CONDITIONAL DISTRIBUTION OF ORDER STATISTICS AND DISTRIBUTION OF THE REDUCED *i*th ORDER STATISTIC OF THE EXPONENTIAL MODEL[1]

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1. Introduction and summary. In case the underlying distribution of a sample is normal, a substantial literature has been devoted to the distribution of quantities such as $(X_{(i)} - u)/v$ and $(X_{(i)} - u)/w$, where $X_{(i)}$ denotes the ith ordered observation, u and v are location and scale statistics of the sample, or one is a location or scale parameter and w is an independent scale statistic.

The case $i = 1$ or n has been frequently studied in view of the great importance of extreme values in physical phenomena and also with a view to testing outlying observations or the normality of the distribution. Bibliographical references will be found in Savage [10] and, as far as the general problem of testing outliers is concerned, in Ferguson [4]; references to recent literature include Dixon [1], [2], Grubbs [5], Pillai and Tienzo [9].

Thompson [12] has studied the distribution of $(X_i - \bar{X})/s$ where X_i is one observation picked at random among the sample, and this statistic has been used in the study of outliers; Laurent has generalized Thompson's distribution to the case of a subsample picked at random among a sample [7], then to the multivariate case and the general linear hypothesis [8]. Thompson's distribution is not only the marginal distribution of $(X_i - \bar{X})/s$ but its conditional distribution, given the sufficient statistic $(\bar{X}, s)$, hence it provides the distribution of X_i given $\bar{X}$, s, and, using the Rao-Blackwell-Lehmann-Scheffé theorem, gives a way of obtaining a minimum variance unbiased estimate of any estimable function of the parameters of a normal distribution for which an unbiased estimate depending on one observation is available, a fact that has been exploited in sampling inspection by variable.

The present paper presents an analogue to Thompson's distribution in case the underlying distribution of a sample is exponential (the exponential model is nowadays widely used in Failure and Queuing Theories). Such a distribution makes it possible to obtain minimum variance unbiased estimates of functions of the parameters of the exponential distribution. Here an estimate is provided for the survival function $P(X > x) = S(x)$ and its powers. As an application of these results the probability distribution of the "reduced" ith ordered observation in a sample and that of the reduced range are derived. For possible applications to testing outliers or exponentiality the reader is invited to refer to the bibliography.

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2. An exponential analogue to Thompson's distribution. Let $X_{(1)}$ and $\bar{X}$ be the smallest observation and the arithmetic mean of a sample $\mathbf{X} = (X_1, \cdots, X_n)$ of n independent observations with probability density

$$(1) \qquad (1/c)e^{-(x-m)/c} \quad \text{for } x \geqq m; \quad 0 \text{ otherwise.}$$

Let $\tilde{\mathbf{X}} = (X_{(1)}, \cdots, X_{(n)})$ be the order statistic of the sample.

It is well known [3] that $(\bar{X}, X_{(1)})$ hence $(X_{(1)}, Y)$, where Y denotes $\bar{X} - X_{(1)}$, constitute a complete and sufficient statistic for the distribution of X. Note that $n(\bar{X} - m) = n(X_{(1)} - m) + nY$. Now $2n(\bar{X} - m)/c$ and

$$2n(X_{(1)} - m)/c$$

are chi-square variates with $2n$ and 2 degrees of freedom respectively. Since Y/c is parameter free and independent of $X_{(1)}$, it follows that $2nY/c$ is chi-square with $2(n-1)$ degrees of freedom and independent of $X_{(1)}$.

Let $\xi = (\xi_1, \cdots, \xi_k)$ be a subsample of X with order statistic $\tilde{\xi} = (\xi_{(1)}, \cdots, \xi_{(k)})$ and sufficient statistic $(\xi_{(1)}, \bar{\xi} - \xi_{(1)})$, and $\mathbf{X}^* = (X_1^*, \cdots, X_{n-k}^*)$ the complementary subsample (so that $\mathbf{X} = (\xi, \mathbf{X}^*)$) with order statistic $\tilde{\mathbf{X}}^* = (X_{(1)}^*, \cdots, X_{(n-k)}^*)$ and sufficient statistic $(X_{(1)}^*, Y^*)$, where $Y^* = \bar{X}^* - X_{(1)}^*$.

The conditional density $f(\tilde{\xi} \mid X_{(1)}, Y)$ of $\tilde{\xi}$, given $X_{(1)}, Y$, is parameter free, and according to the Rao-Blackwell-Lehmann-Scheffé theorem the expected value, with respect to f, of any unbiased estimate $\varphi(\tilde{\xi})$ of an estimable function $\psi(m, c)$ of the parameters m, c is the uniformly minimum variance unbiased estimate of ψ. This density is obtained by writing the joint density of $\tilde{\xi}$, $X_{(1)}^*$, Y^*, making the change of variables $X_{(1)}^* = X_{(1)}^*(X_{(1)}, Y, \tilde{\xi})$, $Y^* = Y^*(X_{(1)}, Y, \tilde{\xi})$, then dividing by the joint density of $X_{(1)}$, Y. The latter is

$$(3) \qquad [n^n/c^n\Gamma(n-1)]Y^{n-2} \exp -(n/c)[(X_{(1)} - m) + Y].$$

The joint density of $\tilde{\xi}$, $X_{(1)}^*$, Y^* is

$$(4) \qquad \begin{aligned} &[k!(n-k)^{n-k}/c^n\Gamma(n-k-1)]Y^{*n-k-2} \exp -[k(\bar{\xi} - X_{(1)}^*) \\ &\qquad\qquad + n(X_{(1)}^* - m) + (n-k)Y^*]/c. \end{aligned}$$

By definition one has

$$(5) \qquad Y^* = nY/(n-k) - [k/(n-k)](\bar{\xi} - X_{(1)}) - (X_{(1)}^*),$$

and there are two cases to consider according to whether $\xi_{(1)}$ is greater than or equal to $X_{(1)}$.

In case $\xi_{(1)} > X_{(1)}$, then $X_{(1)}^* = X_{(1)}$; by substituting the proper values into (4)—the Jacobian of the substitution is

$$J(\tilde{\xi}, X_{(1)}^*, Y^*; \tilde{\xi}, X_{(1)}, Y) = n/(n-k)$$

—and dividing by (3), one obtains the conditional density

$$(6) \quad f(\tilde{\xi} \mid (X_{(1)}, Y) = \frac{k!(n-2)!}{(n-k-2)!}\left(1 - \frac{k}{n}\right)\left[1 - \frac{k}{n}\frac{\bar{\xi} - X_{(1)}}{Y}\right]^{n-k-2} \frac{1}{(nY)^k}$$

in the domain where the bracket is positive.

In case $\xi_{(1)} = X_{(1)}$, let $\tilde{\xi} = (\xi_{(1)}, \tilde{\xi}^-)$, where $\tilde{\xi}^- = (\xi_{(2)}, \cdots, \xi_{(k)})$; using the same technique as above yields the conditional density of $\tilde{\xi}^-$, $X^*_{(1)}$ given $X_{(1)}$, Y. By integrating $X^*_{(1)} - X_{(1)}$ from 0 to $[nY - k(\bar{\xi} - X_{(1)})]/(n - k)$, one obtains

$$(7) \qquad f(\tilde{\xi}^- \mid X_{(1)}, Y) = \frac{k!(n-2)!}{(n-k-1)!}\frac{1}{n}\left[1 - \frac{k}{n}\frac{\bar{\xi} - X_{(1)}}{Y}\right]^{n-k-1}\frac{1}{n^{k-1}Y^{k-1}}$$

in the domain where the bracket is positive. As the reduced variables $U_{(i)} = (\xi_{(i)} - X_{(1)})/nY$ are independent of $X_{(1)}$ and Y, the densities above provide also the marginal densities of $\tilde{U} = (U_{(1)}, \cdots, U_{(k)})$ and $\tilde{U}^- = (U_{(2)}, \cdots, U_{(k)})$.

For $k = 1$, one obtains

$$(8) \qquad \begin{aligned} f(\xi \mid X_{(1)}, Y) &= 1/n && \text{if } \xi = X_{(1)}, \\ &= (n-2)\left(1 - \frac{1}{n}\right)\left[1 - \frac{\xi - X_{(1)}}{Y}\cdot\frac{1}{n}\right]^{n-3}\frac{1}{nY} && \text{if } X_{(1)} < \xi \leqq X_{(1)} + nY, \\ &= 0 && \text{otherwise,} \end{aligned}$$

which provides also the marginal density of the reduced variable

$$U = (\xi - X_{(1)})/nY.$$

The corresponding distribution is the analogue for the exponential case of Thompson's distribution for a normal sample.

3. Best estimates of $S(x)$ and $S^r(x)$. Let $S(x)$ denote the "survival function"

$$(9) \qquad \begin{aligned} S(x) = P(X > x) &= e^{-(x-m)/c} && \text{for } x \geqq m, \\ &= 1 && \text{otherwise.} \end{aligned}$$

An unbiased estimate of $S(x)$ is the characteristic function $I_x(X)$ of the set $(x, +\infty)$, hence the minimum variance unbiased estimate for $S(x)$ is $E[I_x(X) \mid X_{(1)}, Y]$, namely

$$(10) \qquad \begin{aligned} \hat{S}(x) &= \int_x^\infty f(\xi \mid X_{(1)}, Y)\, d\xi \\ &= 1 && \text{if } x < X_{(1)} \\ &= \left(1 - \frac{1}{n}\right)\left[1 - \frac{x - X_{(1)}}{nY}\right]^{n-2} && \text{if } X_{(1)} \leqq x \leqq X_{(1)} + nY \\ &= 0 && \text{if } x > X_{(1)} + nY. \end{aligned}$$

At the same time,

$$\hat{S}(x) = P(\xi > x \mid X_{(1)}, Y) = P(U > u), \quad \text{with} \quad u = (x - X_{(1)})/nY.$$

This result is consistent with the one given by Tate [11].

Similarly, the minimum variance unbiased estimate for $S^r(x)$ is obtained by integrating (6), with $k = r$, over the domain $\xi_1 = (x, \infty), \cdots, \xi_r = (x, \infty)$. The structures of $S(x)$ and $S^r(x)$, however, suggest trying an estimate $T(x; X_{(1)}, Y)$ obtained by replacing $1 - 1/n$ by some k and Y by Y/r in $\hat{S}(x)$. Taking the expected value of $T(\xi; X_{(1)}, Y)$, (integrate Y from $r(x - X_{(1)})/n$ to $+\infty$ and $X_{(1)}$ from m to $+\infty$) one obtains $k = 1 - r/n$, which yields

$$(11)\qquad \begin{aligned} [S^r(x)]^* &= 1 && \text{if } x < X_{(1)}, \\ &= \left(1 - \frac{r}{n}\right)\left[1 - \frac{r}{n}\frac{x - X_{(1)}}{Y}\right]^{n-2} && \text{if } X_{(1)} \leqq x \leqq X_{(1)} + Yn/r, \\ &= 0 && \text{if } x > X_{(1)} + Yn/r. \end{aligned}$$

4. Distribution of the reduced *i*th order statistic.

(i) *Lemmas*: It is well known [6] that if $A_1 \cdots A_n$ are n events, possible outcomes of a trial, the probability of obtaining at least z of these events is

$$(12)\qquad P(Z \geqq z) = \sum_{r=z}^{n} (-1)^{r+z} \binom{r-1}{z-1} \sum P\left(\bigcap_{i=1}^{r} A_{j_i}\right),$$

where $P(\bigcap_{i=1}^{r} A_{j_i})$ denotes the probability of obtaining $A_{j_i}, \cdots, A_{j_r}$ simultaneously and where the summation extends to the set of $\binom{n}{r}$ combinations of n events r by r.

It is also known that

$$(13)\qquad \sum P\left(\bigcap_{i=1}^{r} A_{j_i}\right) = \frac{1}{r!} E[Z^{[r]}],$$

where $Z^{[r]}$ is the factorial moment of order r of the number of successes Z.

(ii) Let N_x be the number of observations with a value at least equal to x in a sample of size n. The probability distribution of the ith order statistic $X_{(i)}$ is $P(X_{(i)} \leqq x) = 1 - P(N_x \geqq n - i + 1)$, therefore is given by (12) with $A_i = [X_i \geqq x]$ (so that $\sum P(\bigcap_{i=1}^{r} A_{j_i}) = \binom{n}{r} S^r(x))$ and

$$z = n - i + 1.$$

It follows from (13) that $P(X_{(i)} \leqq x)$ admits

$$(14)\qquad 1 - \sum_{r=n-i+1}^{n} (-1)^{r+n-i+1} \binom{r-1}{n-i} N_x^{[r]}/r!$$

as an unbiased estimate and

$$\begin{aligned} (15)\qquad & 1 - \sum_{r=n-i+1}^{n} (-1)^{r+n-i+1} \binom{r-1}{n-i} E\left[\frac{N_x^{[r]}}{r!} \,\middle|\, X_{(1)}, Y\right] \\ (16)\qquad & = 1 - \sum_{r=n-i+1}^{n} (-1)^{r+n-i+1} \binom{r-1}{n-i} \sum P\left[\bigcap_{i=1}^{r} A_{j_i} \mid X_{(1)}, Y\right] \\ (17)\qquad & = P(X_{(i)} \leqq x \mid X_{(1)}, Y) \end{aligned}$$

as its minimum variance unbiased estimate.

As $N_x^{[r]}/r!$ is an unbiased estimate of $\sum P(\bigcap_{i=1}^r A_{j_i}) = \binom{n}{r} S^r(x)$ it follows that

$$E\left[\frac{N_x^{[r]}}{r!} \,\middle|\, X_{(1)}, Y\right] = \sum P\left(\bigcap_{i=1}^{r} A_{j_i} \mid X_{(1)}, Y\right)$$

is the minimum variance unbiased estimate $\binom{n}{r}[S^r(x)]^*$ of $\binom{n}{r} S^r(x)$. Making the proper substitution in (16), (17) yields (in view of (11))

$$(18)\quad P(X_{(i)} \leqq x \mid X_{(1)}, Y) = 1 - \sum_{r=n-i+1}^{n} (-1)^{r+n-i+1} \binom{r-1}{n-i} \binom{n}{r} [S^r(x)]^*$$

$$(19)\quad = 1 - \sum_{r=n-i+1}^{k} (-1)^{r+n-i+1} \binom{r-1}{n-i} \binom{n}{r} \left(1 - \frac{r}{n}\right) \cdot \left(1 - \frac{r}{n} \frac{x - X_{(1)}}{Y}\right)^{n-2}$$

for $x \geqq X_{(1)}$ and $k \leqq nY/(x - X_{(1)})$, $([S^r(x)]^* = 0$ for $r > nY/(x - X_{(1)}))$.

Since $U_{(i)} = (X_{(i)} - X_{(1)})/nY$ is independent of $X_{(1)}$, Y, one has

$$(20)\quad P(U_{(i)} \leqq u) = 1 - \sum_{r=n-i+1}^{k} (-1)^{r+n-i+1} \binom{n}{r} \binom{r-1}{n-i} \cdot \left(1 - \frac{r}{n}\right) [1 - ru]^{n-2}$$

for $u \geqq 0$ and $k \leqq 1/u$.

TABLE I

Probability of the reduced range, $P(R \leqq u)$

n	u = .10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
3	.00000	.00000	.00000	.00000	.0000	.20000	.40000	.60000	.80000	1.00000
4	.00000	.00000	.00000	.04000	.25000	.52000	.73000	.88000	.97000	1.00000
5	.00000	.00000	.04800	.18400	.50000	.74400	.89200	.96800	.99600	1.00000
6	.00000	.00000	.05450	.36800	.68750	.87200	.95950	.99200	.99950	1.00000
7	.00000	.00032	.14498	.53824	.81250	.93856	.98542	.99808	.99994	1.00000
8	.00000	.00365	.26244	.67475	.89063	.97133	.99490	.99955	.99999	1.00000
9	.00000	.01724	.38704	.77641	.93750	.98689	.99825	.99990	1.00000	1.00000
10	.00000	.03998	.50476	.84883	.96484	.99410	.99941	.99998	1.00000	1.00000

TABLE II

90 % and 99 % quantiles of the reduced range

n	3	4	5	6	7	8	9	10	11	12
$R_{.90}$	.95000	.81743	.70760	.62394	.55907	.50741				
$R_{.99}$	.99500	.94226	.86428	.78853	.72179	.66440	.61517	.57271	.53584	.51357

In case $i = n$ one obtains the distribution of the reduced range R,

$$P(R \leqq u) = 1 - \sum_{j=i}^{k} (-1)^{j+1} \binom{n-1}{j} (1 - ju)^{n-2} \qquad k \leqq 1/u \tag{21}$$

Table I provides for $n = 3$ to 12 and for some values of u the probability of the reduced range. Table II gives, for $n = 3$ to 12, the 99% point of the same statistic.

Alternatively, by means of the transformation $(\bar{X} - X_{(1)})/(X_{(n)} - X_{(1)}) = 1/nU_{(n)}$, one can use these tables to study the deviation of the smallest value from the mean, studentized by the range. Note also

$$(X_{(n)} - \bar{X})/(X_{(n)} - X_{(1)}) = 1 - 1/nU_{(n)}$$

and $(X_{(i)} - \bar{X})/(X_{(n)} - X_{(1)}) = (nU_{(i)} - 1)/nU_{(n)}$.

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