ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Central Regional Meeting, Madison, Wisconsin, June 14–15, 1963. Additional abstracts appeared in the March and June, 1963 issues.)

8. Two Normally Distributed Distribution-Free Statistics. C. B. BELL and KJELL DOKSUM, San Diego State College.

For the 1-sample $H_0 : F = F_0$, the statistic $\bar{Z} = n^{-1} \sum J(F_0(X_i))$, where J is the inverse of the cpf of a standard normal, is compared with the Birnbaum-Chapman $\bar{U} = n^{-1} \sum F_0(X_i)$ and $\bar{X}$. For the 2-sample $H_0' : F_1 = F_2$, one considers $\bar{c} = n^{-1} \sum W(R(X_i)) - m^{-1} \sum W(R(Y_j))$, where $R(X_i)$ is the rank of X_i in the combined sample; the W_i $(1 \leqq i \leqq n + m)$ constitute an independent random sample from $n(0, 1)$; and the $W(i)$ are the order statistics. $\bar{c}$ is compared to the Fisher-Yates c_1, Van der Waerden c_1', Wilcoxon V and Student T. *Theorem* 1. $\bar{Z}$ is DF wrt the class of continuous cpfs; and under H_0 is exactly $n(0, n^{-1})$ for all n. *Theorem* 2. $\bar{c}$ is DF wrt the class of continuous cpfs; and under H_0' is exactly $n(0, n^{-1} + m^{-1})$ for all n and m. *Theorem* 3. The asymptotic relative efficiencies $A(\cdot, \cdot)$ for translation alternatives (and the appropriate number of samples) satisfy (a) $A(\bar{U}, \bar{X}) = A(V, T)$ $(= 3\pi^{-1}$ for normals); (b) $A(\bar{Z}, \bar{X}) = A(c_1, T) = A(c_1', T) = A(\bar{c}, T) \geqq 1$, with equality iff the cpfs are normals. Inverses of cpfs other than $n(0, 1)$, and generalizations to k-sample and independence hypotheses are considered.

9. A New Test of Fiducial Consistency. ROBERT J. BUEHLER, Iowa State University.

Let y be a (univariate) future observation and $\mathbf{x}$ be a vector of past observations, both having distributions depending on θ. A function $L(\mathbf{x}; \alpha)$ provides a system of "upper prediction limits" for y with "fiducial" probability α if $P(Y \leqq L(\mathbf{X}; \alpha) \mid \theta) = \alpha$ for all θ. *Lemma:* $L(\mathbf{x}; \alpha)$ exists for $0 < \alpha < 1$ if and only if there exists an ancillary statistic $\phi(\mathbf{X}, Y)$ having a uniform distribution on the unit interval $(\phi(\mathbf{X}, Y) \sim U(0, 1))$ for all θ. Formally, $L(\mathbf{x}; \alpha)$ is the value of y for which $\phi(x, y) = \alpha$, and formally $\phi(\mathbf{x}, y) = P_f(Y \leqq y \mid \mathbf{x})$ where P_f denotes fiducial probability. In "Statistical Methods and Scientific Inference," pp 113 and 126, Fisher implicitly indicates that $P_f(Y \leqq y \mid \mathbf{x})$ can be obtained from the fiducial density $f_f(\theta \mid \mathbf{x})$; explicitly, $\phi(\mathbf{x}, y) = P_f(Y \leqq y \mid \mathbf{x}) = \int P(Y \leqq y \mid \theta) f_f(\theta \mid \mathbf{x}) d\theta$. The following consistency test is proposed: Is $\phi(\mathbf{X}, Y) \sim U(0, 1)$ for all θ? This test is different from Lindley's (*J. Roy. Statist. Soc. Ser. B* **20** (1958) 102), although similar in spirit. Suppose $x_i = t_i + \theta$ (or $\sigma t_i + \theta$) and $y = u + \theta$ (or $\sigma u + \theta$) and the joint density $f(t_1, \cdots, t_n) g(u)$ is given. In the (generalized) Pitman case $f_f(\theta \mid \mathbf{x})$ (or $f_f(\theta, \sigma \mid \mathbf{x})$) is the posterior density corresponding to a uniform prior density of θ (or of θ and $\log \sigma$). Consistency is shown to hold. Immediate generalizations follow by separate transformations of the variates and parameters.

10. On Asymptotic Normality of a Class of Statistics Related to Linear Stochastic Processes (Preliminary report). K. C. CHANDA, Iowa State University.

Let $F_n(x)$ denote the sample distribution function of random variables $X(1), \cdots, X(n)$ being a realization from a linear normal process $\{X(t), t = 0, \pm 1, \cdots\}$ defined by $X(t) = \sum_{j=0}^{\infty} g_j \epsilon(t - j)$, $E\{X(t)\} = 0$, $\sum_{j=0}^{\infty} |g_j| < \infty$ where $\{\epsilon(t), t = 0, \pm 1, \cdots\}$ is a pure white noise normal process with mean zero and standard deviation unity. Let $F(x) = P\{X(t) \leqq x\}$. It is then shown that under a set of mild regularity conditions and for arbitrary real numbers $x_1, \cdots, x_k$ the joint distribution of $n^{\frac{1}{2}}\{F_n(x_i) - F(x_i)\}$ $(1 \leqq i \leqq k)$ is, asymptotically, normal with mean vector zero and dispersion matrix $\Sigma = ((\sigma_{ij}))$ where $\sigma_{ij} = \sum_{v=-\infty}^{\infty} h(t_i, t_j, v), h(t_i, t_j, v) = P\{U(1) \leqq t_i, U(v+1) \leqq t_j\} - t_i t_j, U(1) = F\{X(1)/\sigma\}$ etc., $t_i = F(x_i/\sigma)$, and $\sigma^2 = V\{X(t)\}$. Let $K(s, t) = \min(s, t) - st + 2 \sum_{v=1}^{\infty} h(s, t, v)$, $(0 \leqq s, t \leqq 1)$. Then it is conjectured (the author has not been able to prove yet) that for $0 < c < \infty$, $\lim_{n\to\infty} P\{\text{l.u.b.}_{-\infty<x<\infty} n^{\frac{1}{2}} |F_n(x) - F(x)| \leqq c\} = P\{\max_{0 \leqq t \leqq 1} |Y(t)| \leqq c\}$ where $Y(t)$ is a separable normal process with (i) $E\{Y(t)\} = 0$, (ii) $\text{Cov}\{Y(s), Y(t)\} = K(s, t)$. It may be noted that if $I(\xi, x)$ is defined as a function which takes the value 1 whenever $\xi \leqq x$ and 0 otherwise then $F_n(x) = \sum_{t=1}^{n} I\{X(t), x\}/n$. More generally, if $\{W(t), t = 0, \pm 1, \cdots\}$ is a stochastic process defined by $W(t) = G\{X(t)\}$ where $G(x)$ is integrable with respect the normal measure with zero mean then the property of asymptotic normality proved above can be extended to the class of statistics $\sum_{t=1}^{n} W(t)/n$.

11. **On the Consistency of the Two-Sample Empty Cell Test.** M. Csorgo and Irwin Guttman, McGill University and University of Wisconsin.

S. S. Wilks, in the *Proc. Fourth Berkeley Symp.* **I** (1961) and again in his recent book, *Mathematical Statistics* (Wiley, 1962), proposed a two-sample empty cell test. The procedure is as follows. Denoting the order statistics of the first sample of n_1 observations by $X_{(1)}, \cdots, X_{(n_1)}$, define cells $I_1, \cdots, I_{n_1+1}$ by $I_i = (X_{(i-1)}, X_{(i)}), i = 1, \cdots, n_1 + 1$ where $X_{(0)} = -\infty$ and $X_{(n_1+1)} = +\infty$. Let a second sample of n_2 observations be taken and let r_i be the number of observations of the second sample that lies in I_i. Finally, let S_0 be the number of I_i with $r_i = 0$, that is, S_0 is the number of empty cells. A new proof that a test of the hypothesis that the samples come from the same population which rejects if the observed value of S_0 is "significantly" large is given. The method of proof uses the "law of total probability" (see p. 106 of Feller, *An Introduction to Probability Theory and Its Applications*, **1**, 2nd ed.).

12. **Quasi-Martingale Processes.** Donald L. Fisk, Michigan State University. (Introduced by Herman Rubin)

The real valued process $\{X(t), F(t), t \varepsilon T = [a, b]\}$ defined on (Ω, F, P) is called a quasi-martingale process if $X(t)$ has the following decomposition: $X(t) = X_1(t) + X_2(t)$ where $\{X_1(t), F(t), t \varepsilon T\}$ is a martingale process and $X_2(t)$ has almost all sample functions of bounded variation on T. *Theorem:* If $\{X(t), F(t), t \varepsilon T\}$ is a uniformly bounded a.s. sample continuous process such that for some sequence of partitions $\{\Pi_N\}$ of T with $\| \Pi_N \| \to 0$ and $\Pi_N \subset \Pi_{N+1}$ for $N = 1, 2, \cdots$, $\lim E(\sum | E(X(t_{j+1}^N) - X(t_j^N) \mid F(t_j^N)) |) < K < \infty$ then the $X(t)$ process is a quasi-martingale. If $\{X(t), F(t), t \varepsilon T\}$ and $\{Y(t), F(t), t \varepsilon T\}$ are quasi-martingales satisfying certain continuity and boundedness conditions then $Z(t) = \int_a^t Y(s)\, dX(s) = \text{Plim} \sum^t [Y(t_{j+1}^N) + Y(t_j^N)][X(t_{j+1}^N) - X(t_j^N)]/2$ exists for all t in T and furthermore the process $\{Z(t), F(t), t \varepsilon T\}$ is a quasi-martingale process.

13. **Bayesian Bio-Assay.** Charles H. Kraft and Constance van Eeden, University of Minnesota.

Consider the bio-assay problem in which the observation at each dosage level, t, is binomial $[n, F(t)]$. A characterization of the class of all a priori distributions for the distribution function F is given. The corresponding Bayes' estimates are found for a class of loss functions and this class of estimates is shown to be a complete class.

14. Mean and Variance of a Generalized Two-Coin Tossing Problem (Preliminary report). SRI GOPAL MOHANTY, State University of New York.

Let $\mathcal{G}_i(r, a)$ denote the game which starts with coin i, $(i = 1, 2)$ and played with the following rule: (i) the kth trial is made with coin 1 or 2 according as the $(k - 1)$st trial is a tail or a head $(k > 1)$ and (ii) stop making further trials when for the first time the total number of heads is equal to $rn + a$, where n is the number of tails, $r \geqq 0$ and $a > 0$ are integers. Assume p_i to be the probability of obtaining heads in a single trial with coin i such that $p_1 + p_2 \geqq r(q_1 + q_2)$, where $q_i = 1 - p_i$. The mean and variance of the number of trials in $\mathcal{G}_2(r, a)$ are obtained as $a(p_1 + q_2)/(p_1 - rq_2)$ and $a(r + 1)^2 p_1 q_2 (q_1 + p_2)/(p_1 - rq_2)^3$ respectively, and that in $\mathcal{G}_1(r, a)$ as $[(a - 1)(p_1 + q_2) + \{1 + r(q_1 - q_2)\}]/(p_1 - rq_2)$ and $[(a - 1)(r + 1)^2 p_1 q_2 (q_1 + p_2) + (r + 1)^2 q_1 \{p_1 + rq_2(q_1 - q_2)\}/(p_1 - rq_2)^3$ respectively. The solution to the "ballot problem" follows from a special case of the game, when $p_1 = p_2$.

15. λ-Continuous Markov Chains. SHU-TEH C. MOY, Syracuse University.

Let a Markov transition probability function $P(x, A)$ have the representation $P(x, A) = \int_A p(x, y)\lambda(dy)$ for (λ) almost all x. P is *conservative* if every λ-non-null set A is recurrent (the probability that a sample path starting at x will ever meet A is 1 for (λ) almost all $x \varepsilon A$). P may be described as a λ-measurable Markov operator which possesses a density function. Let $p^{(n)}(x, y)$ be the density function of $P^{(n)}(x, A)$. The following theorems are proved. (1) The space is decomposed into at most countably many indecomposable closed sets $C_1, C_2, \cdots$. (2) For each C_i there is a σ-finite invariant measure μ_i which is equivalent to λ on C_i and vanishes outside C_i. Every invariant measure is of the form $\sum \alpha_i \mu_i$. (3) for $(\lambda \times \lambda \times \lambda)$ almost all $(x, y, z) \varepsilon C_i \times C_i \times C_i$ there exists $\lim_{n \to \infty} \sum_{n=1}^{N} p^{(n)}(x, y)/\sum_{n=1}^{N} p^{n} \cdot (x, z) = f(y)/f(z)$ where f is the density of μ_i with respect to λ.

16. On Selecting the Factors for Experimentation. M. S. PATEL, Purdue University.

In this paper, each factor is assigned a prior probability p of being significant and a factorial experiment is conducted with a set of factors, each with two levels using an orthogonal plan for main effects. It is assumed that σ^2 is known. Then using a normal deviate test with a level of significance α, the expected number of correct decisions is obtained. This number is then maximised w.r.t. α and is shown to be minimum for $p = \frac{1}{2}$ which implies that only those factors should be included in the experiment for which $p \neq \frac{1}{2}$.

17. Prediction in Location and Scale Parameter Families. F. L. RAMSEY and R. J. BUEHLER, Iowa State University.

Let $x_i = t_i + \theta$ and $y = u + \theta$ where the joint density $f(t_1, \cdots, t_n)g(u)$ is given. From observations $\mathbf{x} = (x_1, \cdots, x_n)$ it is desired to predict y. Let the fiducial density $f_f(\theta \mid \mathbf{x})$ be defined as the posterior density given a uniform prior density of θ, and let $g_f(y \mid \mathbf{x}) = \int g(y - \theta) f_f(\theta \mid \mathbf{x})\, d\theta$. *Theorem* 1: For any "location invariant" predictor $B = B(x)$, $E_R(B - y)^r = E_f(B - y)^r$ where E_f is expectation with respect to $g_f(y \mid x)$ and E_R is conditional expectation given constant ancillaries $x_2 - x_1, \cdots, x_n - x_1$. This result and those following are analogous to results of Pitman (*Biometrika* **30** (1939) 391), but y here replaces θ. A typical consequence is *Corollary* 1: The mean of $g_f(y \mid \mathbf{x})$ is the minimum mean square error predictor of y. Next let $x_i = \sigma t_i + \theta$, $y = \sigma u + \theta$, and let $f_f(\theta, \sigma \mid \mathbf{x})$ be the posterior density with respect to a uniform prior density of θ and $\log \sigma$. *Theorem* 2: For any location and scale invariant predictor B, $E_R\{\sigma^{-1}(B - y)\}^r = E_f\{\sigma^{-1}(B - y)\}^r$ where E_R is conditional expectation for constant $(x_3 - x_1)/(x_2 - x_1), \cdots, (x_n - x_1)/(x_2 - x_1)$ and E_f is expectation with respect to the trivariate (fiducial (?)) density $\sigma^{-1}g(\sigma^{-1}(y - \theta))f_f(\theta, \sigma \mid x)$ of (y, θ, σ). *Corollary* 2: The minimum mean square error invariant predictor of y is $E_f(\sigma^{-2}y)/E_f(\sigma^{-2})$.

18. Likelihood Ratios of Differential Processes. Herman Rubin, Michigan State University.

Let P_j, $j = 1, 2$, be two separable differential processes on $[0, T]$, such that if X is a sample function, $X(0) = 0$ and X has only discontinuities of the first kind and is everywhere continuous in probability. If the logarithm of the characteristic function of $X(t)$ under P_j is $f_j(\lambda, t) = -\frac{1}{2}\sigma_j^2(t) + im_j(t) + \int_{-\infty}^{\infty}\int_0^t [e^{i\lambda x} - 1 - i\lambda h_j(x, u)]\, dG_j(x, u)$, then the processes are not orthogonal if and only if: (1) $\int_{-\infty}^{\infty}\int_0^t \{[d(G_1 - G_2)/d(G_1 + G_2)](x, t)\}^2 d(G_1 + G_2)(x, t) < \infty$, (2) $\sigma_1^2 = \sigma_2^2 = \sigma^2$, and (3) if the h_j are so modified that $h_1dG_1 = h_2dG_2$, which can always be done under (1), then $\int_0^t [d(m_2 - m_1)]^2/d\sigma^2 < \infty$. In this case the likelihood ratio of P_1 with respect to P_2 can be computed with probability 1, with the values 0 and ∞ occurring only if some factor is 0 or ∞, respectively.

19. Some Applications of the Jiřina Sequential Procedure to Observations With Trend. Sam C. Saunders, Boeing Scientific Research Laboratories, Seattle, Washington.

Let each random variable of a sequence have a density which is a Pólya frequency function of order two. To this sequence we apply the Jiřina sequential procedure to determine a tolerance interval. We find some sufficient conditions on a type of trend permissible for this sequence which enable us to show that in the case of such trend, when the Jiřina procedure is used, the sampling will stop sooner and the tolerance interval cover more of the population (in a stochastic sense) than would occur in the case the sequence was without trend. Similar considerations for one-sided tolerance limits are shown to hold when the sequences of observations have densities which have non-decreasing hazard rates. This work invites some numerical comparisons in simple cases of the expected sample size and coverage of the Jiřina procedure with the Wilks fixed sample procedure in the case of trend.

20. Comparison of Combined Estimators in Balanced Incomplete Blocks. V. Seshadri, McGill University.

In the analysis of balanced incomplete blocks there arise two independent estimates of treatment differences which have been combined by Yates to produce an efficient estimator. Graybill and Weeks have suggested an alternative combined estimator. The aim of the paper is to compare the variances of the two estimators. Yates' estimator is a weighted combination of the two independent estimators where the weights are random when the analysis of variance yields a positive estimate of the block variance, while constant weights are used when a negative estimate of the block variance is obtained. Using theorems on conditional expectation to calculate the variance of Yates' estimator, it is shown that for all values of the ratio σ_b^2/σ^2 (the ratio of the block variance to the error variance), Yates' estimator is superior to Graybill and Weeks estimator, from the point of view of minimum variance.

21. Bio-Assay With Prior Information. Morris Skibinsky, University of Minnesota.

Let n, δ, α be given numbers; n a positive integer, $0 < \delta \leqq \frac{1}{2}$, $0 < \alpha < 1$. Let Θ_i, X_i, $i = 1, 2$, be random variables defined on a measurable space $(\Omega, \mathfrak{A})$ and denote by $\mathfrak{M}(\delta, \alpha)$

the family of all probability measures on $\mathfrak{A}$ such that: (i) $P_p(X_1 = x_1, X_2 = x_2 \mid \Theta_1 = \theta_1, \Theta_2 = \theta_2) = f(x_1, \theta_1)f(x_2, \theta_2)$, $[p]$, for $x_1, x_2 = 0, 1, \cdots, n$ and $0 \leqq \theta_1, \theta_2 \leqq 1$, where $f(x, \theta)$ is the value at x of the binomial frequency function with parameters n and θ; (ii) $p(0 \leqq \Theta_1 \leqq \Theta_2 \leqq 1) = 1$; (iii) $p(\delta \leqq \Theta_1$ and $\Theta_2 \leqq 1 - \delta) \geqq 1 - \alpha$. A two step maximum likelihood procedure suggested by Skibinsky and Cote, *Ann. Math. Statist.* **34** No. 3, for the case of one X and one Θ is applied to obtain a predictor of (Θ_1, Θ_2) from (X_1, X_2). For $\alpha \geqq$ a positive number, c (which is $\leqq \frac{1}{2}$), this predictor specializes to the estimate obtained by Brunk, Ewing, Reid and Silverman, *Ann. Math. Statist.* **26** 641–647 (the case of two events) which is a maximum likelihood estimate when only conditions (i) and (ii) above are taken to hold. Relative to the squared distance between (Θ_1, Θ_2) and predictor, the two-step predictor may be shown, for $0 < \alpha < c$, and $\alpha' > 0$ sufficiently small to be uniformly better over $\mathfrak{M}(\delta, \alpha')$ than this estimate.

22. An Algorithm for the Analysis of Multidimensional Partially Balanced Designs. J. N. Srivastava, University of North Carolina. (By title)

In this paper, the linear associative algebras generated by the association schemes of MDPB (multidimensional partially balanced) designs (defined earlier by the author) are considered. It has been found that such an algebra is non-commutative. However, using the properties of the commutative algebra generated by the association scheme of ordinary PBIB designs, an algorithm has been developed for the analysis of MDPB designs. This algorithm does not involve any matrix inversion, and reduces the analysis to multiplication and addition of matrices of very low order. These results have a direct application to the analysis of (possibly nonorthogonal) balanced or partially balanced factorial fractions. For the balanced case, the matrices involved have orders ranging from 2×2 to 6×6 (under the assumption that 3-factor and higher order interactions are negligible).

(Abstracts of papers presented at the Western Regional Meeting, Eugene, Oregon, June 20–21, 1963. Additional abstracts appeared in the June, 1963 issue.)

5. Exchangeable Processes Which Are Functions of Stationary Markov Chains. S. W. Dharmadhikari, University of California, Berkeley.

Let $\{Y_n, n \geqq 1\}$ be an exchangeable process with a countable state-space J. $\{Y_n\}$ is then a mixture of sequences of independent and identically distributed random variables with values in J. In this paper, it is proved that $\{Y_n\}$ is a function of a stationary countable-state Markov chain if, and only if, the above mixture is a countable mixture. The proof of the "if" becomes straightforward as soon as the sequences of random variables which go into the mixture are treated as Markov chains. The "only if" part is proved by taking the Cesàro limit of powers of the transition probability matrix of the underlying Markov chain.

6. Confidence Bands in Straight Line Regression. Antranig V. Gafarian, System Development Corporation, Santa Monica, California.

This paper develops a method for obtaining confidence bands in polynomial regression when the observations are independently distributed with constant but unknown variance. The bands may be obtained, in principle, over arbitrary sets of the independent variable with exact preassigned confidence coefficients. In general, difficult distribution problems result when specific applications are attempted. Some progress has been made in the case of first degree polynomials. A table is provided to obtain a constant width confidence band which contains the true but unknown straight regression line for values of the independent variable in some arbitrarily selected interval with an exact preassigned confidence coefficient. The method is compared with the well known hyperbolic band for the whole regression line.

7. On the Robustness of Some Non-Parametric Estimates for Shift (Preliminary report). ARNLJOT HØYLAND, University of Oslo.

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be $N(= m + n)$ independent observations from distributions $P(X_i \leqq u) = F(u)$ and $P(Y_j \leqq u) = F(u - \Delta)$ where F is supposed to be continuous but otherwise unknown. The estimates $\tilde{\Delta}_1 = \text{median } (Y_j - X_i)$, [the median of the set of mn differences $(Y_j - X_i)$], and $\tilde{\Delta}_2 = \text{median } [(Y_i + Y_j)/2] - \text{median } [X_r + X_s)/2]$ are considered and compared with $\Delta^* = \bar{Y} - \bar{X}$. Under these assumptions Hodges and Lehmann have shown that the asymptotic relative efficiency of $\tilde{\Delta}_1$ to Δ^*, $e(\tilde{\Delta}, \Delta^*) = 12\sigma_x^2 [\int f^2(x)dx]^2$. Here it is shown that under the same assumptions $e(\tilde{\Delta}_2, \tilde{\Delta}_1) \leqq 1$ with equality only when F is symmetric; furthermore that if in fact an unknown scale parameter c is present and $F(u) = H(u - \theta)$, $G(u) = H[(u - \theta - \Delta)/c]$ where H is symmetric about 0 and θ is unknown, then $e(\tilde{\Delta}_2, \Delta^*)$ is still $12\sigma_x^2[\int f^2(x)dx]^2$ while $e(\tilde{\Delta}_1, \Delta^*)$ depends on c and $\lambda(= \lim_{N\to\infty} m/N)$. In particular if $\lambda = \frac{1}{2}$ and $-\log f(x)$ is convex, $e(\tilde{\Delta}_2, \tilde{\Delta}_1) \geqq 1$ with equality for $c = 1$.

8. On the Efficiency of Optimal Non-Parametric Procedures in the Two Sample Case With Nuisance Parameters. PIOTR WITOLD MIKULSKI, University of Maryland.

Consider Pitman's efficiency of the locally most powerful rank test relative to the likelihood ratio test for the two sample problem when the distribution of random variables in question is assumed to be specified up to location and scale parameters only. Suppose that the information about the distribution is false and Pitman's efficiency for these two tests is computed under some other distribution. It turns out that a necessary and sufficient condition for this lower bound to be equal to unity is that the distribution for which the procedures are derived is normal. This result is obtained under some regularity restrictions similar to those involved in the same study when nuisance parameters were absent.

9. Correlation Models. R. F. TATE, University of Washington. (By title)

Assume X univariate and Y multivariate. The sample multiple correlation coefficient based on (X_n, Y_n) is r; the population multiple correlation is ρ. The distribution of Y given $X = x$ is multivariate normal with mean vector $\mu(x)$ and covariance matrix independent of x. Models are specified by placing various restrictions on $\mu(\cdot)$ and on the distribution of X. Results are related to those of Tate (*Ann. Math. Statist.* **25** (1954) 603–607), Olkin and Tate (*Ann. Math. Statist.* **32** (1961) 448–465), and Das Gupta (*Psychometrika* **25** (1960)). It is shown, for example, that if X has coefficient of excess $\gamma \geqq -4$, and finite eighth moment, then $r \sim \mathfrak{N}(\rho, (1 - \rho^2)^2(1 + \frac{1}{4}\gamma\rho^2)/n)$. Thus, the cases X-normal and X-double exponential lead to the same limiting distribution.

(Abstracts of papers presented at the European Regional Meeting, Copenhagen, Denmark, July 8–10, 1963. Additional abstracts appeared in the June, 1963 issue.)

3. Mean Square Expectations for Orthogonal Contrasts in Mixed Model Analyses of Variance. KLAUS ABT, United States Naval Weapons Laboratory, Dahlgren, Virginia. (Introduced by Victor Chew)

In this paper mean square expectations for orthogonal contrasts in mixed model analyses of variance are discussed for the cases of two-way and three-way crossed classifications with one or two fixed-effects (or Model I) factors and with the other factor(s) of the random-effects (or Model II) type. The formulas are given for the most general orthogonal cases of unequal but proportional cell numbers and for the cases of equal cell numbers. Also included are the appropriate variance ratios for testing all null-hypotheses concerning orthogonal contrasts and their interactions with other factors. In the derivation of the mean square expectations variance components are defined for the interaction effects between (fixed factor) orthogonal contrasts and random factors. The introduction of these interaction variance components is shown to be necessary because it is impossible to maintain the usual assumption that the variance of the corresponding interaction term in the linear model is equal for all levels of the fixed factor. Besides the necessary additions the notation, linear model and method used for deriving the mean square expectations are the same as in the author's "Table of Expectations of Mean Squares in the Analysis of Variance for Crossed Classifications," (NWL Report No. 1833, 2 April 1963).

4. A Monotonicity Property of a Class of Tests of the Equality of Two Covariance Matrices. T. W. ANDERSON and S. DAS GUPTA, Columbia University.

Invariant tests of the hypothesis that $\Sigma_1 = \Sigma_2$ are based on the characteristic roots of $S_1S_2^{-1}$, where Σ_1 and Σ_2 and S_1 and S_2 are the population and sample covariance matrices, respectively, of two multivariate normal distributions, and the power of such a test depends on the characteristic roots of $\Sigma_1 \Sigma_2^{-1}$. The power is an increasing function of each ordered root of $\Sigma_1 \Sigma_2^{-1}$ if the acceptance region of the test has the property that if $(c_1, \cdots, c_p)$ is in the region any point with coordinates not greater than these, respectively, is also in the region; here $c_1 \geqq c_2 \geqq \cdots \geqq c_p$ are the roots of $S_1S_2^{-1}$. Examples of such acceptance regions are $c_1 \leqq a$ and $c_p \leqq a$, with a constant.

5. Quantum Mechanics and Probability Theory: Criticism of Feynman Position. H. BRENY, University of Liège.

It has been contended that the foundations of quantum mechanics are incompatible with classical probability theory (see e.g. Feynman, R. P., *Second Berkeley Symp.* 533–542; also Hibbs, A. R., Appendix II of *Probability and Physical Sciences* by M. Kac). If true, that thesis would have far-reaching consequences, for quantum-mechanical reasoning does make use of classical probability theory (laws of large numbers). In fact, a critical analysis of Feynman's reasoning shows that it is inconclusive, and that quantum mechanics and probability theory are quite compatible.

6. Asymptotic Regression Curves With Different Asymptotes. I. M. CHAKRAVARTI, University of Geneva.

Stevens (1951) gave a fully efficient method of estimating the parameters in the asymptotic regression $y = \alpha + \beta\lambda^t$, $0 < \lambda < 1$. This article provides an operational method for

obtaining least squares estimates when α's are different for different experimental units, but β and λ are the same. The model used here is thus, $y_{ij} = \alpha + \beta\lambda^{t_{ij}}, j = 1, 2, \cdots, n_i$, $i = 1, 2, \cdots, k$, where y_{ij} is the response observed on the ith experimental unit at the dose t_{ij}. Starting with a preliminary estimate r of λ, the first step is to calculate the elements of a square matrix R of order $(k + 2)$ and those of an one-columned matrix Y of $(k + 2)$ elements. The matrix-product $R^{-1}Y$ provides the elements for calculating the improved estimates. The operations are repeated until the desired accuracy in the estimates is achieved.

7. On Optimal Stopping Rules for Maximizing s_n/n in Coin Tossing. Y. S. Chow and Herbert Robbins, Columbia University.

Let $x_1, x_2, \cdots$ be independent random variables with $P(x_n = -1) = P(x_n = 1) = \frac{1}{2}$. We observe $x_1, x_2, \cdots$ sequentially; if we stop with x_n we receive the reward $y_n = (x_1 + \cdots + x_n)/n$. A *stopping rule* is a random variable τ with positive integer values, such that the event $\tau = k$ is measurable with respect to $x_1, \cdots, x_k$ for every $k = 1, 2, \cdots$. Using τ our reward is the random variable y_τ. We show that there exists an optimal stopping rule t; i.e., one for which $E(y_t)$ is a maximum. t is difficult to exhibit explicitly but we give approximations to $E(y_t)$.

8. An Estimator for the Population Regression Coefficient From a Stratified Sample. M. de Vries, A. C. Nielsen (Nederland) N. V. Amsterdam. (Introduced by J. Hemelrijk)

In sample surveys the design of the sampling scheme usually aims at optimal efficiency for enumerative data. However, such samples often have to be used for analytical investigations which were not considered as part of the original sample plan. Usually such investigations require more complicated calculations and are less efficient. In this paper for the case of a stratified sample, the population regression coefficient and correlation coefficient have been derived for a real or assumed linear relationship. (This problem should be clearly distinguished from the regression estimator which serves a different purpose.) In order to simplify calculations, a biased version of the estimator is proposed. This biased estimator is examined in a multiple regression example.

9. Maximum-Likelihood Estimation of the Parameters of a System of Simultaneous Regression Equations. J. Durbin, London School of Economics.

The problem of full-information maximum-likelihood estimation of the parameters of a system of simultaneous regression equations is attacked by transforming the equations of maximum likelihood into a set more amenable to solution. The equations so obtained lend themselves to solution by a straightforward Newton-Raphson iterative procedure. They also show up clearly the relation between maximum-likelihood estimates and three-stage least-squares estimates. Methods are given for dealing with the special problems arising from the presence in the model of identities and of under-identified and just-identified equations.

10. Exact Power Values of the Wilcoxon (Mann-Whitney) Two-Sample Test Against Lehmann's Alternatives. Herbert B. Eisenberg, System Development Corporation, Santa Monica, California.

Tables of the exact power values of the one-sided and two-sided Wilcoxon (Mann-Whitney) two-sample tests against Lehmann's alternatives (i.e., $H_0 : G = F$ against the alternatives $H_1 : G = F^k$, where k is a positive number) have been computed. These cover a range of values of the parameter k from $\frac{1}{2}$ to 4, all combinations of sample sizes $2 \leqq n \leqq m \leqq 10$, and selected values of the significance level.

11. Distribution Functions With Given Marginal Distributions. Hans G. Kellerer, University of Munich.

Let I be a finite index set and $\mathfrak{F}_I$ the class of all distribution functions F defined on the product space $R_I = (R, i \varepsilon I)$. Then each $F \varepsilon \mathfrak{F}_I$ defines a collection of marginal distributions $F^T \varepsilon \mathfrak{F}_T$ corresponding to the different subsets T of I. Now if $\mathfrak{T}$ is any system of such subsets, the following main theorem holds: Necessary and sufficient for the existence of a distribution $F \varepsilon \mathfrak{F}_I$ with the marginals $F_T \varepsilon \mathfrak{F}_T$ for $T \varepsilon \mathfrak{T}$ is the condition (*) $\sum_{T\varepsilon\mathfrak{T}} \int_{R_T} g_T dF_T \geqq 0$ for all bounded continuous functions $g_T \mid R_T$ with $\sum_{T\varepsilon\mathfrak{T}} g_T \geqq 0$ everywhere in R_I. Using this theorem it is possible to give a simple combinatorial characterization of those systems $\mathfrak{T}$, for which the condition (*) may be replaced by the usual consistency condition. In this case of "solubility" explicit solutions of the problem can be stated. Furthermore it is possible to extend the main theorem to the case of an arbitrary index set; this yields in particular a generalization of Kolmogorov's well known theorem.

12. Some Limit Theorems for the Dodge-Romig AOQL Single Sampling Inspection Plans. A. Hald and E. Kousgaard, University of Copenhagen.

In a previous paper by Hald (*Technometrics*, 1962) limit theorems for the Dodge-Romig LTPD single sampling inspection plans have been derived. The purpose of the present paper is to find similar results for the AOQL plans. The main results are that the highest allowable fraction defective in the sample converges to the AOQL, the difference being of order $(\log n)^{\frac{1}{2}}/n^{\frac{1}{2}}$, and that sample size asymptotically is proportional to the logarithm of lot size. It is further shown that the producers risk asymptotically decreases inversely proportional to lot size and that the average amount of inspection for lots of process average quality apart from sampling inspection is independent of lot size. Finally, numerical investigations have shown that the asymptotic formulas for acceptance number and sample size are good approximations to the exact solution also for small lot sizes and a compact graphical representation of the asymptotic solution is given. From a purely probabilistic point of view the most interesting is perhaps the following result regarding the Poisson distribution: The equation $B(c, x) = xb(c, x)$ has the asymptotic solution $c = x + [x \log \cdot (x/2\pi)]^{\frac{1}{2}} + (1/6) \log (x/2\pi) - \frac{1}{2} + o(1)$ where $B(c, x) = \sum_{i=0}^{c} b(i, x)$ and $b(i, x)$ denotes the Poisson probability for the outcome i when the parameter is x.

13. On the Semimartingale Convergence Theorem. S. Johansen and J. Karush, University of Copenhagen.

Let $(\varphi_n, n \geqq 1)$ be a nondecreasing sequence of signed measures on nondecreasing σ-fields of a probability space, and let $X_n = d\varphi_n/dP$. Let $\varphi_0 = \lim \varphi_n$ (defined on the union field) and suppose φ_0^+ is σ-finite. It is shown that the semimartingale convergence theorem (slightly extended) asserting that X_n converges a.s. can be obtained as an immediate consequence of simple semimartingale inequalities. The limiting function X is then characterized measure-theoretically, as follows: $X^+(X^-) = d\mu+/dP(d\mu-/dP)$, where $\mu+(\mu-)$ is the maximal measure contained in the content $\varphi_0^+(\varphi_0^-)$. This provides a completion of the approach of Andersen and Jessen (*Danske Vid. Selsk. Mat.-Fys. Medd.* **25** No. 5 (1948) 8 pp). A simple illustrative example is given of the decomposition of a content into its maximal σ-additive and "purely finitely additive" parts.

14. Analysis of Computer Failure Patterns. I: A Branching Poisson-Process Model. Peter A. W. Lewis, University of London.

Models for computer failures generally assume that the sequence of failures in each component position constitute a renewal process, with failures in each component position being independent of the failures in other component positions. The superposition of these renewal processes—the failure pattern of the computer—should then form a Poisson-process. Experience has shown however, that the times-between-failures of a computer are not exponentially distributed and also that they tend to be correlated with one another. These deviations from a Poisson-process are explained by the following model. Initial failures of components constitute a main process, which is assumed to be Poisson. Independently, each of these initial failures is repaired with probability q. Otherwise the failure recurs some time Y_1 later, is then repaired with probability $1 - p$, and so forth. The random variables $\{Y_i\}$ are assumed to be independent and identically distributed, but no assumptions are made as to the form of their distribution. The non-zero times between the recurrences of failures in these subsidiary processes are due either to the fact that the component failure is intermittent rather than catastrophic, or to the fact that the failed but unlocated component is not used in computing for some time. The pooled main and subsidiary processes constitute the computer failure pattern. Conditions for stationarity and a complete probabilistic description of this branching Poisson-process are given.

15. A Comparison Between the Variability of the Partial Regression Coefficients of x_1 on x_2 and Those of x_2 on x_1. Ejnar Lyttkens, University Institute of Statistics, Uppsala, Sweden. (Introduced by H. Wold)

For samples from different k-dimensional populations a large sample test is designed for the hypothesis that all partial regression coefficients of x_1 on x_2 and partial regression coefficients of x_2 on x_1 are equal against the hypothesis that one and only one of the two sets of regression coefficients are not equal in all populations considered. Instead of an approximately F-distributed test variable, used in my paper on the corresponding bivariate problem at the conference in Dublin 1962, a test variable distributed approximately as the difference between two independent χ^2-distributed variables is used. The distribution function of such a difference is tabulated by K. Pearson, S. A. Stouffer and F. N. David (1932) in "Further application of the $T_m(x)$ Bessel function", *Biometrika* **24** 293. Furthermore we meet also in the multi-dimensional normal case the difficulty that the underlying normally distributed variables appearing in the χ^2-sums do not form a bivariate normal distribution, although the marginal distributions are normal, but in the large sample case the deviation from the bivariate normal distribution does not in general seem too serious.

16. A Test of Whether Two Regression Lines Are Parallel When the Variances May Be Unequal. Richard F. Potthoff, University of North Carolina.

The principal topic covered in this paper is the development of a test of the hypothesis that two regression lines are parallel under the conditions that the two sets of error terms are normally distributed but with (possibly) different variances. An incidental topic which is covered concerns a test for the slope of a single regression line; no normality assumption is required for this second test. Both tests are analogous to the Wilcoxon test: the test statistic for each test is based on a symmetric sum of correlated binomial variates.

17. Nonsensical Maximum Likelihood Estimators. OLAV REIERSØL, University of Oslo.

All references of this abstract are to T. Koopmans: *Linear Regression Analysis of Economic Time Series*, De Erven F. Bohn N.V., Haarlem, 1937. We consider the specification given in Section 7 with the following modifications: The matrix ε is supposed to be diagonal and we assume known that each ε_{kk} has a lower positive bound when the ε_{kk} are normalized by (7.5). Using the results of Section 8 on maximum likelihood estimation when ε is given, we derive maximum likelihood estimators when ε is not supposed to be known. The main result is that any local maximum and the absolute maximum of the likelihood function must occur when all ε_{kk} except one are equal to their respective lower bounds. Any solution of the system of equations which we get when the partial derivatives of the likelihood function are equated to zero will give a saddle point of the likelihood function. Neither the solutions giving maxima of the likelihood function nor the solutions giving saddle points have any relation to the true values of the parameters ε_{kk}.

18. Random Division of an Interval. F. W. STEUTEL, Mathematisch Centrum, Amsterdam. (Introduced by W. R. van Zwet)

An interval of length t is divided into n parts by $n - 1$ random points. The relation that exists between the distribution function $P_n(z_1, \cdots, z_n; t)$ of the n intervals and the distribution function $Q_n(z_1, \cdots, z_n; \tau)$ of n independent exponentially distributed random variables with mean τ^{-1} may be written as $\int_0^\infty P_n(z_1, \cdots, z_n; t)t^{n-1}e^{-\tau t}dt = [(n-1)!/\tau^n]Q_n(z_1, \cdots, z_n; \tau)$. By interpreting the integral as a Laplace-transform the probability of any event concerning the n intervals may be obtained by Laplace-inversion of the probability of the corresponding event concerning n independent exponentially distributed random variables. A special case of this relation has been used by M. Dwass (1961), *Trabajos Estadist.* **12** (1, 2). Some applications are given.

19. Simultaneous Confidence Intervals for Weighted Sums of Classification Probabilities. PAIRLEE J. STINSON and JOHN E. WALSH, Veterans Administration Hospital, Sepulveda, California; System Development Corporation, Santa Monica, California.

Several (K in all) large-sized groups of individuals are considered, where each individual belongs to exactly one of a given set of U classifications and the groups are independent. Thus, the data are expressible in the form of a $K \times U$ table with independent rows and each row representing independent trials from a multinomial distribution with the classifications as categories. For each classification, relative comparisons of the corresponding multinomial probabilities for the groups are of interest. Specified weighted sums of the group probabilities (for a fixed classification) are used for the comparisons. $K - 1$ comparisons can be made for each of a stated $U - 1$ of the classifications. A separate interval based on the observations is developed for each of the $(K - 1)(U - 1)$ weighted sums of probabilities. These intervals are such that the probability is (approximately) at least a specified amount that all these interval relations are simultaneously satisfied. Basis of results is development of a suitable statistic with a large-sample chi-square distribution. These confidence regions yield tests which have advantage of indicating which comparisons led to significance. Results are useful for bio-medical retrospective studies, such as investigating possible causes for lung cancer.

20. Progressive Correction (Preliminary report). C. S. VAN DOBBEN DE BRUYN, N. V. Philips' Gloeilampenfabrieken, Eindhoven, Holland. (Introduced by L. C. A. Corsten)

A new method for extrapolating time series, of which the mean is not a constant but exhibits long-term drifts or sudden shifts, is presented. Whereas being very responsive to changes in mean level the technique yields a stable and unbiased estimate of the mean if it is a constant; this combination of stability and adaptivity is achieved by reacting slowly when the error of prediction is small compared with the variation in the series (noise) and quickly whenever the error becomes large (thence the name *progressive* correction). A special case is chosen and its adaptivity and stability are compared with those of exponential smoothing. The mean square error (m.s.e.) of the predictor is compared with the m.s.e. of exponential smoothing and the Wiener predictor, for the case of a Markov series. The need for criteria other than the m.s.e. of prediction for choosing a predictor arises if an inventory-control system utilizes the predictions; the variance of predictions (not of prediction errors) and the speed and manner of response are possible successors of the m.s.e. and they are computed for the chosen form of progressive correction.

21. Non-Parametric Regression Analysis. G. S. WATSON, Johns Hopkins University.

Let $(X_i, Y_i)(i = 1, \cdots, n)$ be identically and independently distributed and suppose that $E(Y \mid X = x) = m(x)$ exists. The problem is to estimate the regression function $m(x)$ with the minimum of assumptions about the joint distribution of X and Y. A class of estimators that is suitable when $m(x)$ and the marginal density of X are continuous is $\tilde{m}(x) = \sum_{i=1}^{n} Y_i\delta_n(x - X_i)/ \sum_{i=1}^{n} \delta_n(x - X_i)$ where the function $\delta_n(\cdot)$ depends on the configuration of $(X_1, \cdots, X_n)$. $1/n$ times the denominator provides an estimator of the density of X. Large sample theory and Monte Carlo investigations have been made.

(Abstracts of papers presented at the Annual Meeting of the Institute, Ottawa, August 27–29, 1963. Additional abstracts appeared in the March and June, 1963 issues.)

1. Distribution of the Number of Admissible Points (Preliminary report). O. BARNDORFF-NIELSEN and MILTON SOBEL, University of Minnesota.

Let $X_i = (X_{i1}, X_{i2}, \cdots, X_{id})$ denote mutually independent d-dimensional vector random variables with a common absolutely continuous distribution function $F = F_1(x_1) F_2(x_2) \cdots F_d(x_d)$, i.e., the d components are mutually independent but the marginals need not be common. Define X_i to be an admissible point in the set $\{X_1, X_2, \cdots, X_n\}$ if there is no X_j in the set such that $X_{j\alpha} \geqq X_{i\alpha}$ $(\alpha = 1, 2, \cdots, d)$ with strict inequality for at least one α. The number of admissible points, A_n, in a sample of size n is distribution-free, i.e., its distribution does not depend on F. For small values of n, the exact distribution of A_n is derived, as well as the mean and variance, for arbitrary $d \geqq 1$. For small values of d, the exact distribution of A_n is obtained for arbitrary $n \geqq 1$. For $d = 2$ if the points X_i $(i = 1, 2, \cdots)$ are taken sequentially then the sequence $\{A_i\}$, (A_i being the number of admissible points among the first i points) is a realization of a Markov Chain with non-stationary transition probabilities. It is also shown that for $d = 2$ and $n \to \infty$, the distribution of A_n is asymptotically Normal and similar results are conjectured for $d \geqq 3$.

2. **On the Limit Behaviour of Extreme Order Statistics.** Ole Barndorff-Nielsen, University of Minnesota. (Invited)

The aim of the paper is to indicate the current state of research in theory of limit behaviour of extreme order statistics. Some new results due to the author will be mentioned, the main of which is a criterion for almost sure stability of the maximal order statistic of a sequence of independent, identically distributed random variables.

3. **Some Statistical Results in Renewal Theory.** T. N. Bhargava, Kent State University. (By title)

The notations in this paper are that of Feller (*An Introduction to Probability Theory and Its Applications*, **1,** (1st ed.), Chapter 12). Maximum likelihood estimates of μ and σ^2 are obtained in terms of the random variables N_{ri} $(i = 1, 2, \cdots, n)$, where N_{ri} denotes the number of occurrences of ϵ in the first r trials in the ith sample. It is found that these estimates are unbiased and the expressions for their variances are obtained. It is shown that with certain modifications the standard tests of significance for the mean and difference of two means, and the analysis of variance tests can be applied to the renewal theory. Some of these tests are explicitly derived.

4. **Enumeration of Cyclic Paired-Comparison Designs.** H. A. David, Virginia Polytechnic Institute.

Suppose that n "objects" $0, 1, 2, \cdots, n - 1$ are to be compared in pairs. The totality of $\frac{1}{2}n\ (n - 1)$ paired comparisons can be divided into $m = \frac{1}{2}\ (n - 1)$ cyclic sets of n pairs if n is odd and into $\frac{1}{2}n - 1$ sets of n together with a set of $\frac{1}{2}n$ if n is even. A typical set is $s = 0,\ s\ 1,\ s + 1 \cdots t, s + t \cdots n - 1, s + n - 1$, where s is a positive integer less than $\frac{1}{2}n$ and $s + t$ has to be reduced modulo n when necessary. Cyclic paired-comparison designs are made up of combinations of cyclic sets. A design of size (n, r) involves n objects each of which is compared r times $(r = 1, 2, \cdots, n - 2)$. It is shown that, for n prime, the number of distinct (non-isomorphic) designs of size (n, r) is equal to the number of arrangements, remaining distinct under rotation, of $\frac{1}{2}r$ white and $m - \frac{1}{2}r$ black beads on a necklace. The case where n is not prime is also treated.

5. **Stochastic Give-and-Take.** M. H. DeGroot and M. M. Rao, Carnegie Institute of Technology. (Invited)

The following process, initially considered by C. C. Li, is studied. Two players share a fixed amount of some commodity and at the nth stage of the process they exchange random proportions X_n and Y_n of their shares. If $\{(X_n, Y_n): n = 1, 2, \cdots\}$ is an independent and identically distributed sequence of random pairs then the process is a Markov process whose state space is the unit interval and whose state R_{n+1} at the $(n + 1)$th stage is given by the nonlinear relation $R_{n+1} = (1 - X_n)\ R_n + Y_n\ (1 - R_n)$. The asymptotic, steady state distributions are obtained in the general case and a number of special distributions of (X_n, Y_n) are considered. (The paper will appear in *J. Mathematical Analysis and Applications.*)

6. **Double Limit and Run Control Charts: Exact Statistical Properties.** J. Tiago de Oliveira and S. B. Littauer, Columbia University. (By title)

The authors develop a new approach to control chart usage under conditions of stability. Two control charts for sample averages and their properties are studied in this paper. The chief tool for this study is the already used notion of mean action time as a substitute for the power function which is not adaptable to the present criteria. The exact expression of the mean action time is obtained. One of the control charts (with warning and action limits) deals with the following rule for action: look for assignable causes when two successive sample averages fall between the upper (lower) warning and action limits or when one sample average falls outside the (larger) action limits. The second control chart uses the following action rule: when one of the observed sample means falls outside the control limits or a prescribed run of observed sample averages lies entirely above or entirely below the grand mean of the sample average, look for assignable causes.

The effect of using estimates from a previous large sample is accounted for.

Tables and graphs for measuring the effectiveness of those control charts are being prepared as well as a study of the economic choice of the control chart parameters.

7. On General Birth and Death Processes and Mission Reliability (Preliminary report). RONALD S. DICK, International Electric Corporation, Paramus, New Jersey.

Continuing the writer's previous work on reliability models containing maintenance time constraints and restoration time constraints, (Dick, R. S., "The Reliability of Repairable Complex Systems—Part A: The Similar Machine Case" *5th MIL-E-CON Symposium on Military Electronics*, Washington, D. C., 1961), these concepts are generalized from fixed time periods to random variables. Also the relationship of the models to Semi-Markov Processes is shown. Extensions of the basic model to the case where there are partial absorbing barriers in each state of the model is also included in the paper.

8. Single Sampling Inspection Plans Based on a Specified Acceptance Probability and Minimum Costs. ANDERS HALD, University of Copenhagen.

Let $K(p) = nk_s(p) + (N - n)(k_a(p)P(p) + k_r(p)Q(p))$ denote the costs of a sampling plan (n, c) for lots of size N and quality p. The sampling plans discussed in the present paper are defined by specifying the acceptance probability for one quality level and minimizing the costs for another quality level. Three special cases of particular interest are discussed: (1) LTPD plans with minimum producers costs. (2) AQL plans with minimum consumers costs. (3) IQL plans with minimum producers or consumers costs. The cost function may always be reduced to one of the two standard forms: $n + (N - n)\gamma_1 Q(p_1) + N\delta_1$ or $n + (N - n)\gamma_2 P(p_2) + N\delta_2$ where p_1 and p_2 denote the two quality levels and (γ, δ) are corresponding cost constants. The general solution to the minimization problem is given, a corresponding program for an electronic computer has been constructed, and some tables are provided. An approximate solution has also been obtained by deriving limit theorems and afterwards correcting the asymptotic formulas so that they become valid for small N. Asymptotically sample size increases proportional to the logarithm of lot size. An important new (asymptotic) result is the following: The sampling plan corresponding to lot size N and cost constant γ is found as the sampling plan for lot size $N\gamma$ and cost constant 1.

9. Confounding $3(2^{5-2})$ Designs of Resolution *V*. PETER W. M. JOHN, University of California, Davis.

The $3(2^{5-2})$ fractional factorials occur in six basic designs. These correspond to confounding patterns given by the following sets of defining contrasts (together with I): A, BCD, $ABCD$; AB, ACD, BCD; AB, $ACDE$, $BCDE$; A, $BCDE$, $ABCDE$; AB, CDE, $ABCDE$; ABC, ADE, $BCDE$. Any of the designs may be split into two blocks of twelve runs each or into one block of sixteen runs and one of eight runs. The first three designs can be blocked into two blocks of eight runs and two blocks of four runs. The last three designs can be divided into four blocks of six runs each.

10. Minimal Sufficient Statistics for the Group Divisible Partially Balanced Incomplete Block Design (GD-PBIB) With Interaction Under an Eisenhart Model II. C. H. KAPADIA and DAVID L. WEEKS, Southern Methodist University, Oklahoma State University.

In this paper, an Eisenhart Model II with interaction for a GD-PBIB design with p replicates per cell is considered. Specifically the model $Y_{ijk} = \mu + \beta_i + \tau_j + (\beta\tau)_{ij} + e_{ijk}$ is assumed, where $i = 1, 2, \cdots, b; j = 1, 2, \cdots, t; k = n_{ij}$ and $n_{ij} = 0$ if treatment j does not appear in block i, and $n_{ij} = 1, 2, \cdots, p$ if treatment j appears in block i.

If β_i, τ_j, $(\beta\tau)_{ij}$ and e_{ijk} are normally and independently distributed, then a minimal sufficient (Vector-valued) statistic for the class of densities for this model is found, together with the distribution of each component in the minimal sufficient statistic. It is also shown that the minimal sufficient statistic for this class of densities is not complete. Hence the solution of the problem of finding minimum variance unbiased estimators of the variance components is not straightforward. If minimum variance unbiased etimators exist independent of the parameter, they must be explicit functions of the elements in the minimal sufficient statistic found in this paper.

11. Use of Behavioristic Models in Analysing Special (Pertaining to Space) Data—Part II. S. K. KATTI, Florida State University. (By title)

Part I of this paper has been delivered at the International Symposium on Classical and Contagious Distributions at Montreal, Canada, August 15–20, 1963. Therein, a model for the distribution of corn borers was developed and frequency functions for one plant per plot, two plants per plot and four plants per plot were derived. The study being preliminary, certain assumptions were made to simplify algebra. In Part II, frequency functions have been obtained without the simplifying assumptions. Empirical frequencies are being obtained for two more fields and the fits of the theoretical frequency functions are being studied. Limiting forms have been obtained. Preliminary results indicate that the new model gives substantial improvement over the old model, but all conclusions must wait until all the results have been obtained.

12. Simultaneous Tests for Equality of Covariance Matrices Against Certain Alternatives. P. R. KRISHNAIAH, Wright-Patterson Air Force Base, Ohio.

Consider K multivariate normal populations with covariance matrices $\Sigma_1, \cdots, \Sigma_K$. In the present paper, some procedures are proposed to test the hypothesis $H: \Sigma_1 = \cdots = \Sigma_K$ against the alternative hypotheses A_1, A_2 and A_3 where $A_1 = \bigcup_{i \neq j=1}^{K} A_{ij}$, $A_2 = \bigcup_{i=1}^{K-1} A_{iK}$, $A_3 = \bigcup_{i=1}^{K-1} A_{i,i+1}$ and $A_{ij} : \Sigma_i \neq \Sigma_j$. They are based upon expressing the total hypothesis as a finite intersection of several elementary hypotheses and testing these elementary hypotheses simultaneously. These procedures are generalizations of the "Step-Down Procedure" proposed by J. Roy (these *Annals* **29** 1177–1187) for testing the equality of two covariance matrices. The procedure proposed in the present paper for testing H against A_1 is restricted to the situations where the sample sizes are equal. In the univariate case, the present procedures for testing H against A_1 and A_2 are respectively equivalent to Hartley's test (*Biometrika* **37** 308–312) and Gnanadesikan's test (these *Annals* **30** 177–184) for the equality of variances.

13. **Probability Distribution of the Radial Error.** ANDRE G. LAURENT, Wayne State University.

Let the coordinates $\mathbf{X}$, with covariance matrix $\mathbf{\Sigma}$, of the points of impact M be normally distributed $N(\mathbf{0}, \mathbf{\Sigma})$ around the origin. Then the probability of a hit within a circle of radius r centered at the target, that is, the probability distribution of the radial error R is $P(\mathbf{X}'\mathbf{X} \leqq r^2) = P(R \leqq r) = \sum_{j=0}^{\infty} [(-1)^j/(j+1)!] \, (r^2|\Sigma|^{-\frac{1}{2}}/2)^{j+1} \, P_j \, [(\text{trace } \Sigma) \, |\Sigma|^{-\frac{1}{2}}/2]$, where P_j is a Legendre polynomial. Different expressions for the moments of R and the distribution of the sample quadratic mean of R are also given.

14. **Bayesian Inference for Contingency Tables.** D. V. LINDLEY, University College of Wales. (Invited)

A discussion of approximate methods of making Bayesian analyses (i.e. analyses using prior distributions) with binomial samples leads to a generalization to multinomial samples and contingency tables. The main result is that if θ_i are the multinomial parameters and a_i a set of constants such that $\sum a_i = 0$ then $\sum a_i \log \theta_i$ has a posterior distribution which is approximately normal with mean $\sum a_i \log n_i$ and variance $\sum a_i^2 n_i^{-1}$ where n_i is the observed number in the class of parameter θ_i. Extensions of this result enable contingency tables to be analyzed using non-orthogonal analyses of variance. It is suggested that the breakdown of the sums of squares in the analysis should not follow the conventional lines of main effects, interactions, etc. but should correspond to the independence of the classifications under different conditions. Comparisons with other methods of analysis are given.

15. **A Ratio Limit Theorem for Cascade Processes.** P. E. NEY, Cornell University.

Given an initial particle of unit energy which after a time T, splits into N particles of energies $X_1, \cdots, X_N$ respectively, where $T, N, X_1, \cdots, X_N$ are random variables. Assume that $P\{X_1 + \cdots + X_N \leqq 1\} = 1$. Let $N(x, t)$ denote the number of particles of energy at least x at time t, and $p_n(x, t) = P\{N(x, t) = n\}$. Under certain regularity conditions on the distributions of the above random variables it is shown that for $0 < x \leqq 1$ and $m < n$ we have $p_n(x, t)/p_n(x, t) \to 0$. This generalizes a result of Lopuszanski and Urbanik (*Nuovo Cimento*, Ser. 10, **2**, Suppl. 4, 1147–1167).

16. **The Limit Distribution of a Binary Cascade Process** (Preliminary report). P. E. NEY, Cornell University. (By title)

The following conjecture of T. E. Harris is proved. Use the same notation as in the previous abstract. Assume that T has an exponential distribution with parameter λ, that $P\{N = 2\} = 1$, and that (X_1, X_2) have a symmetric d.f. Let $\mu = E(-\log X_1)$ and $\sigma^2 =$ var $(-\log X_1)$. Let $x_t = \exp\{-2\lambda\mu t - k[2\lambda t \, (\mu^2 + \sigma^2)]^{\frac{1}{2}}\}$, where k is a constant. Let $\Phi(\cdot)$ denote the Gaussian d.f. *Theorem:* $N(x_t, t)/N(0, t) \to \Phi(k)$ in probability.

17. **Ordering of Probabilities of Rank Orders: Fine Structure** (Preliminary report). I. RICHARD SAVAGE and MILTON SOBEL, University of Minnesota.

Assume $X = (X_1, \cdots, X_m)$ and $Y = (Y_1, \cdots, Y_n)$ are independent samples drawn from the densities $f(\cdot)$ and $g(\cdot)$ respectively. Let $Z = (Z_1, \cdots, Z_{m+n})$ be a random vector of 0's and 1's such that $Z_i = 0$ or 1 according as the ith smallest of the combined sample of $m + n$ observations is an X or a Y. Let $P(z) = \Pr(Z = z)$. *Define:* $z^c = 1 - z = (1 - z_i)$ and $z^t = (z_i^t)$ where $z_i^t = z_{m+n+1-i}$. *Assumptions:* ST: $f(x) = f(-x)$ and $g(x) = f(x - \theta)$;

U: $f(x) \geqq f(x')$ if $0 \leqq x < x'$; MLR: $g(y/f(y) \geqq g(x)/f(x)$ if $y > x$; N: $f(\cdot)$ and $g(\cdot)$ are normal distributions with common variance 1 and means 0 and θ, respectively. Under ST and /or N, we use the notations $P(z)$ and $P(z \mid \theta)$ interchangeably. *Theorem* 1. If ST holds then $P(z) = P(z^{tc})$ for all θ. See *Ann. Math. Statist.* **28** (1957) p. 975. *Theorem* 2. If ST holds then $P(z \mid \theta) = P(z^t \mid -\theta) = P(z^c \mid -\theta)$ for all θ. *Theorem* 3. If MLR holds and z and z' have a common number of 0's and 1's such that $\sum_{j=1}^{i} (z_j' - z_j) \geqq 0$ for $i = 1, \cdots, m + n$, then $P(z) \geqq P(z')$ for all $\theta > 0$. Equality holds if and only if $z = z'$ or $f(\cdot) = g(\cdot)$. See *Ann. Math. Statist.* **27** (1956) p. 597. *Theorem* 4. If N holds then $P(1, 0^{r+2}, 1) > P(0, 1, 0^r, 1, 0)$ and if N holds then $P(0^{r+1}, 1, 1, 0^{r+1}) > P(0^r, 1, 0, 0, 1, 0^r)$ for $\theta \neq 0$ and $r = 1, 2, \cdots$. *Theorem* 5. If N holds then $P(z, 1, 0, 0, 1) > P(z, 0, 1, 1, 0)$ and $P(0, 1, 1, 0, z) > P(1, 0, 0, 1, z)$ for any z and all $\theta > 0$. *Theorem* 6. If ST and U hold then $P(0, 0, 1, 1, 0) > P(1, 0, 0, 0, 1)$ for all $\theta > 0$. *Theorem* 7. If ST and MLR hold then $P(0, 0, 1, 1, 0, z^{tc}) > P(z, 1, 0, 0, 0, 1)$ for all $\theta > 0$, provided the number of 1's equals the number of 0's in z. In the following $z(abc)z'$ means $P(z) > P(z')$ for all $\theta > 0$ under conditions *abc*. 00011 (MLR) 00101 (MLR) 01001 (N) 00110 (ST and U) 10001 (N) 01010 (MLR) 01100 (N) 10010 (MLR) 10100 (MLR) 11000.

18. A Method of Fitting the Regression Curve $E(y) = \alpha + \delta x + \beta\rho^x$. B. K. Shah and C. G. Khatri, M.S. University of Baroda; Gujarat University. (By title)

In fitting this curve, estimation of ρ plays an important role in such a nonlinear curve. In a previous paper Shah and Khatri described the quadratic estimators and Hartley's modified estimators for ρ. As n increases the efficiencies of these estimates decreases in an systematic order. The least squares estimate $\hat{r}$ can be expressed as $\hat{r} = \sum_1^{n-1} w_x(\hat{r})\, y_x / \sum_1^{n-1} w_x\, (\hat{r})\, y_{x-1}$, $(\sum_1^{n-1} w_x = \sum x w_x = 0)$. Where $w_x(\hat{r})$ are polynomials of degree $3n$–12 in $\hat{r}$. In this paper $w_x(\hat{r})$ is replaced by functions $u_x + r v_x$. When $n = 5$, the u_x and v_x can be chosen so that $u_x + r_i v_x \alpha w_x(r_i)$ for three different values of $r_i = r_1\, r_2$, and r_3. The estimate r is then equal to $\hat{r}$ for each of the three values. The efficiency is very high (over 99.9%) throughout the entire range of ρ. For $n > 5$, the method of obtaining u_x and v_x is described in such a way that the efficiencies at $\rho = 0$ and at $\rho = 1$ are nearly the same. Thus at $n = 14$, the overall efficiencies have been found to be over 93%, while in quadratic and Hartley's modified method they are found to be 93% at $n = 8$.

19. Further Investigation in Fitting the Regression Curve of the Type $E(y) = \alpha + \delta x + \beta\rho^x$. B. K. Shah and C. G. Khatri, M.S. University of Baroda; Gujarat University. (By title)

In this paper detailed study, for providing the initial estimates for the method of Shah and Patel (1961) and to enable rapid checks on the assumed values of ρ, has been made. In this paper, the estimate of ρ is considered under two alternative methods: (i) Patterson's (1958) method of estimating ρ by considering a ratio of two "Quadratic functions of y's" which he calls the "Quadratic Estimates", (ii) Modified Hartley's method suggested by considering the internal regression of y_{x+1} on $kS_x + lS_{x+1}$, x and x^2, as described by Khatri and Shah (1959). It is interesting to note that the efficiencies in the case of quadratic estimators are about 97% for $r < 0.4$, while in the case of Hartley's modified method they are found to be 99.9% for $r > 0.5$. The biases in both the methods are also considered. The construction of a matrix in quadratic estimator is made such that the quadratic estimate of ρ has minimum asymptotic variance when ρ takes some particular value, ρ_0 say.

20. The Distribution of Products of Independent Random Variables (Preliminary report). MELVIN D. SPRINGER and WILLIAM E. THOMPSON, General Motors Corporation, Santa Barbara; University of New Mexico.

Fundamental methods for the derivation of p.d.f.'s of independent random variables are developed, using a modification of the Mellin transform. This represents an extension to n variables of a method presented by Epstein (*Ann. Math. Statist.*, **19** (1948), 370–380). It is shown that a minor extension of this procedure, allows the derivation of the p.d.f. of the geometric means. The p.d.f.'s of products and geometric means of rectangular, Cauchy, and normal random variables are then derived for specific cases including some which, to the best knowledge of the authors, are hitherto unknown. The limiting forms of the p.d.f.'s, as the number of factors increases without limit, are obtained. In particular, it is shown that the p.d.f.'s tend to unusual but very simple forms. A suggested computation procedure uses the method of contour integration to evaluate a general form of the Mellin inversion integral which is applicable beyond the specific cases treated in this paper. The procedure is especially convenient for numerical evaluation and tabulation by electronic digital computers.

21. An Experimental Study of the Power of Goodness-of-Fit Tests. RICHARD C. TAEUBER, CLAIR J. BECKER, BENJAMIN CRON, and BEVERLY C. HASSELL, C-E-I-R, Inc.; United States Navy Underwater Sound Laboratory, New London, Connecticut.

The problem giving rise to this study is the determination of the parent distribution of various sets of acoustical data, and the sample size needed to give specified power to the test used. 100 samples at each of several different sample sizes, from 20 to 300, were drawn from a normal population. The Kolmogorov-Smirnov and chi-square tests were then used to test the hypotheses that the sample data came from the normal, log-normal, uniform and Rayleigh distributions in turn. Various specified values of the parameters involved were used, as well as the maximum likelihood estimates (these giving that member of a given distribution-family closest to the sample data). Graphs of the experimental power function of the K.S. and χ^2 tests versus sample size are given for the various hypothetical assumptions. In addition, the χ^2 test is compared under various groupings of class intervals.

22. The Relation Between Pitman's Asymptotic Relative Efficiency of Two Tests and the Correlation Coefficient Between Their Test Statistics. CONSTANCE VAN EEDEN, University of Minnesota.

Let T'_n be a test for H_0 based on n observations. Let T_n be an asymptotically locally most powerful test for H_0 and let t'_n and t_n be the test statistics for these two tests. In this paper it is shown that, under certain regularity conditions, Pitman's asymptotic relative efficiency of the test T'_n with respect to the test T_n equals the limit (for $n \to \infty$) of the correlation coefficient between t'_n and t_n under H_0.

23. Sequential Optimum Procedures for Unbiased Estimation of a Binomial Parameter. M. T. WASAN, Queen's University.

Let $x_1, x_2, \cdots x_n, \cdots$ be a sequence of independent random variables with common density function $P(x = 1) = p, P(x = 0) = 1 - p, 0 < p < 1$. The non-randomized sequential procedures δ's are considered for estimating p and the following two kinds of problems on choice of δ are considered subject to suitable regularity conditions; (A) subject to $E_p(Z_\delta - p)^2 \leqq a$ (where a is a positive real number and Z_δ is an unbiased estimate of p), choose δ to minimize E_pN_δ ; (B) choose δ to minimize $CE_pN_\delta + E_p(Z_\delta - p)^2$ where C is a positive real number as cost of an observation. In each case the minimization is to be done uniformly in p if possible; otherwise the supremum over p of the risk in question is to be minimized. The fixed sample size procedure is shown to be admissible and minimax for problem (A) and (B). A procedure is constructed which asymptotically uniformly better than the fixed sample size for the problem (A). Some other optimum procedures are constructed for problem (B).

24. Censored Least Squares Unbiased Linear Estimation for the Log Weibull (Extreme Value) Distribution. JOHN S. WHITE, General Motors Research Laboratories, Warren, Michigan.

Let T be a random variable having a two parameter Weibull distribution with parameters β and θ. Then Prob $(T \leqq t) = F(t) = 1 - \exp(-(t/\theta)^\beta)$. The distribution function of $x = \log T$ is then $F(x) = 1 - \exp(-\exp(\beta(x - \log\theta)))$ and the reduced variable $Z = \beta(X - \log\theta)$ has a form of the extreme value distribution $F(z) = 1 - \exp(-\exp z)$. Setting $A = \log\theta$, $B = 1/\beta$ gives the relation $\log T = X = A + BZ$. Since the Weibull distribution is a model for fatigue or wearout life, it is of interest to have estimators of A and B (or θ and β) for samples censored on the right. Using the generalized least squares theorem of Lloyd (*Biometrika* **39** (1952)) coefficients $A(I, J, N)$, $B(I, J, N)$, $(1 \leqq I \leqq J \leqq N \leqq 20)$ are computed such that the estimators $A(J, N) = \sum_{I=1}^{J} A(I, J, N)X(I, N)$ and $B(J, N) = \sum_{I=1}^{J} B(I, J, N)X(I, N)$ are unbiased and have minimum variance in the class of all linear estimators depending only on the first J order statistics $X(1, N) \leqq \cdots \leqq X(J, N)$. This paper extends the results of Lieblein (NACA, TN 3053, 1954) from $N = 6$ to $N = 20$. Tables of the variances and covariances of the order statistics $X(I, N)$, $X(K, N)$ are also included.

25. Limiting Distributions of Random Sums of Independent Random Variables. HELEN WITTENBERG, University of California, Berkeley.

Given a sequence $\{X_k\}$ of independent and identically distributed random variables and a sequence $\{\tau_k\}$ of nonnegative integer-valued random variables, the limiting behavior of $S(\tau_n) = \sum_1^{\tau_n} X_k$ is investigated. The first case considered is of random variables $\{X_k\}$ symmetric about zero and $\{\tau_k\}$ such that for some sequence $c_n \to \infty$, $\tau_n/c_n \to_p 1$. For such variables the Kolmogorov distance—the greatest vertical distance between distribution functions—between $S(\tau_n)$ and $S(c_n)$ tends to zero. When, instead, $\tau_n/c_n \to_p \xi$ then $S(\tau_n)$ and $S[\xi c_n]$ differ little. Moreover, if γ is any random variable distributed as ξ and independent of $\{X_k\}$, the distance between $S(\tau_n)$ and $S[\gamma c_n]$ also tends to zero. For more general summands $S(\tau_n)$ and $S(c_n)$ may differ widely even when $\tau_n/c_n \to_p 1$. However, for such τ_n the distance between $S(c_n)$ and $S(\tau_n)$ suitably centered tends to zero. A class of variables is defined which behaves as variables distributed symmetrically about zero, for the purposes of this problem. Included in this class are all variables with the property that for some sequence $b(n)$, $S(n)/b(n)$ converges in law to a proper stable law Y of index α. It is shown that for such variables, if $\tau_n/c_n \to_p \xi$ then $\mathfrak{L}(S(n)/b(c_n)) \to \mathfrak{L}(\xi^{1/\alpha}Y)$.

(Abstracts not connected with any meeting of the Institute.)

1. Non-Linear Regression Made Computationally Easy. ARTHUR ALBERT, Arcon Corporation, Lexington, Massachusetts.

Let $\{X_n\}$, $(n = 1, 2, \cdots)$ be a stochastic process of the form $X_n = F_n(\theta) + V_n$, where $\{V_n\}$ is zero mean process with bounded variances. The functional form of the mean value function $F(\theta)$ is known, except for the real valued parameter, θ, which is to be estimated from the observations $X_1, X_2, \cdots$. We discuss sequences of estimates of the form: θ_1 arbitrary, $\theta_{n+1} = \theta_n + k_n[X_n - F_n(\theta_n)]$. In particular, we deal with the case where k_n depends upon the past history of the process $(k_n = F'_n(\theta_n)/\sum_{j=1}^n F'_j(\theta_j)^2)$ and another case where the k_n are deterministic, $(k_n = b_n/\sum_{j=1}^n b_j^2, b_n = \inf_\theta |F'_n(\theta)|)$. Under reasonable regularity conditions, we demonstrate that $\theta_n \rightarrow \theta$ w.p.1 and in the mean square. Investigations of the efficiency of these estimation procedures and of the case where θ is a vector parameter, are currently in progress.

2. **Correlation Coefficient Between Ranges in Samples From a Bivariate Population With Applications to Normal and Pareto Type 1 Populations** (Preliminary report). K. V. MARDIA, University of Rajasthan. (Introduced by B. D. Tikkiwal)

Let (x_i, y_i), $i = 1, \cdots, n$, be a random sample of size n from a bivariate continuous population. Let $X_1 = \min(x_i)$, $X_2 = \max(x_i)$, $Y_1 = \min(y_i)$, $Y_2 = \max(y_i)$, $R_1 = X_2 - X_1$ and $R_2 = Y_2 - Y_1$. Tippett (*Biometrika* **17** (1925) 364–387) gives formulae for the expectations and variances of R_1 and R_2. In this paper, similar formula for $E(R_1R_2)$ is derived by help of the density function of (X_1, X_2, Y_1, Y_2) (Mardia, a paper under consideration in *Ann. Math. Statist.*). The correlation between R_1 and R_2 can now be obtained. The form of the correlation and its properties have been studied in details for the bivariate normal and Pareto type 1 populations (Mardia, *Ann. Math. Statist.* **33** (1962) 1008–1015). In the normal case, this correlation is found to be an even function of ρ where ρ is the population correlation coefficient. In this case, the table of Cor (R_1, R_2) is under preparation for $n = 2(1)20$, 30, 60, 100, 200, 500, 1000 and $\rho = .01(.01)1$.

3. **Exact Distribution of Order Statistics in Samples From a Multivariate Population With Applications to Pareto Type 1 Population** (Preliminary report). K. V. MARDIA, University of Rajasthan. (Introduced by B. D. Tikkiwal)

Let $(x_{1r}, \cdots, x_{kr})$, $r = 1, \cdots, n$, be a random sample of size n from a k-variate continuous population. Let the ordered sample values of the ith variate be $x_i(1) < \cdots < x_i(n)$, $i = 1, \cdots, k$. In this paper, the exact density function of $(x_1(n_1), \cdots, x_k(n_k))$; $1 \leqq n_i \leqq n$, $i = 1, \cdots, k$, is obtained by application of the multinomial theorem. It is noted that the density of $(x_1(1), \cdots, x_k(1))$ in Pareto type 1 population (Mardia, *Ann. Math. Statist.* **33** (1962) 1008–1015) is again of the Pareto type 1 form. The necessary and the sufficient condition for preserving the form of the distribution is that the Prob $(X_1 > x_1, \cdots, X_k > x_k)$ of population should be of the form $(g(x_1, \cdots, x_k))^a$. The well known identity, $n^k = \sum_{r=1}^n n^{(r)}S(k, r)$ where $S(k, r)$ is a Stirling number of the second kind, and an expansion of $(np + k - 1)^{(k)}$, $p > 0$, are obtained by considering some special cases of the density function of $(x_1(1), \cdots, x_k(1))$.

4. **Admissibility of Some Tests of Manova.** M. N. GHOSH, Institute of Agricultural Research Statistics, New Delhi.

We consider the canonical form of the test of multivariate linear hypothesis, where we have row-vectors $Z(q)$ $(q = 1, \cdots Q + m)$ and $Y(u)$ $(u = 1, \cdots n)$ with p-variate normal distributions $N(b(q), \Sigma)$ and $N(0, \Sigma)$ respectively and the hypothesis to be tested is $b(q\} = 0(q = 1, \cdots Q)$. Several criteria have been proposed for the test of this hypothesis, which are functions of the roots $\theta_1, \theta_2, \cdots, \theta_p$ of the equation $|Z - \theta Y| = |\sum_{q=1}^Q Z(q)'Z(q) - \theta\sum_{u=1}^n y(u)'y(u)| = 0$, e.g., (i) $\sum \theta_i = \text{Tr}(ZY^{-1})$, (ii) θ_1, (iii) θ_p, (iv) $\prod(1 + \theta_i)^{-1}$.

Following the method of Stein for the admissibility of Hotelling's T^2 (*Ann. Math. Statist.* **27** 616–623), it is shown that the test criteria based on the sum of roots and on the largest root are admissible.

5. **Hotelling's Generalised T^2 in the Multivariate Analysis of Variance.** M. N. Ghosh, Institute of Agricultural Research Statistics, New Delhi.

In the canonical form of the tests of multivariate linear hypothesis, we have row-vectors $Z(q)$, $(q = 1, \cdots Q)$ and $y(u)(u = 1, \cdots n)$ with p-variate normal distributions $N(b(q), \Sigma)$ and $N(0, \Sigma)$ respectively and the hypothesis to be tested in $b(q) = 0$, $(q = 1, \cdots Q)$. Hotelling's generalised T^2 is $1/n$ times the sum of the roots of the determinantal equation $|Z - \theta Y| = |\sum_{q=1}^{Q} Z(q)'Z(q)' - \theta \sum_{u=1}^{n} y(u)'y(u)| = 0$, i.e., $T_Q^2 = (1/n) \text{ Tr } (ZY^{-1})$. The distribution of T_Q^2 is shown to be a monotonic increasing function of each of the population characteristic roots. The mean of T_Q^2 for all p and variance for $p = 3$ and $p = 4$ are calculated in the noncentral case and an unbiased estimate of a linear function of the sum of population roots, which may be considered as a convenient measure of noncentrality is obtained from this statistic. Simultaneous confidence intervals of all linear functions of the means $b_i(q)$ are also obtained by using the T_Q^2-statistic in a more general form than by Roy and Bose (*Ann. Math. Statist.* **24** 513–536).