THE LIMITING POWER OF CATEGORICAL DATA CHI-SQUARE TESTS ANALOGOUS TO NORMAL ANALYSIS OF VARIANCE[1]

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1. Introduction. Mitra [5] has derived the Pitman limiting power [10] of the frequency chi-square test. He considers the case in which it is hypothesized that the cell probabilities are specified functions of unknown parameters which are to be estimated from the sample. For completeness and because it is needed for subsequent proofs, his theorem is presented without proof. In the usual situation of making tests for categorical data tables of two or more dimensions, where the hypotheses to be tested are of the forms presented by Roy and Mitra [12] and Diamond, Mitra and Roy [3], the hypothesis is that the cell probabilities are specified functions of unknown parameters which are to be estimated from the sample and which are subject to specified functional conditions. In Section 3 of this paper a theorem covering this latter case is presented without proof. The proof which is exactly analogous to that for Mitra's theorem covering the simpler case is presented elsewhere by Mitra [6], Ogawa [7] and Diamond [2].

Another type of test, analogous to those of normal analysis of variance, might be considered. One assumes that the cell probabilities are specified functions of unknown parameters. This assumption, together with the initial sampling distribution, form the "model". The hypothesis to be tested is that the parameters satisfy specified functional relationships. In Section 4 two theorems are proved. The first, a necessary preliminary, covers the situation in which the hypothesis is that some of the unknown parameters have specified values. The second theorem in Section 4 covers the situation in which the hypothesis is that the parameters satisfy specified functional relationships. In both theorems, the limiting distribution is shown to be a non-central chi-square with certain degrees of freedom and a specific non-centrality parameter in the non-null case.

2. Mitra's theorem in frequency chi-square [5]. Suppose we have $R = \sum_{i=1}^{q} r_i$ functions $p_{ij}(\alpha_1, \alpha_2, \cdots \alpha_s)$ $(i = 1, 2, \cdots, q; j = 1, 2, \cdots, r_i)$ of $s < R - q$ parameters $\alpha_1, \alpha_2, \cdots, \alpha_s$ such that for all points of a nondegenerate interval A in the s-dimensional space of the α_k's the p_{ij} satisfy the following conditions:

(a) $\sum_{j=1}^{r_i} p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s) = 1$ for $i = 1, 2, \cdots, q$,
(b) $p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s) > c^2 > 0$ for all ij,
(c) Every p_{ij} has continuous derivatives $\partial p_{ij}/\partial \alpha_k$ and $\partial^2 p_{ij}/\partial \alpha_k \partial \alpha_l$,
(d) The $R \times s$ matrix $\{\partial p_{ij}/\partial \alpha_k\}$ is of rank s.

(It is assumed that the index pairs (i, j) indicating the rows of the above matrix

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or of any such matrix defined subsequently, are arranged in lexicographic order.)

For $n = 1, 2, \cdots$, ad. inf., let $(N_1^{(n)}, N_2^{(n)}, \cdots, N_q^{(n)})$ be a sequence of row vectors such that for $i = 1, 2, \cdots, q$, and every n, (i) $N_i^{(n)}$ is a natural number, (ii) $N_i^{(n+1)} > N_i^{(n)}$, (iii) if $N_n = \sum_{i=1}^q N_i^{(n)}$, then $N_i^{(n)}/N_n = Q_i$ independent of n.

Let $\boldsymbol{\alpha}_0' = (\alpha_1^0, \alpha_2^0, \cdots, \alpha_s^0)$ be an inner point of A (the "true" parameter point) and let c_{ij} $(i = 1, 2, \cdots, q; j = 1, 2, \cdots, r_i)$ be a given set of numbers such that

$$\sum_{j=1}^{r_i} c_{ij} = 0 \quad \text{for} \quad i = 1, 2, \cdots, q. \tag{2.1}$$

Put

$$p_{ij}^0 = p_{ij}(\alpha_1^0, \alpha_2^0, \cdots, \alpha_s^0) \tag{2.2}$$

and

$$p_{ijn} = p_{ij}^0 + c_{ij}N_n^{-\frac{1}{2}}. \tag{2.3}$$

Let n_0 be a positive integer such that for $n \geqq n_0$, $p_{ijn} > 0$ for all i, j. For $n = n_0, n_0 + 1, \cdots$, ad. inf., let $\{v_{ijn}\}$ $(i = 1, 2, \cdots, q, j = 1, 2, \cdots, r_i)$ be a sequence of R-dimensional random variables such that

$$\text{Prob. } \{v_{ijn}\} = \prod_{i=1}^{q} \left[N_i^{(n)}! \Big/ \prod_{j=1}^{r_i} v_{ijn}! \right] \prod_{j=1}^{r_i} (p_{ijn})^{v_{ijn}} \tag{2.4}$$

if v_{ijn} are any set of non-negative integers (some of which might be zero) and

$$\sum_{j=1}^{r_i} v_{ijn} = N_i^{(n)}, \quad i = 1, 2, \cdots, q,$$

$$= 0, \qquad \text{otherwise.}$$

Consider the system of equations

$$\sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{v_{ijn} - N_nQ_ip_{ij}}{p_{ij}} \cdot \frac{\partial p_{ij}}{\partial \alpha_k} = 0, \qquad k = 1, 2, \cdots, s. \tag{2.5}$$

Mitra [5] proves

THEOREM 2.1.

(i) *The system of equations* (2.5) *have exactly one system of solutions* $\hat{\boldsymbol{\alpha}}_n' = (\hat{\alpha}_{1n}, \hat{\alpha}_{2n}, \cdots, \hat{\alpha}_{sn})$ *such that* $\hat{\boldsymbol{\alpha}}_n'$ *converges in probability to* $\boldsymbol{\alpha}_0'$ *as* $n \to \infty$.

(ii) *The value of* χ^2 *obtained by inserting* $\alpha_k = \hat{\alpha}_{kn}$ *in*

$$\chi^2 = \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{(v_{ijn} - N_nQ_ip_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s))^2}{N_nQ_ip_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s)} \tag{2.6}$$

is, in the limit as $n \to \infty$, *distributed in a non-central* χ^2*-distribution* ([4], [9]), *with* $R - q - s$ *degrees of freedom and a non-centrality parameter)* $\Delta = \boldsymbol{\delta}'[\mathbf{I} - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}']\boldsymbol{\delta}$, *where*

$$\boldsymbol{\delta}(R \times 1) = \{(Q_i/p_{ij}^0)^{\frac{1}{2}} c_{ij}\},$$

and

$$\mathbf{B}(R \times s) = \{(Q_i/p_{ij}^0)^{\frac{1}{2}} (\partial p_{ij}/\partial \alpha_k)_0\},$$

where the notation $(\ )_0$ *indicates the derivative is evaluated at* $\boldsymbol{\alpha}_0$. *(This notation is used throughout the paper.)*

3. The limiting power function of the chi-square contingency test. Mitra's theorem develops the limiting power function for tests of hypotheses of the form

$$(3.1) \quad H_0 : p_{ij} = p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s), \quad (i = 1, 2, \cdots, q;\ j = 1, 2, \cdots, r_i)$$

against alternatives of the form

$$(3.2) \quad H_n : p_{ijn} = p_{ij}(\alpha_1^0, \alpha_2, \cdots, \alpha_s^0) + c_{ij} N_n^{-\frac{1}{2}}$$

where the α_k are to be estimated from a sample of N_n observations.

In making a contingency test we are concerned with hypotheses of the form

$$(3.3) \quad H_0 : p_{ij} = p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s), \quad (i = 1, 2, \cdots, q;\ j = 1, 2, \cdots, r_i),$$
$$\text{subject to } f_m(\alpha_1, \alpha_2, \cdots, \alpha_s) = 0, \quad (m = 1, 2, \cdots, t < s),$$

against alternatives of the form

$$(3.4) \quad H_n : p_{ijn} = p_{ij}(\alpha_1^0, \alpha_2^0, \cdots, \alpha_s^0) + c_{ij} N_n^{-\frac{1}{2}}$$
$$\text{subject to } f_m(\alpha_1^0, \alpha_2^0, \cdots, \alpha_s^0) = 0$$

where the α_k are to be estimated from a sample of N_n observations.

Suppose that in addition to the conditions and definitions of Section 2 the $t < s$ functions $f_m(\alpha_1, \alpha_2, \cdots, \alpha_s)$, $(m = 1, 2, \cdots, t)$ of the s parameters $\alpha_1, \alpha_2, \cdots, \alpha_s$ are such that for all points of A the f_m satisfy the following conditions:

(e) Every f_m has continuous derivatives $\partial f_m/\partial \alpha_k$ and $\partial^2 f_m/\partial \alpha_k \partial \alpha_l$,

(f) The $t \times s$ matrix $\{\partial f_m/\partial \alpha_k\}$ is of rank t.

Consider the system of equations

$$(3.5) \quad \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{v_{ijn} - N_n Q_i p_{ij}}{p_{ij}} \cdot \frac{\partial p_{ij}}{\partial \alpha_k} + \sum_{m=1}^{t} \lambda_m \frac{\partial f_m}{\partial \alpha_k} = 0, \quad k = 1, 2, \cdots, s,$$
$$f_m = 0, m = 1, 2, \cdots, t.$$

In a manner exactly analogous to that given by Mitra [5] for Theorem 2.1 we can prove (see [2], [6], [7])

THEOREM 3.1.

(i) *The system of equations* (3.5) *have exactly one system of solutions* $\tilde{\boldsymbol{\alpha}}_n' = (\tilde{\alpha}_{1n}, \tilde{\alpha}_{2n}, \cdots, \tilde{\alpha}_{sn})$, $\tilde{\boldsymbol{\lambda}}' = (\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_t)$ *such that* $\tilde{\boldsymbol{\alpha}}_n'$ *converges in probability to* $\boldsymbol{\alpha}_0'$ *as* $n \to \infty$.

(ii) *The value of* χ^2 *obtained by inserting* $\alpha_k = \tilde{\alpha}_{kn}$ *in* (2.6) *is, in the limit as*

$n \to \infty$, distributed in a non-central χ^2-distribution with $R - q - s + t$ degrees of freedom and a non-centrality parameter $\Delta = \boldsymbol{\delta}'[\mathbf{I} - \mathbf{C}(\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}']\boldsymbol{\delta}$ where

$$\boldsymbol{\delta}(R \times 1) = \{(Q_i/p_{ij}^0)^{\frac{1}{2}} c_{ij}\}$$

and

$$\mathbf{C}(R \times (s - t)) = \{(Q_i/p_{ij}^0)^{\frac{1}{2}} (\partial p_{ij}^*/\partial \alpha_{k_2})_0\}$$

where p_{ij}^* is p_{ij} expressed in terms of the $s - t$ α_k that are independent under $f_m = 0$ $(m = 1, 2, \cdots, t)$ and α_{k_2} are the $s - t$ independent α_k.

4. The limiting power function of chi-square tests analogous to normal analysis of variance tests. Suppose it is given as a part of the "model" that $p_{ij} = p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s)$, $(i = 1, 2, \cdots, q; j = 1, 2, \cdots, r_i)$. We may then test a hypothesis of the form

$$H_0: f_{m_1}(\alpha_1, \alpha_2, \cdots, \alpha_s) = 0, \qquad m_1 = 1, 2, \cdots, t < s \tag{4.1}$$

against an alternative of the form

$$H_n: f_{m_1}(\alpha_1^0, \alpha_2^0, \cdots, \alpha_s^0) = d_{m_1} N_n^{-\frac{1}{2}} \text{ where not all } d_{m_1} = 0. \tag{4.2}$$

In order to develop the limiting power function of this type of test we must first consider the simpler special case where we are testing the hypothesis

$$H_0: \alpha_{k_1} = a_{k_1}, \qquad k_1 = 1, 2, \cdots, t < s \tag{4.3}$$

against the alternative

$$H_n: \alpha_{k_1}^0 = a_{k_1} + d_{k_1} N_n^{-\frac{1}{2}} \text{ where not all } d_{k_1} = 0. \tag{4.4}$$

Suppose that the conditions of Sections 2 and 3 hold. Let $\hat{\boldsymbol{\alpha}}$ be the unique consistent solution of equations (2.5) and let χ^2 be the corresponding test statistic formed by inserting $\alpha_k = \hat{\alpha}_k$ in (2.6).

Let $\tilde{\boldsymbol{\alpha}}$ be the unique consistent solution of the equations

$$\sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{v_{ijn} - N_n Q_i p_{ij}}{p_{ij}} \cdot \frac{\partial p_{ij}}{\partial \alpha_k} + \sum_{k_1=1}^{t} \lambda_{k_1} \frac{\partial \alpha_{k_1}}{\partial \alpha_k} = 0, \tag{4.5}$$
$$k = 1, 2, \cdots, s, \alpha_{k_1} = a_{k_1}, \qquad k_1 = 1, 2, \cdots, t,$$

and let χ_*^2 be the corresponding test statistic formed by inserting $\alpha_k = \tilde{\alpha}_k$ in (2.6). (Note that $\partial\alpha_{k_1}/\partial\alpha_k = 1$ or 0 according as $k_1 = k$ or $k_1 \neq k$ and hence (4.5) may be reduced to $s - t$ equations in $\alpha_{t+1}, \alpha_{t+2}, \cdots, \alpha_s$).

We shall prove

THEOREM 4.1. *$\chi_0^2 = \chi_*^2 - \chi^2$ is, in the limit, independent of χ^2 in probability and distributed in a non-central χ^2-distribution with t degrees of freedom and a non-centrality parameter $\Delta = \mathbf{d}'\mathbf{B}_1'[\mathbf{I} - \mathbf{B}_2(\mathbf{B}_2'\mathbf{B}_2)^{-1}\mathbf{B}_2']\mathbf{B}_1\mathbf{d}$* where $\mathbf{d}(t \times 1) = \{d_{k_1}\}$,

$$\mathbf{B}_1(R \times t) = \{(Q_i/p_{ij}^0)^{\frac{1}{2}} (\partial p_{ij}/\partial \alpha_{k_1})_0\}, \quad k_1 = 1, 2, \cdots, t$$

and

$$\mathbf{B}_2(R \times (s-t)) = \{(Q_i/p_{ij}^0)^{\frac{1}{2}}(\partial p_{ij}/\partial\alpha_{k_2})_0\}, \qquad k_2 = t+1, t+2, \cdots, s.$$

PROOF. We put

$$y_{ijn} = [v_{ijn} - N_n Q_i p_{ij}(\hat{\boldsymbol{\alpha}})]/[N_n Q_i p_{ij}(\hat{\boldsymbol{\alpha}})]^{\frac{1}{2}}$$

$$y_{ijn}^* = [v_{ijn} - N_n Q_i p_{ij}(\tilde{\boldsymbol{\alpha}})]/[N_n Q_i p_{ij}(\tilde{\boldsymbol{\alpha}})]^{\frac{1}{2}}$$

$$x_{ijn} = [v_{ijn} - N_n Q_i p_{ij}^0]/[N_n Q_i p_{ij}^0]^{\frac{1}{2}}$$

$$\mathbf{y}_{(n)}(R \times 1) = \{y_{ijn}\}, \quad \mathbf{y}_{(n)}^*(R \times 1) = \{y_{ijn}^*\}, \quad \mathbf{x}_{(n)}(R \times 1) = \{x_{ijn}\}.$$

Ogawa [7] has shown that in the limit the equations (2.5) may be written

$$\sum_{k=1}^{s} (\alpha_k - \alpha_k^0) \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{Q_i}{p_{ij}^0} \left[\frac{\partial p_{ij}}{\partial \alpha_k}\right]_0 \left[\frac{\partial p_{ij}}{\partial \alpha_m}\right]_0 = \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{(v_{ijn} - N_n Q_i p_{ij}^0) Q_i}{N_n Q_i p_{ij}^0} \left[\frac{\partial p_{ij}}{\partial \alpha_m}\right]_0, \qquad m = 1, 2, \cdots, s.$$

In matrix form this becomes $\mathbf{B}'\mathbf{B}(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0) = N_n^{-\frac{1}{2}}\mathbf{B}'\mathbf{x}_{(n)}$ where

$$\underset{R\times s}{\mathbf{B}} = [\underset{t}{\mathbf{B}_1} : \underset{s-t}{\mathbf{B}_2}]\, R.$$

Similarly, in the limit, the equations (4.5) may be written

$$\sum_{k_1=1}^{t} (a_{k_1} - \alpha_{k_1}^0) \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{Q_i}{p_{ij}^0} \left[\frac{\partial p_{ij}}{\partial \alpha_{k_1}}\right]_0 \left[\frac{\partial p_{ij}}{\partial \alpha_{m_2}}\right]_0 + \sum_{k_2=t+1}^{s} (\alpha_{k_2} - \alpha_{k_2}^0) \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{Q_i}{p_{ij}^0} \left[\frac{\partial p_{ij}}{\partial \alpha_{k_2}}\right]_0 \left[\frac{\partial p_{ij}}{\partial \alpha_{m_2}}\right]_0 = \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{(v_{ijn} - N_n Q_i p_{ij}^0) Q_i}{N_n Q_i p_{ij}^0} \left[\frac{\partial p_{ij}}{\partial \alpha_{m_2}}\right]_0, \qquad m_2 = t+1, t+2, \cdots, s,$$

and in matrix notation $\mathbf{B}_2'\mathbf{B}_1(\mathbf{a} - \boldsymbol{\alpha}_1^0) + \mathbf{B}_2'\mathbf{B}_2(\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_2^0) = N_n^{-\frac{1}{2}}\mathbf{B}_2'\mathbf{x}_{(n)}$ where $\mathbf{a}' = (a_1, a_2, \cdots, a_t)$, $\boldsymbol{\alpha}_1' = (\alpha_1, \alpha_2, \cdots, \alpha_t)$ and $\boldsymbol{\alpha}_2' = (\alpha_{t+1}, \alpha_{t+2}, \cdots, \alpha_s)$.

As a solution of equations (2.5) we have, in the limit, $\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0 = N_n^{-\frac{1}{2}}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{x}_{(n)}$ and as a solution of equations (4.5) we have, in the limit, $\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_2^0 = N_n^{-\frac{1}{2}}(\mathbf{B}_2'\mathbf{B}_2)^{-1}\mathbf{B}_2'\mathbf{x}_{(n)} - (\mathbf{B}_2'\mathbf{B}_2)^{-1}\mathbf{B}_2'\mathbf{B}_1(\mathbf{a} - \boldsymbol{\alpha}_1^0)$. Mitra [6] shows that $\mathbf{y}_{(n)} - [\mathbf{x}_{(n)} - N_n\mathbf{B}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0)] \rightarrow_p 0$, or $\mathbf{y}_{(n)} - [\mathbf{I} - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}']\mathbf{x}_{(n)} \rightarrow_p 0$. Similarly $\mathbf{y}_{(n)}^* - [\mathbf{x}_{(n)} - N_n^{\frac{1}{2}}\mathbf{B}_1(\mathbf{a} - \boldsymbol{\alpha}_1^0) - N_n^{\frac{1}{2}}\mathbf{B}_2(\tilde{\boldsymbol{\alpha}}_2 - \boldsymbol{\alpha}_2^0)] \rightarrow_p 0$, or $\mathbf{y}_{(n)}^* - [\mathbf{I} - \mathbf{B}_2(\mathbf{B}_2'\mathbf{B}_2)^{-1}\mathbf{B}_2'] \cdot (\mathbf{x}_{(n)} + \mathbf{B}_1\mathbf{d}) \rightarrow_p 0$.

We can find a matrix $\mathbf{K}$ such that $\mathbf{J} = \mathbf{BK}$ where

$$\mathbf{J} = [\underset{t}{\mathbf{J}_1} : \underset{s-t}{\mathbf{J}_2}]R \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{Q} & \mathbf{S} \end{bmatrix}\begin{matrix} t \\ s-t \end{matrix},$$

where $\mathbf{T}$ and $\mathbf{S}$ are non-singular, and such that $\mathbf{J}_1'\mathbf{J}_2 = \mathbf{0}$ (Ogawa [8]). Then $\mathbf{J}_1 = \mathbf{B}_1\mathbf{T} + \mathbf{B}_2\mathbf{Q}$, $\mathbf{J}_2 = \mathbf{B}_2\mathbf{S}$ and $\mathbf{B}_2'\mathbf{B}_1\mathbf{T} + \mathbf{B}_2'\mathbf{B}_2\mathbf{Q} = \mathbf{0}$. Since $\mathbf{K}$ is non-singular we may find its inverse

$$\mathbf{K}^{-1} = \begin{bmatrix} \mathbf{T}^{-1} & \mathbf{0} \\ -\mathbf{S}^{-1}\mathbf{Q}\mathbf{T}^{-1} & \mathbf{S}^{-1} \end{bmatrix} \begin{matrix} t \\ s-t \end{matrix} .$$
$$\qquad\quad t \qquad\qquad s-t$$

Then $\mathbf{B} = \mathbf{J}\mathbf{K}^{-1}$, $\mathbf{B}_1 = \mathbf{J}_1\mathbf{T}^{-1} - \mathbf{J}_2\mathbf{S}^{-1}\mathbf{Q}\mathbf{T}^{-1}$ and $\mathbf{B}_2 = \mathbf{J}_2\mathbf{S}^{-1}$. Thus, in the limit,

$$\begin{aligned} \mathbf{y}_{(n)} &= [\mathbf{I} - \mathbf{J}\mathbf{K}^{-1}(\mathbf{K}'^{-1}\mathbf{J}'\mathbf{J}\mathbf{K}^{-1})^{-1}\mathbf{K}'^{-1}\mathbf{J}']\mathbf{x}_{(n)} , \\ &= [\mathbf{I} - \mathbf{J}(\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}']\mathbf{x}_{(n)} \\ &= [\mathbf{I} - \mathbf{J}_1(\mathbf{J}_1'\mathbf{J}_1)^{-1}\mathbf{J}_1' - \mathbf{J}_2(\mathbf{J}_2'\mathbf{J}_2)^{-1}\mathbf{J}_2']\mathbf{x}_{(n)} , \end{aligned}$$

and

$$\begin{aligned} \mathbf{y}^*_{(n)} &= [\mathbf{I} - \mathbf{J}_2\mathbf{S}^{-1}(\mathbf{S}'^{-1}\mathbf{J}_2'\mathbf{J}_2\mathbf{S}^{-1})^{-1}\mathbf{S}'^{-1}\mathbf{J}_2][\mathbf{x}_{(n)} + (\mathbf{J}_1\mathbf{T}^{-1} - \mathbf{J}_2\mathbf{S}^{-1}\mathbf{Q}\mathbf{T}^{-1})\mathbf{d}], \\ &= [\mathbf{I} - \mathbf{J}_2(\mathbf{J}_2'\mathbf{J}_2)^{-1}\mathbf{J}_2'][\mathbf{x}_{(n)} + (\mathbf{J}_1\mathbf{T}^{-1} - \mathbf{J}_2\mathbf{S}^{-1}\mathbf{Q}\mathbf{T}^{-1})\mathbf{d}], \\ &= [\mathbf{I} - \mathbf{J}_2(\mathbf{J}_2'\mathbf{J}_2)^{-1}\mathbf{J}_2']\mathbf{x}_{(n)} + \mathbf{J}_1\mathbf{T}^{-1}\,\mathbf{d}. \end{aligned}$$

Let $\mathbf{W} = \mathbf{M}'\mathbf{x}_{(n)}$ where

$$\mathbf{M} = [\underset{q}{\mathbf{P}} : \underset{t}{\mathbf{M}_1} : \underset{s-t}{\mathbf{M}_2} : \underset{R-q-s}{\mathbf{M}_3}\ ]\,R,$$

such that $\mathbf{M}$ is orthogonal and where $\mathbf{P}(R \times q) = \{\delta_{il}p_{ij}^{0\frac{1}{2}}\}$, $(i = 1, 2, \cdots, q;$ $j = 1, 2, \cdots, r_i)$, $(l = 1, 2, \cdots, q)$ and δ_{il} is the Kronecker symbol, $\mathbf{M}_1\mathbf{M}_1' = \mathbf{J}_1(\mathbf{J}_1'\mathbf{J}_1)^{-1}\mathbf{J}_1'$, $\mathbf{M}_2\mathbf{M}_2' = \mathbf{J}_2(\mathbf{J}_2'\mathbf{J}_2)^{-1}\mathbf{J}_2'$.

Mitra [5] shows that $\mathbf{x}_{(n)}$ is asymptotically normal with mean $\mathbf{0}$ and variance-covariance matrix $\mathbf{I} - \mathbf{P}\mathbf{P}'$. It follows that $w_1, w_2, \cdots, w_q$ equal zero with probability one and $w_{q+1}, w_{q+2}, \cdots, w_R$ are asymptotically independent with zero means and unit standard deviations. Thus $\mathbf{x}_{(n)} = \mathbf{M}_1\mathbf{w}_1 + \mathbf{M}_2\mathbf{w}_2 + \mathbf{M}_3\mathbf{w}_3$ where $\mathbf{w}_1' = (w_{q+1}, \cdots, w_{q+t})$, $\mathbf{w}_2' = (w_{q+t+1}, \cdots, w_{q+s})$, $\mathbf{w}_3' = (w_{q+s+1}, \cdots, w_R)$.

Then, in the limit,

$$\begin{aligned} \mathbf{y}_{(n)} &= (\mathbf{I} - \mathbf{M}_1\mathbf{M}_1' - \mathbf{M}_2\mathbf{M}_2')(\mathbf{M}_1\mathbf{w}_1 + \mathbf{M}_2\mathbf{w}_2 + \mathbf{M}_3\mathbf{w}_3), \\ &= \mathbf{M}_1\mathbf{w}_1 + \mathbf{M}_2\mathbf{w}_2 + \mathbf{M}_3\mathbf{w}_3 - \mathbf{M}_1\mathbf{w}_1 - \mathbf{M}_2\mathbf{w}_2 , \\ &= \mathbf{M}_3\mathbf{w}_3 , \end{aligned}$$

and

$$\begin{aligned} \mathbf{y}^*_{(n)} &= (\mathbf{I} - \mathbf{M}_2\mathbf{M}_2')(\mathbf{M}_1\mathbf{w}_1 + \mathbf{M}_2\mathbf{w}_2 + \mathbf{M}_3\mathbf{w}_3) + \mathbf{J}_1\mathbf{T}^{-1}\,\mathbf{d}, \\ &= \mathbf{M}_1\mathbf{w}_1 + \mathbf{M}_2\mathbf{w}_2 + \mathbf{M}_3\mathbf{w}_3 - \mathbf{M}_2\mathbf{w}_2 + \mathbf{J}_1\mathbf{T}^{-1}\,\mathbf{d}, \\ &= \mathbf{M}_1\mathbf{w}_1 + \mathbf{M}_3\mathbf{w}_3 + \mathbf{J}_1\mathbf{T}^{-1}\,\mathbf{d}. \end{aligned}$$

Therefore, since $\chi^2 = \mathbf{y}'_{(n)}\mathbf{y}_{(n)}$ and $\chi^2_* = \mathbf{y}^{*\prime}_{(n)}\mathbf{y}^*_{(n)}$,

$$\chi^2 - \mathbf{w}_3'\mathbf{M}_3'\mathbf{M}_3\mathbf{w}_3 \to_p 0 \quad \text{or} \quad \chi^2 - \mathbf{w}_3'\mathbf{w}_3 \to_p 0,$$

and

$$\chi_*^2 - [\mathbf{w}_1'\mathbf{M}_1'\mathbf{M}_1\mathbf{w}_1 + \mathbf{w}_1'\mathbf{M}_1'\mathbf{J}_1\mathbf{T}^{-1}\,\mathbf{d} + \mathbf{d}'\mathbf{T}^{-1}\mathbf{J}_1'\mathbf{M}_1\mathbf{w}_1$$
$$+ \mathbf{w}_3'\mathbf{M}_3'\mathbf{M}_3\mathbf{w}_3 + \mathbf{w}_3'\mathbf{M}_3'\mathbf{J}_1\mathbf{T}^{-1}\,\mathbf{d} + \mathbf{d}'\mathbf{T}'^{-1}\mathbf{J}_1'\mathbf{M}_3\mathbf{w}_3 + \mathbf{d}'\mathbf{T}'^{-1}\mathbf{J}_1'\mathbf{J}_1\mathbf{T}^{-1}\,\mathbf{d}] \rightarrow_p 0.$$

But $\mathbf{J}_1'\mathbf{M}_3 = (\mathbf{J}_1'\mathbf{J}_1)^{\frac{1}{2}}(\mathbf{J}_1'\mathbf{J}_1)^{-\frac{1}{2}}\mathbf{J}_1'\mathbf{M}_3 = (\mathbf{J}_1'\mathbf{J}_1)^{\frac{1}{2}}\mathbf{M}_1'\mathbf{M}_3 = 0$ and $\mathbf{J}_1'\mathbf{M}_1 = \mathbf{J}_1'\mathbf{J}_1(\mathbf{J}_1'\mathbf{J}_1)^{-\frac{1}{2}} = (\mathbf{J}_1'\mathbf{J}_1)^{\frac{1}{2}}$, where the square root of a symmetric positive definite matrix is defined as in Ogawa [8].

Then $\chi_*^2 - [\mathbf{w}_1'\mathbf{w}_1 + 2\,\mathbf{d}'\mathbf{T}'^{-1}(\mathbf{J}_1'\mathbf{J}_1)^{\frac{1}{2}}\mathbf{w}_1 + \mathbf{d}'\mathbf{T}'^{-1}\mathbf{J}_1'\mathbf{J}_1\mathbf{T}^{-1}\,\mathbf{d} + \mathbf{w}_3'\mathbf{w}_3] \rightarrow_p 0$. Thus, in the limit,

$$\chi_0^2 = \chi_*^2 - \chi^2 = \mathbf{w}_1'\mathbf{w}_1 + 2\,\mathbf{d}'\mathbf{T}'^{-1} - (\mathbf{J}_1'\mathbf{J}_1)^{\frac{1}{2}}\mathbf{w}_1 + \mathbf{d}'\mathbf{T}'^{-1}\mathbf{J}_1'\mathbf{J}_1\mathbf{T}^{-1}\,\mathbf{d}$$
$$= [\mathbf{w}_1 + (\mathbf{J}_1'\mathbf{J}_1)^{\frac{1}{2}}\mathbf{T}^{-1}\,\mathbf{d}]'[\mathbf{w}_1 + (\mathbf{J}_1'\mathbf{J}_1)^{\frac{1}{2}}\mathbf{T}^{-1}\,\mathbf{d}]$$
$$= \mathbf{z}'\mathbf{z} \text{ (say) where } \mathbf{z}' = (z_1, \cdots, z_t).$$

Since $\chi^2 - \mathbf{w}_3'\mathbf{w}_3 \rightarrow_p 0$ while χ_0^2 does not involve $\mathbf{w}_3$, it is seen that χ_0^2 is asymptotically independent of χ^2 in probability whether H_0 or H_n is true.

It is seen that $\mathbf{z}$ is asymptotically normal with mean vector $(\mathbf{J}_1'\mathbf{J}_1)^{\frac{1}{2}}\mathbf{T}^{-1}\,\mathbf{d}$ and variance covariance matrix $\mathbf{I}$. But under H_0, $\mathbf{d} = \mathbf{0}$. Thus, under H_0, χ_0^2 is, in the limit, distributed as a central chi-square variate with t degrees of freedom, and under H_n as a non-central chi-square variate with t degrees of freedom and a non-centrality parameter $\Delta = \mathbf{d}'\mathbf{T}'^{-1}(\mathbf{J}_1'\mathbf{J}_1)\mathbf{T}^{-1}\,\mathbf{d}$. But

$$\mathbf{T}'^{-1}(\mathbf{J}_1'\mathbf{J}_1)\mathbf{T}^{-1} = \mathbf{T}'^{-1}(\mathbf{T}'\mathbf{B}_1' + \mathbf{Q}'\mathbf{B}_2')(\mathbf{B}_1\mathbf{T} + \mathbf{B}_2\mathbf{Q})\mathbf{T}^{-1}$$
$$= (\mathbf{B}_1' + \mathbf{T}'^{-1}\mathbf{Q}'\mathbf{B}_2')(\mathbf{B}_1 + \mathbf{B}_2\mathbf{Q}\mathbf{T}^{-1})$$
$$= \mathbf{B}_1'\mathbf{B}_1 + \mathbf{B}_1'\mathbf{B}_2\mathbf{Q}\mathbf{T}^{-1} + \mathbf{T}'^{-1}\mathbf{Q}'\mathbf{B}_2'\mathbf{B}_1 + \mathbf{T}'^{-1}\mathbf{Q}'\mathbf{B}_2'\mathbf{B}_2\mathbf{Q}\mathbf{T}^{-1}$$
$$= \mathbf{B}_1'\mathbf{B}_1 + \mathbf{B}_1'\mathbf{B}_2\mathbf{Q}\mathbf{T}^{-1} + \mathbf{T}'^{-1}\mathbf{Q}'\mathbf{B}_2'\mathbf{B}_1 - \mathbf{T}'^{-1}\mathbf{Q}'\mathbf{B}_2'\mathbf{B}_1$$
$$= \mathbf{B}_1'\mathbf{B}_1 + \mathbf{B}_1'\mathbf{B}_2\mathbf{Q}\mathbf{T}^{-1} = \mathbf{B}_1'\mathbf{B}_1 - \mathbf{B}_1'\mathbf{B}_2(\mathbf{B}_2'\mathbf{B}_2)^{-1}\mathbf{B}_2'\mathbf{B}_1.$$

Therefore, $\Delta = \mathbf{d}'\mathbf{B}_1'[\mathbf{I} - \mathbf{B}_2(\mathbf{B}_2'\mathbf{B}_2)^{-1}\mathbf{B}_2']\mathbf{B}_1\,\mathbf{d}$.

Returning to the hypothesis that functions of the parameters are equal to zero, we shall prove,

THEOREM 4.2. *Suppose that the conditions of Sections* 2 *and* 3 *hold and suppose it is given as a part of the model that* $p_{ij} = p_{ij}(\alpha_1, \cdots, \alpha_s)$.

Let $\hat{\boldsymbol{\alpha}}$ *be the unique consistent solution of equations* (2.5) *and let* χ^2 *be the corresponding test statistic obtained by inserting* $\alpha_k = \hat{\alpha}_k$ *in* (2.6).

Let $\tilde{\boldsymbol{\alpha}}$ *be the unique consistent solution of equations* (3.5) *and let* χ_*^2 *be the corresponding test statistic obtained by inserting* $\alpha_k = \tilde{\alpha}_k$ *in* (2.6).

Then $\chi_0^2 = \chi_*^2 - \chi^2$, *the test statistic for testing the hypothesis* $H_0{:}f_{m_1}(\alpha_1, \cdots, \alpha_s) = 0$ for $m_1 = 1, 2, \cdots, t < s$ *against the alternative,* $H_n{:}f_{m_1}(\alpha_1^0, \cdots, \alpha_s^0) = d_{m_1}N_n^{-\frac{1}{2}}$ *where not all* $d_{m_1} = 0$, *is, in the limit, independent of* χ^2 *in probability and distributed in a non-central* χ^2*-distribution with t degrees of freedom and a non-centrality parameter*

$$\Delta = \mathbf{d}'\mathbf{F}_1'^{-1}\mathbf{B}_1'[\mathbf{I} - \mathbf{C}(\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}']\mathbf{B}_1\mathbf{F}_1^{-1}\,\mathbf{d}$$

where **d** *and* $\mathbf{B}_1$ *are as in Theorem* 4.1, **C** *is as in Theorem* 3.1 *and* $\mathbf{F}_1(t \times t) = \{(\partial f_{m_1}/\partial \alpha_{k_1})_0\}, m_1, k_1, = 1, 2, \cdots, t$.

Note that we assume, without loss of generality, that α_{k_1} ($k_1 = 1, \cdots, t$) denote the t parameters made dependent under H_0 and α_{k_2} ($k_2 = t + 1, \cdots, s$) denote the $s - t$ parameters remaining independent.

PROOF. Denote by

$$\mathbf{F} = \underset{s \quad s-t}{[\mathbf{F}_1 : \mathbf{F}_2]}\, t$$

the matrix $(\partial f_{m_1}/\partial \alpha_k)_0$ where $\mathbf{F}_1 = (\partial f_{m_1}/\partial \alpha_{k_1})_0$ and $\mathbf{F}_2 = (\partial f_{m_1}/\partial \alpha_{k_2})_0$. Since rank $(\mathbf{F}) = t$, we may assume without loss of generality that $|\mathbf{F}_1| \neq 0$. Define $f_{m_2}(\boldsymbol{\alpha}) = \alpha_{m_2}$ for $m_2 = t + 1, \cdots, s$ so that $f_m(\boldsymbol{\alpha})$ where $m = 1, \cdots, s$ are continuously differentiable and the jacobian of the transformation from $\boldsymbol{\alpha}' = (\alpha_1, \cdots, \alpha_s)$ to $(f_1(\boldsymbol{\alpha}), \cdots, f_s(\boldsymbol{\alpha}))$ at $\boldsymbol{\alpha}_0$ is given by $|\mathbf{F}_1|^{-1}$.

By a well known theorem on the inversion of a transformation (Courant [1], p. 152) it is seen that, in a neighborhood A of $\boldsymbol{\alpha}_0$, the system of equations $\beta_m = f_m(\boldsymbol{\alpha})$ for $m = 1, \cdots, s$ has a unique inverse $\alpha_k = g_k(\beta_1, \cdots, \beta_s)$ for $k = 1, \cdots, s$, and if $\beta_m^0 = f_m(\alpha_1^0, \cdots, \alpha_s^0)$, then in a neighborhood B of $\boldsymbol{\beta}_0' = (\beta_1^0, \cdots, \beta_s^0)$ the inverse functions g_k possess continuous first and second order derivatives and the first order derivatives are given by

$$(\partial \alpha_k/\partial \beta_m) = (\partial \beta_m/\partial \alpha_k)^{-1} \quad \text{for } k, m = 1, \cdots, s.$$

Put $p_{ij}(\alpha_1, \cdots, \alpha_s) = p_{ij}(g_1(\boldsymbol{\beta}), \cdots, g_s(\boldsymbol{\beta})) = q_{ij}(\beta_1, \cdots, \beta_s)$. Then, by Theorem (2.1), the equations

$$\sum_{i=1}^{q} \sum_{j=1}^{r_i} [(v_{ijn} - N_n Q_n q_{ij})/q_{ij}] \cdot (\partial q_{ij}/\partial \beta_m) = 0 \quad \text{for} \quad m = 1, \cdots, s$$

have a unique solution $\hat{\boldsymbol{\beta}}$ such that $\hat{\boldsymbol{\beta}} \to \boldsymbol{\beta}_0$ in probability as $n \to \infty$ and Mitra [6] shows that $\hat{\boldsymbol{\alpha}} = \mathbf{g}(\hat{\boldsymbol{\beta}})$ is asymptotically a solution of the equations (2.5).

By Theorem (3.1), the equations

$$\sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{v_{ijn} - N_n Q_n q_{ij}}{q_{ij}} \cdot \frac{\partial q_{ij}}{\partial \beta_m} + \sum_{m_1=1}^{t} \lambda_{m_1} \frac{\partial \beta_{m_1}}{\partial \beta_m} = 0, \qquad m = 1, \cdots, s$$

$$\text{and } \beta_{m_1} = 0, \qquad m_1 = 1, \cdots, t$$

have a unique solution

$$\tilde{\boldsymbol{\beta}}' = \underset{t \quad s-t}{[\mathbf{0}' : \tilde{\boldsymbol{\beta}}_2']}\, 1,$$

where $\tilde{\boldsymbol{\beta}}_2' = (\tilde{\beta}_{t+1}, \cdots, \tilde{\beta}_s)$, such that $\tilde{\boldsymbol{\beta}}' \to [\mathbf{0}' : \boldsymbol{\beta}_2^{0\prime}]$ in probability as $n \to \infty$ and Mitra [6] shows that $\tilde{\boldsymbol{\alpha}} = \mathbf{g}(\tilde{\boldsymbol{\beta}})$ is asymptotically a solution of the equations (3.5).

Thus it is seen that the problem of testing the hypothesis $H_0 : f_{m_1}(\boldsymbol{\alpha}) = 0$ formally reduces to that of testing $H_0 : \beta_{m_1} = 0$ for $m_1 = 1, \cdots, t$. Where $\boldsymbol{\beta}_0' = (\beta_1^0, \cdots, \beta_s^0)$ is now the "true" parameter point with respect to the new frame of reference $\beta_1, \cdots, \beta_s$. Define

$$\mathbf{G} = [\underset{t}{\mathbf{G}_1}:\underset{s-t}{\mathbf{G}_2}]R,$$

where

$$\mathbf{G}_1(R \times t) = \{(Q_i/q^0_{ij})^{\frac{1}{2}}(\partial q_{ij}/\partial \beta_{m_1})_0\}, \quad \text{where} \quad m_1 = 1, \cdots, t;$$

and

$$\mathbf{G}_2(R \times (s-t)) = \{(Q_i/q^0_{ij})^{\frac{1}{2}}(\partial q_{ij}/\partial \beta_{m_2})_0\}, \quad \text{where} \quad m_2 = t+1, \cdots, s.$$

Then, by Theorem 4.1, we have that $\chi_0^2 = \chi_*^2 - \chi^2$ is, in the limit, independent of χ^2 in probability and distributed in a non-central χ^2-distribution with t degrees of freedom and a non-centrality parameter $\Delta = \mathbf{d}'\mathbf{G}_1'[\mathbf{I} - \mathbf{G}_2(\mathbf{G}_2'\mathbf{G}_2)^{-1}\mathbf{G}_2']\mathbf{G}_1\,\mathbf{d}$.

Let

$$\mathbf{B} = [\underset{t}{\mathbf{B}_1}:\underset{s-t}{\mathbf{B}_2}]R$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are defined as in Theorem 4.1. It is seen that

$$\mathbf{B} = \mathbf{G}(\partial\beta_m/\partial\alpha_k)_0 \quad \text{for} \quad k, m = 1, \cdots, s$$

or

$$[\mathbf{B}_1:\mathbf{B}_2] = [\mathbf{G}_1:\mathbf{G}_2]\cdot\begin{bmatrix}\left(\frac{\partial\beta_{m_1}}{\partial\alpha_{k_1}}\right)_0 & \left(\frac{\partial\beta_{m_1}}{\partial\alpha_{k_2}}\right)_0 \\ \left(\frac{\partial\beta_{m_2}}{\partial\alpha_{k_1}}\right)_0 & \left(\frac{\partial\beta_{m_2}}{\partial\alpha_{k_2}}\right)_0\end{bmatrix} = [\mathbf{G}_1:\mathbf{G}_2]\cdot\begin{bmatrix}\mathbf{F}_1 & \mathbf{F}_2 \\ \mathbf{0} & \mathbf{I}\end{bmatrix}.$$

Inverting, we get

$$[\mathbf{G}_1:\mathbf{G}_2] = [\mathbf{B}_1:\mathbf{B}_2]\cdot\begin{bmatrix}\mathbf{F}_1^{-1} & -\mathbf{F}_1^{-1}\mathbf{F}_2 \\ \mathbf{0} & \mathbf{I}\end{bmatrix}$$

or $\mathbf{G}_1 = \mathbf{B}_1\mathbf{F}_1^{-1}$ and $\mathbf{G}_2 = \mathbf{B}_2 - \mathbf{B}_1\mathbf{F}_1^{-1}\mathbf{F}_2$. However, from a theorem on jacobians due to Roy ([11], page 166),

$$\begin{aligned}\mathbf{C} = \left\{\left(\frac{Q_i}{p^0_{ij}}\right)^{\frac{1}{2}}\left(\frac{\partial p^*_{ij}}{\partial\alpha_{k_2}}\right)_0\right\} &= \left\{\left(\frac{Q_i}{p^0_{ij}}\right)^{\frac{1}{2}}\left(\frac{\partial p_{ij}}{\partial\alpha_{k_2}}\right)_0\right\} - \left\{\left(\frac{Q_i}{p^0_{ij}}\right)^{\frac{1}{2}}\left(\frac{\partial p_{ij}}{\partial\alpha_{k_1}}\right)_0\right\}\left(\frac{\partial f_{m_1}}{\partial\alpha_{k_1}}\right)_0^{-1}\left(\frac{\partial f_{m_1}}{\partial\alpha_{k_2}}\right)_0 \\ &= \mathbf{B}_2 - \mathbf{B}_1\,\mathbf{F}_1^{-1}\mathbf{F}_2 \\ &= \mathbf{G}_2.\end{aligned}$$

Therefore, the non-centrality parameter may be expressed as

$$\Delta = \mathbf{d}'\mathbf{F}_1'^{-1}\mathbf{B}_1'[\mathbf{I} - \mathbf{C}(\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}']\mathbf{B}_1\mathbf{F}_1^{-1}\,\mathbf{d}.$$

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