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A TEST FOR EQUALITY OF MEANS WHEN COVARIANCE MATRICES ARE UNEQUAL[1]

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Let $x_\alpha^{(g)}$ be an observation from the p-variate normal distribution $N(\mu^{(g)}, \Sigma_g)$, $\alpha = 1, \cdots, N_g$, $g = 1, \cdots, q$. Consider testing the null hypothesis[2]

$$H:\mu^{(1)} = \cdots = \mu^{(q)}. \tag{1}$$

When the covariance matrices Σ_g are equal, the hypothesis is a form of the so-called general linear hypothesis, and a number of tests are available. (See Chapter 8 of Anderson (1958), for example.) When $q = 2$, Bennett (1951) has extended the procedure of Scheffé (1943) to give an exact test based on Hotelling's generalized T^2. (See Section 5.6 of Anderson (1958).) In this note we extend previous procedures to $q > 2$.

As an example, let $q = 3$ and $N_1 = N_2 = N_3 = N$, say. Let

$$\begin{aligned} y_\alpha &= a_1 x_\alpha^{(1)} + a_2 x_\alpha^{(2)} + a_3 x_\alpha^{(3)}, \\ z_\alpha &= b_1 x_\alpha^{(1)} + b_2 x_\alpha^{(2)} + b_3 x_\alpha^{(3)}, \end{aligned} \tag{2}$$

where $\sum_{g=1}^3 a_g = 0, \sum_{g=1}^3 b_g = 0$ and (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly independent. (In practice the indexing of the observations in each sample would be done randomly.) Then the hypothesis (1) is equivalent to the hypothesis

$$\mathcal{E}y_\alpha = \sum_{g=1}^3 a_g\mu^{(g)} = 0, \qquad \mathcal{E}z_\alpha = \sum_{g=1}^3 b_g\mu^{(g)} = 0. \tag{3}$$

The covariance matrix of $(y_\alpha' \quad z_\alpha')$ is

$$\begin{pmatrix} a_1^2 \Sigma_1 + a_2^2 \Sigma_2 + a_3^2 \Sigma_3 & a_1 b_1 \Sigma_1 + a_2 b_2 \Sigma_2 + a_3 b_3 \Sigma_3 \\ a_1 b_1 \Sigma_1 + a_2 b_2 \Sigma_2 + a_3 b_3 \Sigma_3 & b_1^2 \Sigma_1 + b_2^2 \Sigma_2 + b_3^2 \Sigma_3 \end{pmatrix}. \tag{4}$$

The hypothesis (3) can be tested by a T^2-statistic

$$T^2 = N(\bar{y}' \; \bar{z}')S^{-1}\begin{pmatrix}\bar{y}\\ \bar{z}\end{pmatrix}, \tag{5}$$

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[2] Dr. Charles V. Riche called this problem to my attention.

where

$$S = \frac{1}{N-1} \sum_{\alpha=1}^{N} \begin{pmatrix} y_\alpha - \bar{y} \\ z_\alpha - \bar{z} \end{pmatrix} \begin{pmatrix} y_\alpha - \bar{y} \\ z_\alpha - \bar{z} \end{pmatrix}' \tag{6}$$

and $\bar{y}$ and $\bar{z}$ are the sample mean vectors. When the null hypothesis is true, $(N - 2p)T^2/(N - 1)2p$ has the F-distribution with $2p$ and $N - 2p$ degrees of freedom.

It does not matter what linear combinations (2) are used for the test because the T^2-statistic is invariant with regard to linear transformations; indeed, the linear combinations may be chosen as some contrasts of special interest. The fact that the test is based on a sample covariance matrix S with only $N - 1$ degrees of freedom is a characteristic also of the case $q = 2$, for which Scheffé (1943) showed that this was the maximum number of degrees of freedom for a t-test when $p = 1$. Here N must be greater than $2p$. This extension to $q = 3$ neglects the fact that the off-diagonal submatrices in (4) are symmetric; if such symmetry is imposed on the estimate of (4), the resulting test criterion will not be T^2 and may not have a distribution simply related to the F-distribution.

For any $q > 3$ with equal N_g (2) may be replaced by any $q - 1$ linearly independent linear combinations, the coefficients of each summing to 0,

$$y_\alpha^{(i)} = \sum_{g=1}^{q} a_g^{(i)} x_\alpha^{(g)}, \qquad i = 1, \cdots, q-1, \quad \alpha = 1, \cdots, N. \tag{7}$$

A T^2-statistic may be constructed from the resulting N vectors of $(q - 1)p$ components. If not all N_g are equal, suppose N_1 to be the smallest; define

$$\begin{aligned} y_\alpha^{(i)} &= a_1^{(i)} x_\alpha^{(1)} + \sum_{g=2}^{q} a_g^{(i)} (N_1/N_g)^{\frac{1}{2}} \\ &\quad \cdot \left[x_\alpha^{(g)} - (1/N_1) \sum_{\beta=1}^{N_1} x_\beta^{(g)} + (N_1 N_g)^{-\frac{1}{2}} \sum_{\beta=1}^{N_g} x_\beta^{(g)} \right], \qquad \alpha = 1, \cdots, N_1. \end{aligned} \tag{8}$$

Then

$$\bar{y}^{(i)} = \sum_{g=1}^{N} a_g^{(i)} \bar{x}^{(g)}, \tag{9}$$

and the sample covariance matrix is computed from $y_\alpha^{(i)}$. (See Section 5.6 of Anderson (1958) for details.)

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