SAMPLING VARIANCES OF THE ESTIMATES OF VARIANCE COMPONENTS IN THE UNBALANCED 3-WAY NESTED CLASSIFICATION

By D. M. Mahamunulu[1]

Sri Venkateswara University

1. Introduction. Sampling variances of estimates of components of variance obtained from data, that are unbalanced, are difficult to obtain compared with similar derivation when the data is balanced. Matrix methods of deriving expressions for the sampling variances of the variance component estimates for the unbalanced case are developed in [3] and they are applied to some special cases in [3], [4] and [5]. Here we extend those results to the case of 3-way hierarchial (nested) classification.

2. Model and analysis of variance. In the earlier work [5] the sampling variances of variance component estimates are obtained by Henderson's Method 1 [2] from data having unequal subclass numbers, assuming the completely random model, namely Eisenhart's Model II, [1]. Here we consider the same situation for the 3-way nested classification.

The linear model for an observation x_{ijlm} is

$$x_{ijlm} = \mu + a_i + b_{ij} + c_{ijl} + e_{ijlm}$$

where μ is the general mean, a_i is the effect due to the ith first stage class A_i, b_{ij} is the effect due to the jth second stage class B_{ij} within A_i, c_{ijl} is the effect of lth third stage class C_{ijl} within B_{ij}, and e_{ijlm} is the residual error of the observation x_{ijlm}. We assume the number of first stage classes A_i is α so that $i = 1, \cdots, \alpha$. Within each A-class A_i there are β_i B-classes so that $j = 1, \cdots, \beta_i$. Further within each B_{ij} class there are γ_{ij} C-classes so that $l = 1, \cdots, \gamma_{ij}$. The number of observations in the third stage class C_{ijl} is n_{ijl}. All terms of the model (except μ) are assumed to be independent and normally distributed random variables with zero means and variances σ_α^2, σ_β^2, σ_γ^2 and σ_e^2 respectively. These are the variance components which are to be estimated. The sampling variances of these estimates are to be found.

The usual analysis of variance is given in Table I where $\beta = \sum_i \beta_i$, $\gamma = \sum_i \sum_j \gamma_{ij}$, $N = \sum_i \sum_j \sum_l n_{ijl}$ and with usual notation for totals and means.

The components of variance are estimated by equating each sum of squares of the ANOVA (except for "total") to its expected value. Denoting the resulting

Received April 10, 1962.

[1] Now at the University of Minnesota.

TABLE I

Analysis of Variance (ANOVA)

Source of Variation	d.f.	Sums of squares
Between A-classes	$\alpha - 1$	$(\sum_i n_{i..}\bar{x}_{i...}^2) - N\bar{x}^2_{....} = T_a - T_f$
Between B-classes within A-classes	$\beta - \alpha$	$(\sum_i\sum_j n_{ij.}\bar{x}_{ij..}^2) - T_a = T_{ab} - T_a$
Between C-classes within B-classes	$\gamma - \beta$	$(\sum_i\sum_j\sum_l n_{ijl}\bar{x}_{ijl.}^2) - T_{ab} = T_{abc} - T_{ab}$
Within C-classes	$N - \gamma$	$(\sum_i\sum_j\sum_l\sum_m x_{ijlm}^2) - T_{abc} = T_0 - T_{abc}$
Total	$N - 1$	$T_0 - T_f$

estimates as $\hat{\sigma}_\alpha^2$, $\hat{\sigma}_\beta^2$, $\hat{\sigma}_\gamma^2$, and $\hat{\sigma}_e^2$, the equations giving them are

$$T_a - T_f = v_1\hat{\sigma}_\alpha^2 + v_2\hat{\sigma}_\beta^2 + v_3\hat{\sigma}_\gamma^2 + v_4\hat{\sigma}_e^2$$
$$T_{ab} - T_a = v_5\hat{\sigma}_\beta^2 + v_6\hat{\sigma}_\gamma^2 + v_7\hat{\sigma}_e^2$$
$$T_{abc} - T_{ab} = v_8\hat{\sigma}_\gamma^2 + v_9\hat{\sigma}_e^2$$
$$T_0 - T_{abc} = v_{10}\hat{\sigma}_e^2$$

where

$$v_1 = N - k_1 \quad v_2 = k_4 - k_2 \quad v_3 = k_5 - k_3 \quad v_4 = \alpha - 1$$
$$v_5 = N - k_4 \quad v_6 = k_6 - k_5 \quad v_7 = \beta - \alpha \quad v_8 = N - k_6$$
$$v_9 = \gamma - \beta \quad v_{10} = N - \gamma.$$

The k's that appear in the above relations are functions of n_{ijl}'s namely

$$k_1 = \textstyle\sum_i n_{i..}^2/N \qquad k_2 = \sum_i \sum_j n_{ij.}^2/N$$
$$k_3 = \textstyle\sum_i \sum_j \sum_l n_{ijl}^2/N \qquad k_4 = \sum_i \sum_j n_{ij.}^2/n_{i..}$$
$$k_5 = \textstyle\sum_i \sum_j \sum_l n_{ijl}^2/n_{i..} \qquad k_6 = \sum_i \sum_j \sum_l n_{ijl}^2/n_{ij.}\,.$$

3. The required variances and covariances. The within C-classes sum of squares $(T_0 - T_{abc})/\sigma_e^2$ has a chi-square distribution with $(N - \gamma)$ degrees of freedom. Hence the variance of $\hat{\sigma}_e^2$ is

$$\text{(3.1)} \qquad \text{var}(\hat{\sigma}_e^2) = 2\sigma_e^4/(N - \gamma) = 2\sigma_e^4/v_{10}\,.$$

Further $T_0 - T_{abc}$ is distributed independently of T_a, T_{ab}, T_{abc} and T_f so that covariances of $\hat{\sigma}_\alpha^2$, $\hat{\sigma}_\beta^2$ and $\hat{\sigma}_\gamma^2$ with $\hat{\sigma}_e^2$ are obtained directly as

$$\text{(3.2)} \qquad \text{cov}(\hat{\sigma}_\gamma^2, \hat{\sigma}_e^2) = -(v_9/v_8)\,\text{var}(\hat{\sigma}_e^2)$$

(3.3) $\operatorname{cov}(\hat{\sigma}_\beta^2, \hat{\sigma}_e^2) = (v_6 v_9/v_8 - v_7) \operatorname{var}(\hat{\sigma}_e^2)/v_5$

(3.4) $\operatorname{cov}(\hat{\sigma}_\alpha^2, \hat{\sigma}_e^2) = [v_3 v_5 v_9 + v_2(v_7 v_8 - v_6 v_9) - v_4 v_5 v_8] \operatorname{var}(\hat{\sigma}_e^2)/(v_1 v_5 v_8).$

This property of independence can also be used to derive the variances of $\hat{\sigma}_\alpha^2$, $\hat{\sigma}_\beta^2$ and $\hat{\sigma}_\gamma^2$ and the covariances between them in terms of var $(\hat{\sigma}_e^2)$ and the variances and covariances of T_a, T_{ab}, T_{abc} and T_f. Now

(3.5) $v_8^2 \operatorname{var}(\hat{\sigma}_\gamma^2) = \operatorname{var}(T_{abc} - T_{ab}) + v_9^2 \operatorname{var}(\hat{\sigma}_e^2)$

(3.6)
$$v_5^2 v_8^2 \operatorname{var}(\hat{\sigma}_\beta^2) = \operatorname{var}[v_8 T_a - (v_8 + v_6) T_{ab} + v_6 T_{abc}] + [v_8 \alpha - (v_8 + v_6)\beta + v_6 \gamma]^2 \operatorname{var}(\hat{\sigma}_e^2)$$

(3.7)
$$\begin{aligned} v_1^2 v_5^2 v_8^2 \operatorname{var}(\hat{\sigma}_\alpha^2) = {} & \operatorname{var}[v_8(v_2 + v_5) T_a - (v_2 v_8 + v_2 v_6 - v_3 v_5) T_{ab} \\ & + (v_2 v_6 - v_3 v_5) T_{abc} - v_5 v_8 T_f] + [v_8(v_2 + v_5)\alpha \\ & - (v_2 v_8 + v_2 v_6 - v_3 v_5)\beta + (v_2 v_6 - v_3 v_5)\gamma - v_5 v_8]^2 \operatorname{var}(\hat{\sigma}_e^2) \end{aligned}$$

Further we have

(3.8)
$$\begin{aligned} v_5 v_8 \operatorname{cov}(\hat{\sigma}_\beta^2, \hat{\sigma}_\gamma^2) = {} & \operatorname{cov}(T_{ab} - T_a, T_{abc} - T_{ab}) + v_7 v_9 \operatorname{var}(\hat{\sigma}_e^2) \\ & - v_6 v_8 \operatorname{var}(\hat{\sigma}_\gamma^2) \end{aligned}$$

(3.9)
$$\begin{aligned} v_1 v_5 v_8 \operatorname{cov}(\hat{\sigma}_\alpha^2, \hat{\sigma}_\gamma^2) = {} & [v_5 \operatorname{cov}(T_a - T_f, T_{abc} - T_{ab}) \\ & - v_2 \operatorname{cov}(T_{ab} - T_a, T_{abc} - T_{ab})] \\ & + v_9(v_4 v_5 - v_2 v_7) \operatorname{var}(\hat{\sigma}_e^2) - v_8(v_3 v_5 - v_2 v_6) \operatorname{var}(\hat{\sigma}_\gamma^2) \end{aligned}$$

(3.10)
$$\begin{aligned} v_1 v_5 v_8 \operatorname{cov}(\hat{\sigma}_\alpha^2, \hat{\sigma}_\beta^2) = {} & [v_8 \operatorname{cov}(T_a - T_f, T_{ab} - T_a) \\ & - v_6 \operatorname{cov}(T_a - T_f, T_{abc} - T_{ab}) \\ & - v_3 \operatorname{cov}(T_{ab} - T_a, T_{abc} - T_{ab})] \\ & + [v_4 v_7 v_8 - v_9(v_2 v_6 + v_3 v_7)] \operatorname{var}(\hat{\sigma}_e^2) \\ & - v_2 v_5 v_8 \operatorname{var}(\hat{\sigma}_\beta^2) + v_3 v_6 v_8 \operatorname{var}(\hat{\sigma}_\gamma^2). \end{aligned}$$

The second term in each of these expressions can be obtained from Equation (3.1). The first terms on the right side of Equations (3.5) to (3.10) can be expressed as linear functions of variances and covariances of T_a, T_{ab}, T_{abc} and T_f.

4. Variances and covariances of T_a, T_{ab}, T_{abc} and T_f. By adopting exactly the same methods as in Searle [5] we can get the variances and covariances of T_a, T_{ab}, T_{abc} and T_f. [Details are omitted.] The results are given below. The constants "k_i" that appear in Equations (4.1) through (4.10) are defined in (4.11).

(4.1)
$$\begin{aligned} \operatorname{var}(T_a) = {} & 2(Nk_1\sigma_\alpha^4 + k_{22}\sigma_\beta^4 + k_{21}\sigma_\gamma^4 + \alpha\sigma_e^4 + 2Nk_2\sigma_\alpha^2\sigma_\beta^2 \\ & + 2Nk_3\sigma_\alpha^2\sigma_\gamma^2 + 2N\sigma_\alpha^2\sigma_e^2 + 2k_{20}\sigma_\beta^2\sigma_\gamma^2 + 2k_4\sigma_\beta^2\sigma_e^2 \\ & + 2k_5\sigma_\gamma^2\sigma_e^2) \end{aligned}$$

$$(4.2)\quad \operatorname{var}(T_{ab}) = \operatorname{var} T_a + 2[(Nk_2 - k_{22})\sigma_\beta^4 + (k_{19} - k_{21})\sigma_\gamma^4 + (\beta - \alpha)\sigma_e^4 + 2(Nk_3 - k_{20})\sigma_\beta^2\sigma_\gamma^2 + 2(N - k_4)\sigma_\beta^2\sigma_e^2 + 2(k_6 - k_5)\sigma_\gamma^2\sigma_e^2]$$

$$(4.3)\quad \operatorname{var}(T_{abc}) = \operatorname{var}(T_{ab}) + 2[(Nk_3 - k_{19})\sigma_\gamma^4 + (\gamma - \beta)\sigma_e^4 + 2(N - k_6)\sigma_\gamma^2\sigma_e^2]$$

$$(4.4)\quad \operatorname{var}(T_f) = 2(k_1\sigma_\alpha^2 + k_2\sigma_\beta^2 + k_3\sigma_\gamma^2 + \sigma_e^2)^2$$

$$(4.5)\quad \operatorname{cov}(T_a, T_{ab}) = \operatorname{var}(T_a) + 2[(k_{12} - k_{22})\sigma_\beta^4 + (k_{18} - k_{21})\sigma_\gamma^4 + 2(k_{16} - k_{20})\sigma_\beta^2\sigma_\gamma^2]$$

$$(4.6)\quad \operatorname{cov}(T_a, T_{abc}) = \operatorname{var}(T_a) + 2[(k_{12} - k_{22})\sigma_\beta^4 + (k_{10} - k_{21})\sigma_\gamma^4 + 2(k_{16} - k_{20})\sigma_\beta^2\sigma_\gamma^2] = \operatorname{cov}(T_a, T_{ab}) + 2(k_{10} - k_{18})\sigma_\gamma^4$$

$$(4.7)\quad \operatorname{cov}(T_a, T_f) = 2(k_7\sigma_\alpha^4 + k_{13}\sigma_\beta^4 + k_{14}\sigma_\gamma^4 + N\sigma_e^4 + 2k_{23}\sigma_\alpha^2\sigma_\beta^2 + 2k_{24}\sigma_\alpha^2\sigma_\gamma^2 + 2Nk_1\sigma_\alpha^2\sigma_e^2 + 2k_{17}\sigma_\beta^2\sigma_\gamma^2 + 2Nk_2\sigma_\beta^2\sigma_e^2 + 2Nk_3\sigma_\gamma^2\sigma_e^2)/N$$

$$(4.8)\quad \operatorname{cov}(T_{ab}, T_{abc}) = \operatorname{var}(T_{ab}) + 2(k_{11} - k_{19})\sigma_\gamma^4$$

$$(4.9)\quad \operatorname{cov}(T_{ab}, T_f) = \operatorname{cov}(T_a, T_f) + 2[(k_8 - k_{13})\sigma_\beta^4 + (k_{15} - k_{14})\sigma_\gamma^4 + 2(k_{25} - k_{17})\sigma_\beta^2\sigma_\gamma^2]/N$$

$$(4.10)\quad \operatorname{cov}(T_{abc}, T_f) = \operatorname{cov}(T_{ab}, T_f) + 2(k_9 - k_{15})\sigma_\gamma^4/N.$$

The k's that appeared in the relations (4.1)–(4.10) are functions of n_{ijl}'s and are defined as follows

$$k_7 = \textstyle\sum_i n_{i..}^3 \qquad k_8 = \sum_i \sum_j n_{ij.}^3$$

$$k_9 = \textstyle\sum_i \sum_j \sum_l n_{ijl}^3 \qquad k_{10} = \sum_i (\sum_j \sum_l n_{ijl}^3)/n_{i..}$$

$$k_{11} = \textstyle\sum_i \sum_j (\sum_l n_{ijl}^3)/n_{ij.} \qquad k_{12} = \sum_i (\sum_j n_{ij.}^3)/n_{i..}$$

$$k_{13} = \textstyle\sum_i (\sum_j n_{ij.}^2)^2/n_{i..} \qquad k_{14} = \sum_i (\sum_j \sum_l n_{ijl}^2)^2/n_{i..}$$

$$k_{15} = \textstyle\sum_i \sum_j (\sum_l n_{ijl}^2)^2/n_{ij.} \qquad k_{16} = \sum_i \{\sum_j n_{ij.}(\sum_l n_{ijl}^2)\}/n_{i..}$$

$$(4.11)\quad k_{17} = \textstyle\sum_i (\sum_j n_{ij.}^2)(\sum_j \sum_l n_{ijl}^2)/n_{i..}$$

$$k_{18} = \textstyle\sum_i \{\sum_j (\sum_l n_{ijl}^2)^2/n_{ij.}\}/n_{i..}$$

$$k_{19} = \textstyle\sum_i \sum_j (\sum_l n_{ijl}^2)^2/n_{ij.}^2 \qquad k_{20} = \sum_i (\sum_j n_{ij.}^2)(\sum_j \sum_l n_{ijl}^2)/n_{i..}^2$$

$$k_{21} = \textstyle\sum_i (\sum_j \sum_l n_{ijl}^2)^2/n_{i..}^2 \qquad k_{22} = \sum_i (\sum_j n_{ij.}^2)^2/n_{i..}^2$$

$$k_{23} = \textstyle\sum_i n_{i..} (\sum_j n_{ij.}^2) \qquad k_{24} = \sum_i n_{i..}(\sum_j \sum_l n_{ijl}^2)$$

$$k_{25} = \textstyle\sum_i \sum_j n_{ij.}(\sum_l n_{ijl}^2)$$

5. Results. Using the expressions for variances and covariances of T_a, T_{ab}, T_{abc} and T_f from (3.7) we get

$$v_1^2v_5^2v_8^2 \text{ var } (\hat{\sigma}_\alpha^2) = 2[g_1\sigma_\alpha^4 + g_2\sigma_\beta^4 + g_3\sigma_\gamma^4 + g_4\sigma_e^4 + 2g_5\sigma_\alpha^2\sigma_\beta^2 + 2g_6\sigma_\alpha^2\sigma_\gamma^2 + 2g_7\sigma_\alpha^2\sigma_e^2 + 2g_8\sigma_\beta^2\sigma_\gamma^2 + 2g_9\sigma_\beta^2\sigma_e^2 + 2g_{10}\sigma_\gamma^2\sigma_e^2]$$

where

$$g_1 = v_5^2v_8^2[k_1(N + k_1) - 2k_7/N]$$

$$g_2 = v_5^2v_8^2(k_{22} + k_2^2 - 2k_{13}/N) + v_2^2v_8^2(Nk_2 + k_{22} - 2k_{12}) - 2v_2v_5v_8^2\{(k_{12} - k_{22}) - (k_8 - k_{13})/N\}$$

$$g_3 = v_5^2v_8^2(k_{21} + k_3^2 - 2k_{14}/N) + v_2^2v_8^2(k_{19} + k_{21} - 2k_{18}) + (v_2v_6 - v_3v_5)^2(Nk_3 + k_{19} - 2k_{11}) - 2v_2v_5v_8^2[(k_{18} - k_{21}) - (k_{15} - k_{14})/N] + 2v_5v_8(v_2v_6 - v_3v_5)[(k_{10} - k_{18}) - (k_9 - k_{15})/N] - 2v_2v_8[(k_{11} - k_{19}) - (k_{10} - k_{18})]$$

$$g_4 = v_5^2v_8^2(\alpha + 1 - 2N) + v_2^2v_8^2(\beta - \alpha) + (v_2v_6 - v_3v_5)^2(\gamma - \beta) + [v_5v_8(\alpha - 1) + v_2v_8(\alpha - \beta) + (v_2v_6 - v_3v_5)(\gamma - \beta)]^2/v_{10}$$

$$g_5 = v_5^2v_8^2[k_2(N + k_1) - 2k_{23}/N], \qquad g_6 = v_5^2v_8^2[k_3(N + k_1) - 2k_2/N],$$

$$g_7 = v_5^2v_8^2(N - k_1)$$

$$g_8 = v_5^2v_8^2(k_{20} + k_2k_3 - 2k_{17}/N) + v_2^2v_8^2(Nk_3 - k_{16}) - 2v_2v_5v_8^2[(k_{16} - k_{20}) - (k_{25} - k_{17})/N]$$

$$g_9 = v_5^2v_8^2(k_4 - k_2) + v_2^2v_8^2(N - k_4)$$

$$g_{10} = v_5^2v_8^2(k_5 - k_3) + v_2^2v_8^2(k_6 - k_5) + (v_2v_6 - v_3v_5)^2(N - k_6).$$

Similarly (3.6) reduces to

$$v_5^2v_8^2 \text{ var } (\hat{\sigma}_\beta^2) = 2(d_1\sigma_\beta^4 + d_2\sigma_\gamma^4 + d_3\sigma_e^4 + 2d_4\sigma_\beta^2\sigma_\gamma^2 + 2d_5\sigma_\beta^2\sigma_e^2 + 2d_6\sigma_\gamma^2\sigma_e^2)$$

where

$$d_1 = v_8^2(Nk_2 + k_{22} - 2k_{12})$$

$$d_2 = v_8^2(k_{19} + k_{21} - 2k_{18}) + v_6^2(Nk_3 + k_{19} - 2k_{11}) + 2v_6v_8(k_{10} - k_{18} + k_{11} - k_{19})$$

$$d_3 = v_8^2(\beta - \alpha) + v_6^2(\gamma - \beta) + [v_8(\alpha - \beta) + v_6(\gamma - \beta)]^2/v_{10}$$

$$d_4 = v_8^2(Nk_3 + k_{20} - 2k_{16}), \qquad d_5 = (N - k_6)^2(N - k_4)$$

$$d_6 = (N - k_6)(N - k_5)(k_6 - k_5).$$

The relation (3.5) reduces to

$$v_8^2 \text{ var } (\hat{\sigma}_\gamma^2) = 2(Nk_3 + k_{19} - 2k_{11})\sigma_\gamma^4 + 4(N - k_6)\sigma_\gamma^2\sigma_e^2 + 2(\gamma - \beta)(N - \beta)\sigma_e^4/(N - \gamma)$$

and (3.8) simplifies to

$$v_5 v_8 \operatorname{cov}(\hat\sigma_\beta^2, \hat\sigma_\gamma^2) = 2(k_{11} - k_{19} + k_{18} - k_{10})\sigma_\gamma^4 + 2(v_7 v_9/v_{10})\sigma_e^4 - v_6 v_8 \operatorname{var}(\hat\sigma_\gamma^2).$$

Further from (3.9) we get

$$\begin{aligned} v_1 v_5 v_8 \operatorname{cov}(\hat\sigma_\alpha^2, \hat\sigma_\gamma^2) &= 2[v_5\{(k_{10} - k_{18}) - (k_9 - k_{15})/N\} \\ &\quad - v_2\{(k_{11} - k_{19}) - (k_{10} - k_{18})\}]\sigma_\gamma^4 + 2(v_9/v_{10})(v_4 v_5 - v_2 v_7)\sigma_e^4 \\ &\quad - v_8(v_3 v_5 - v_2 v_6) \operatorname{var}(\hat\sigma_\gamma^2) \end{aligned}$$

Finally from (3.10) we have

$$\begin{aligned} v_1 v_5 v_8 \operatorname{cov}(\hat\sigma_\alpha^2, \hat\sigma_\beta^2) &= 2[(k_{12} - k_{22}) - (k_8 - k_{13})/N]\sigma_\beta^4 \\ &\quad + 2[(k_{18} - k_{21}) - (k_{15} - k_{14})/N - v_6\{(k_{10} - k_{18}) - (k_9 - k_{15})/N\} \\ &\quad - v_3(k_{11} - k_{19} + k_{18} - k_{10})]\sigma_\gamma^4 + 2[(k_{16} - k_{20}) - (k_{25} - k_{17})/N]\sigma_\beta^2\sigma_\gamma^2 \\ &\quad + (2/v_{10})[v_4 v_7 v_8 - v_9(v_4 v_6 + v_3 v_7)]\sigma_e^4 - v_2 v_5 v_8 \operatorname{var}(\hat\sigma_\beta^2) + v_3 v_6 v_8 \operatorname{var}(\hat\sigma_\gamma^2). \end{aligned}$$

The expressions for variances and covariances of the variance components' estimates involve products of the unknown variance components σ_α^2, σ_β^2, σ_γ^2 and σ_e^2. If one is interested in estimating these variances and covariances, one substitutes the estimates $\hat\sigma_\alpha^2$, $\hat\sigma_\beta^2$, $\hat\sigma_\gamma^2$ and $\hat\sigma_e^2$ for the parameters σ_α^2, σ_β^2, σ_γ^2 and σ_e^2 respectively. The estimates thus obtained will in general be biased. In order to obtain unbiased estimates one has to proceed as follows.

In the formulae for variances and covariances of $\hat\sigma_\alpha^2$, $\hat\sigma_\beta^2$, $\hat\sigma_\gamma^2$ and $\hat\sigma_e^2$, every product of the type $\sigma_\theta^2\sigma_\varphi^2$ is to be replaced by $\hat\sigma_\theta^2\hat\sigma_\varphi^2 - \operatorname{cov}(\hat\sigma_\theta^2, \hat\sigma_\varphi^2)$ whenever θ and φ are different. The terms of the type σ_θ^4 are to be replaced by $(\hat\sigma_\theta^2)^2 - \operatorname{var}(\hat\sigma_\theta^2)$. Then one can rewrite those formulae as 10 simultaneous equations for the estimates of variances and covariances of variance components' estimates. Solving these equations one can get the required estimates, which are unbiased.

6. Balanced data. The formulae derived in the previous section reduce to the already known results for balanced data when all the n_{ijl} are put equal to n, say. Suppose that every first stage class contains b second stage classes which in turn each contains c third stage classes. Then we can replace β and γ in the earlier formulae by ab and abc respectively, where a is the number of first stage classes.

For example, we have, then

$$\operatorname{var}(\hat\sigma_\gamma^2) = \frac{2(abcn^2 + abn^2 - 2abn^2)\sigma_\gamma^4 + 4abn(c-1)\sigma_\gamma^2\sigma_e^2 + 2ab(c-1)(cn-1)\sigma_e^4/c(n-1)}{a^2 b^2 n^2 (c-1)^2}.$$

This reduces to

$$\operatorname{var}(\hat\sigma_\gamma^2) = \frac{2}{n^2}\left[\frac{(n\sigma_\gamma^2 + \sigma_e^2)^2}{ab(c-1)} + \frac{\sigma_e^4}{abc(n-1)}\right].$$

The same result we get directly for the balanced case using the fact that

$$(T_{abc} - T_{ab})/(n\sigma_\gamma^2 + \sigma_e^2) \qquad \text{and} \qquad (T_0 - T_{abc})/\sigma_e^2$$

are then distributed independently as χ^2 with $ab(c - 1)$ and $abc(n - 1)$ degrees of freedom respectively. So we have $E(T_{abc} - T_{ab}) = ab(c - 1)(n\sigma_\gamma^2 + \sigma_e^2)$ and $E(T_0 - T_{abc}) = abc(n - 1)\sigma_e^2$ and their variances are equal to twice the square of their expectations divided by their degrees of freedom. Thus the variance of the estimate of σ_γ^2, namely

$$\hat{\sigma}_\gamma^2 = \frac{1}{n}\left[\frac{T_{abc} - T_{ab}}{ab(c - 1)} - \frac{T_0 - T_{abc}}{abc(n - 1)}\right]$$

is same as the expression obtained above. The other results can also be verified.

7. Acknowledgment. The author takes this opportunity to express his thanks to Professor H. S. Konijn, Professor I. R. Savage and the referee for very helpful suggestions and comments.

REFERENCES

[1] EISENHART, CHURCHILL (1947). The assumptions underlying the analysis of variance. *Biometrics* **3** 1–21.

[2] HENDERSON, C. R. (1953). Estimation of variance and covariance components. *Biometrics* **9** 226–252.

[3] SEARLE, S. R. (1956). Matrix methods in variance and covariance components analysis. *Ann. Math. Statist.* **27** 737–748.

[4] SEARLE, S. R. (1958). Sampling variances of estimates of components of variances. *Ann. Math. Statist.* **29** 167–178.

[5] SEARLE, S. R. (1961). Variance components in the unbalanced 2-way nested classification. *Ann. Math. Statist.* **32** 1161–1166.