations are considered to be fixed and probability enters only in dividing the totality of observations into two sets of the sizes of the samples. Under the null hypothesis, all functions that are symmetrical in the totality of observations are constants for this permutation probability model, and can be used in selecting categories; the empirical distribution function is especially useful. The median test and generalizations are examples of this method. Similar approaches apply to the several-sample problem, investigation of randomness, independence of entries of a bivariate variable, etc.; also to some situations where observations are obtained in independent groups.

(Abstracts of papers to be presented at the Annual Meeting of the Institute, Ottawa, August 27–29, 1963. An additional abstract appeared in the March, 1963 issue, and others will appear in the September, 1963 issue.)

2. The Use of Quasi-Range in Setting Exact Confidence Bounds for the Standard Deviation of a Normal Population. H. Leon Harter, Wright-Patterson Air Force Base, Ohio.

For a normal population, reasonably good interval estimates of the population standard deviation σ may be obtained from one suitably chosen sample quasi-range. The coefficients of the rth quasi-range w_r in exact confidence bounds for σ are found by taking the reciprocals of percentage points of the (standardized) quasi-range $W_r = w_r/\sigma$, which are themselves found by inverse interpolation in a table of the probability integral. The interval between exact lower and upper confidence bounds, each associated with confidence $1 - P$, is, of course, an exact central interval (confidence $1 - 2P$). Results have been computed for $r = 0(1)8$, sample size $n = (2r + 2)(1)20(2)40(10)100$, and $P = .0001, .0005, .001, .005, .01, .025, .05, .1\ (.1)\ .5$. The definition of efficiency commonly used for point estimators is extended to confidence bounds and confidence intervals. The following tables are included, together with a description of the method of computation: (1) a table of upper confidence bounds and central confidence intervals for σ, based on one quasi-range, together with their efficiencies, for that value of r which maximizes the efficiency of the upper confidence bound for each combination of n and $1 - P$ and (2) a similar table for that value of r which maximizes the efficiency of the central confidence interval, when the two values of r differ.

3. Renewal Processes Based on Distributions With Increasing Failure Rate. Richard E. Barlow and Frank Proschan, San Jose State College and Boeing Scientific Research Laboratories, Seattle, Washington. (Invited)

Let $M(t) = E[N(t)]$ denote the expected number of renewals, $N(t)$, in $[0, t)$ for a renewal process with underlying distribution F where $F(0^-) = 0$. We say F has increasing (decreasing) failure rate, denoted IFR(DFR), if $\log(1 - F)$ is concave when finite (is convex on $[0, \infty)$). We show that the ordinary moments, the binomial moments, and the variance of $N(t)$ with underlying IFR(DFR) distribution F having mean μ are dominated (subordinated) by the corresponding moments and variance of a Poisson process for which the underlying exponential distribution also has mean μ. We show that if F is IFR(DFR) with mean μ, then (a) $M(t) \geqq (\leqq)\ kM(t/k)$, $k = 1, 2, \cdots$; $t \geqq 0$, (b) $M(t) \leqq (\geqq)\ tF(t)/\int_0^t [1 - F(x)]\cdot dx \leqq (\geqq)\ t/\mu$, (c) $M(h) \leqq (\geqq)\ M(t + h) - M(t)$ for $h \geqq 0$ and all $t \geqq 0$. Dropping the

IFR(DFR) restriction, we show that (a) $M(t) \leqq kM(t/k) + k$, (b) $M(t) \geqq t/\int_0^t [1 - F(x)] \cdot dx - 1 \geqq t/\mu - 1$ for $k = 1, 2, \cdots$ and $t \geqq 0$. Inequalities for generalized renewal processes are also obtained. The results have application to the comparison of replacement policies.

4. **Optimality and the OC Curve for the Wald SPRT.** JAMES A. LECHNER, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania.

This paper examines "conjugate pairs" of points on the OC curve of a Wald Sequential Probability Ratio Test (SPRT). Such pairs are shown to exist for certain classes of SPRT's. For these classes, every point on the OC curve is a member of exactly one such pair. The well-known optimality property of a SPRT, which states that no other test of equal size and power (at the defining values of the parameter) can have a smaller expected sample size for either of the defining values, is extended to each conjugate pair. The OC curve is shown to be obtainable in a simple way, without the necessity of a parametric representation, by use of this notion of conjugate pairs.

(Abstracts not connected with any meeting of the Institute.)

1. **On Estimation and Transforms.** S. EHRENFELD, Columbia University.

Numerical methods of differentiation and for the inversion of generating functions and Laplace transforms, using Laguerre polynomials, are applied to some estimation problems. One problem considered involves the estimation of probabilities related to compound distributions. Let $X_1, X_2, \cdots$ be independent random variables with known distributions. Furthermore, let N be a random variable with an unknown distribution. The problem is to make inferences about the compound random variable $Z = X_1 + X_2 + \cdots + X_N$ on the basis of k independent observations, $N_1, N_2, \cdots, N_k$, on N. Unbiased estimates for $Q(z) = \text{Prob } (Z > z)$ are obtained and studied. Another problem considered involves the estimation of the busy period distribution in a Queueing system with known Poisson arrivals but unknown service distribution, on the basis of independent observations $S_1, S_2, \cdots, S_k$ of the service time. The method uses a functional equation relating the Laplace transforms of the service time and busy period distributions. The Laplace transform of the busy period is then estimated and numerically inverted to obtain estimates of Prob $(B > b)$ where B denotes the busy period random variable. The same type of approach can be applied to a variety of estimation problems in Queueing and Renewal theory. Where the distribution of service time is known the method may be viewed as a numerical method for solving functional equations of the type frequently occurring in probability theory.

2. **Functional Equations in Information Theory.** DAVID G. KENDALL, University of Cambridge.

An axiomatic characterisation of Shannon's entropy $\Sigma\, p_j \log(1/p_j)$ has been given by Fadeev. His axioms require functional equations to be satisfied by the entropy $H_k(p_1, p_2, \cdots, p_k)$ for *all* k, and further require $h(t) = H_2(t, 1 - t)$ to be continuous on the compact interval [0, 1]. In this paper Fadeev's algebraic axioms are used only in so far as they concern the functions H_2 and H_3, and his continuity axiom is replaced by the equally natural requirement that the function $h(\cdot)$ is to be monotonic increasing on the half-open interval $0 < t \leqq \frac{1}{2}$. In this way a pair of functional equations is obtained which are shown to have the Shannon entropy as their only admissible solution.

3. Information Theory and the Limit Theorem for Markov Chains and Processes With a Countable Infinity of States. David G. Kendall, University of Cambridge.

A. Rényi at the 4th Berkeley Symposium showed how the limit theorem for Markov chains with finitely many states and having strictly positive transition probabilities could be proved by information-theoretic methods. In this paper his method is adapted so as to work for chains (discrete time) and processes (continuous time) with a countable infinity of states, when there exists a non-negative totally-finite sub-invariant measure not identically zero. It turns out that the best results are obtained when the Shannon functional $\Sigma\, p \log (p/\pi)$ is replaced by the Fisher functional $\Sigma\, p^2/\pi$.

4. Bivariate Normal Test With Two-Sided Alternative Hypothesis (Preliminary report). Akio Kudo and Hideo Fujisawa, Kyushu University and Nagasaki University.

Suppose a bivariate normal distribution with known variance matrix, which we can assume, without loss of generality, variances be "one". Our problem is to test the null hypothesis that their means are zero, H_0 ; $\theta_1 = \theta_2 = 0$, against the alternative H_1 ; $(\theta_1 \geqq 0, \theta_2 \geqq 0)$ or $(\theta_1 \leqq 0, \theta_2 \leqq 0)$ where one of the inequality is strict in both cases.

We shall show that the likelihood ratio test is based on the statistic, $\bar{\chi}^2 = K$ if $\bar{x}_1 \bar{x}_2 > 0$ and $\bar{\chi}^2 = K - \bar{x}_i^2$ if $\bar{x}_1 \bar{x}_2 \leqq 0$ and $\bar{x}_i^2 < \bar{x}_j^2$ $(i \neq j)$ $(i = 1, 2)$ where $K = (\bar{x}_1^2 - 2\rho\bar{x}_1 \bar{x}_2 + \bar{x}_2^2)/(1 - \rho^2)$, $\bar{x}_i$ $(i = 1, 2)$ are sample means, and ρ is the correlation coefficient. The distribution of this statistic is given by $P_r(\bar{\chi}^2 > R^2) = P_r(\chi_2 > R^2) \text{ arc cos } (-\rho) + 2\Phi\ (R) - 2\Phi\ (R, R; \rho)$ where the probability in the right is that of a χ^2 variable with d.f. 2 exceeds R^2, $\Phi(\cdot)$ standardized normal distribution function, and $\Phi(\cdot, \cdot; \rho)$ is a bivariate normal distribution function with means zero, variances one, and correlation ρ. This result can be generalized to the case of unknown common variances and known correlation. Our result can be applied to derive the exact distribution for a two sided test (when $k = 3$) in: Bartholomew, D. J. (1959). A test of homogeneity for ordered alternatives II. *Biometrika* **49** 328–335.

5. On Comparing Coefficients of Variation in the Case of Two Independent Samples From Normal Populations (Preliminary report). Nico F. Laubscher and D. E. W. Schumann, National Research Institute for Mathematical Sciences, Pretoria, and University of Stellenbosch.

In this paper we derive a statistic by which the null-hypothesis of equal population coefficients of variation, in the case of the two sample problem, is tested. The test statistic contains a nuisance parameter. Methods of estimating this parameter are given and its effect on the distribution is under investigation. Percentage points of the distribution function (as a function of the sample sizes) are computed for various values of the nuisance parameter. Approximations to the exact distribution are also given.

6. On a Property of the Cauchy Distribution. Nico F. Laubscher, National Research Institute for Mathematical Sciences, Pretoria.

In this note the following result is obtained. *Theorem.* The Cauchy distribution, with p.d.f. given by $f(x) = k\pi^{-1} (k^2 + (x - \mu)^2)$, $k > 0$ $-\infty < x < \infty$, is the only family of distributions which has the property that the distribution of the sample mean is identical to the population distribution.

7. Effective Entropy Rate and Transmission of Information Through Channels With Additive Random Noise. K. R. PARTHASARATHY, Indian Statistical Institute, Calcutta. (Introduced by Ingram Olkin)

The famous McMillan's theorem regarding ergodic sources can be reformulated as follows. Consider the minimum number of n-length sequences which have a total probability exceeding $1 - \epsilon$. If μ is the measure describing the source denote this minimum by $N_n(\epsilon, \mu)$. Then $\lim_n n^{-1} \log N_n(\epsilon, \mu)$ exists and is equal to the entropy rate of source as defined by Shannon. In this paper it is proved that for any not necessarily ergodic but stationary source the same limit exists except for a countable set of ϵ's. Two functions $A(\epsilon)$ and $B(\epsilon)$, $(0 < \epsilon < 1)$ are constructed in such a way that $A(\epsilon) = B(\epsilon)$ except on a countable set and the lim sup and lim inf of $[n^{-1} \log N_n(\epsilon, \mu)]$ lie between $A(\epsilon)$ and $B(\epsilon)$. Further, as $n \to \infty$ both the functions $A(\epsilon)$ and $B(\epsilon)$ converge to a unique limit $\bar{H}(\mu)$. The precise description of $\bar{H}(\mu)$ is also given. Finally the coding theorem and its converse are proved for all stationary channels with additive noise.

8. Probability of a Positive Sample Correlation. HAROLD RUBEN, University of Sheffield.

It is well-known that the probability integral for the correlation coefficient, r, in normal samples cannot be expressed in terms of elementary functions. However, the probability integral for the special case $r = 0$ is readily expressible as a probability integral for t, though this does not appear to have been noted previously. Thus, since $r \geqq 0$ if, and only if, $\sum_1^n (x_i - \bar{x}) y_i \geqq 0$, and since $\sum_1^n (x_i - \bar{x}) y_i$, conditionally on fixed $x_1, \cdots, x_n$, is normal with mean $(\rho\sigma_y/\sigma_x)\sum_1^n (x_i - \bar{x})^2$ and variance $\sigma_y^2(1 - \rho^2) \cdot \sum_1^n (x_i - \bar{x})^2$, prob $[r \geqq 0 \mid x_1, \cdots, x_n] = \Phi[(n-1)^{\frac{1}{2}} \rho(1-\rho^2)^{-\frac{1}{2}} \cdot \{u/(n-1)\}^{\frac{1}{2}}]$, where $u = \sum_1^n (x_i - \bar{x})^2/\sigma_x^2$, and $\Phi(\cdot)$ is the standardized normal distribution function. Again, u is a chi-square with $n - 1$ degrees of freedom, whence prob $[r \geqq 0] = \int_0^\infty \Phi[(n-1)^{\frac{1}{2}} \rho(1-\rho^2)^{-\frac{1}{2}} \cdot \{u/(n-1)\}^{\frac{1}{2}}] d F_{n-1}(u)$, where $F_{n-1}(\cdot)$ is the distribution function of chi-square with $n - 1$ degrees of freedom. The latter integral is immediately identifiable with

$$\text{prob } [t_{n-1} \geqq -(n-1)^{\frac{1}{2}} \rho(1-\rho^2)^{-\frac{1}{2}}],$$

where t_{n-1} is a Student variable with $n - 1$ degrees of freedom, defined formally as $t_{n-1} = z/\{u/(n-1)\}^{\frac{1}{2}}$, z being an $N(0, 1)$ which is independent of u. The result is then

$$\text{prob } [r \geqq 0] = \text{prob } [t_{n-1} \leqq (n-1)^{\frac{1}{2}} \rho(1-\rho^2)^{-\frac{1}{2}}].$$

Incidentally, this also follows from the known distribution of b, the sample regression coefficient of y on x. The latter distribution is of Pearson Type VII form centred at $\beta = \rho\sigma_y/\sigma_x$, the population regression coefficient of y on x (Kendall and Stuart, *The Advanced Theory of Statistics*, **1** p. 392), or equivalently, $(\sigma_x/\sigma_y)\{(n-1)/(1-\rho^2)\}^{\frac{1}{2}}(b - \beta)$ is a t_{n-1}, the required result then following on noting (as indicated previously) that $r \geqq 0$ if, and only if, $b \geqq 0$.

The result obtained finds application in the evaluation of the probability that a random sample, supposedly drawn from a bivariate normal population, shall give a misleading result in the sense of exhibiting a positive (negative) correlation when the population correlation has a given negative (positive) value. It also provides a useful spot check on F. N. David's 1938 tables of the correlation coefficient, and may be of some value in possible future extensions of these tables.

9. Approximate Methods for Computing Elliptical Probability Coverages. HANS K. URY, University of California, Berkeley.

Several approximation procedures are investigated for evaluating the bivariate normal probability integral over the area of an offset ellipse. The problem is as usual transformed to evaluating the circular normal integral with unit standard deviation over an offset ellipse oriented parallel to the principal axes of the distribution. Regions are defined in terms of four variables: A, the area of the ellipse; e, the ratio of minor to major axis; and (d, θ), the two polar center coordinates. Approximations accurate to within 0.01 and 10% are obtained over the following regions by using the circular coverage function over the equal-area, equi-centered circle, with correction factors depending on θ: (1) $A \leqq 0.2\pi$, $e \geqq 0.1$, $d \leqq 1.0$; (2) $A \leqq \pi$, $e \geqq 0.5$, d arbitrary. Over (3) A arbitrary, $e \geqq 0.5$, $d \leqq 1.0$, an error of less than 0.02 and 5% is attained. For coverage probabilities below 0.1, the accuracy is in general greater than that attainable through standard interpolation methods in existing tables, and region (2) goes beyond the center-limits of such tables. When the elliptical coverage does not exceed 0.05 [0.03], the uncorrected circular coverage will give virtually no error for ellipses with $e \geqq 0.5$ [0.1] and $d \leqq 1.0$. For larger and "thinner" ellipses with $d > 1$, some other approximation methods are discussed. The behavior of the error as a function of the four variables is also investigated.

10. On a Generalized Weighing Problem (Preliminary report). HANS K. URY, Stanford University.

A well-known weighing problem consists of finding the one false coin in a set of $\sum_{j=1}^{n-1} 3^j$, and whether it is heavy or light, in n weighings using only a balance. For the case of k (=1 2, $\cdots$) equally false coins formulas are derived for the maximum number from which these defectives can be extracted in n (=1, 2, $\cdots$) weighings (a) under similar conditions, (b) with known good coins available, (c) direction of defectiveness known, (d) direction known and good coins available.

A group testing application of the above problem is the following: if a system containing m similar components has exactly k known defectives and if analyses to determine these must be carried out using the components themselves [(a), (c)] or perhaps similar components outside the system [(b), (d)], then the above formulas in turn give the number of analyses required under the minimax approach. For (c) and (d) this is the smallest integer $\geqq [\log m! - \log k! - \log (m-k)!]/\log 3$, for (b) it is the smallest integer $\geqq [\log 2 + \log m! - \log k! - \log (m-k)!]/\log 3$, and for (a) it appears to be the same as for (b) provided $k > 1$. For $k = 2$, the optimal methods without recombinations are described. It is shown that these rather simple procedures will require at most one additional analysis.

11. Spearman's Footrule—an Alternative Rank Statistic. H. K. URY, D. C. KLEINECKE and L. F. WAGNER, University of California, Berkeley.

A known but neglected rank statistic, Spearman's Footrule (*British J. Psychology*, 1906), is based on the sum of the absolute values of the differences between two rankings. Formulas are derived for its mean and variance and the covariance with Spearman's ρ and Kendall's τ. A thorough numerical description is given, including the exact sampling distribution for up to ten ranks when all permutations are equally likely and approximate results, obtained by Monte Carlo methods, for 11 to 15 and 20 ranks. Exact formulas for the frequencies of the seven smallest and the two largest values have been obtained. The statistic appears to be asymptotically normal, but this has not been proven.