TWO THIRD ORDER ROTATABLE DESIGNS IN FOUR DIMENSIONS[1]

BY AGNES M. HERZBERG

University of Saskatchewan

1. Introduction. A 128 point third order rotatable design in four dimensions was presented by Gardiner, Grandage and Hader (1959). If we define $S(x_1, x_2, x_3, x_4)$ to be the set of all permutations of $(\pm x_1, \pm x_2, \pm x_3, \pm x_4)$, then this design is formed by combining the point sets $S(f, f, g, g)$, $S(c, 0, 0, 0)$, $S(d, 0, 0, 0)$ and $S(a, a, a, a)$.

Subsequently Draper (1960b) gave a design using only 96 points. This design is formed from two main pieces. Each piece consists of the three point sets $S(p, p, 0, 0)$, $S(c, 0, 0, 0)$ and $S(a, a, a, a)$, where $p = 2^{\frac{1}{4}}a$, $c = 2a$. A different value for a is given to each portion of the design.

Among the designs listed by Das and Narasimham (1962) is one which contains 72 points. This design consists of the four point sets $S(a, a, a, 0)$, $S(b, 0, 0, 0)$, $S(c, 0, 0, 0)$ and $S(d, d, 0, 0)$, where $a^2 = 0.793701d^2$, $b^2 = 2.577472d^2$ and $c^2 = 0.957168d^2$.

Here we present two other 72 point third order rotatable designs in four dimensions.

2. The first design. Consider the following point sets in four dimensions: $S(p, p, 0, 0)$, $S(e, e, 0, 0)$, $S(d, 0, 0, 0)$ and $S(a, a, a, a)$. The excess functions for each of these point sets will be similar to those evaluated by Draper (1960b). These functions appear in Table 1.

These point sets together will form a third order rotatable arrangement if

$$Ex(S) = Fx(S) = Gx(S) = Hx(S) = Ix(S) = 0,$$

where $S = S(p, p, 0, 0) + S(e, e, 0, 0) + S(d, 0, 0, 0) + S(a, a, a, a)$ (Draper (1960a)). In order that these excess functions equal zero, $2d^4 = 32a^4$, $12p^6 + 12e^6 + 2d^6 = 224a^6$ and $4p^6 + 4e^6 = 32a^6$. From the first of these equations $d = 2a$, whereupon the second equation reduces to the third. There are thus an infinite number of solutions. If we select the particular solution for which $e = a$, we find that $p = 7^{\frac{1}{6}}a$. Since at least two of the point sets have different radii, S forms a third order rotatable design of 72 points, by the theorem in Draper (1960a). If p (or e) $= 0$, we will obtain one of the pieces of Draper's design with centre points.

If we let $S_1 = S(7^{\frac{1}{6}}a, 7^{\frac{1}{6}}a, 0, 0)$, $S_2 = S(a, a, 0, 0)$, $S_3 = S(2a, 0, 0, 0) + S(a, a, a, a)$, then S_1, S_2 and S_3 will each form second order rotatable arrangements of 24 points in four dimensions. Points must be added at the centre

Received 28 August 1963.

[1] This work was done while the author held a National Research Council of Canada Bursary and Studentship at the University of Saskatchewan.

TABLE 1

Excess functions for certain point sets

	$S(p, p, 0, 0)$	$S(d, 0, 0, 0)$	$S(a, a, a, a)$
Number of Points	24	8	16
Ax	$12p^2$	$2d^2$	$16a^2$
Ex	0	$2d^4$	$-32a^4$
Fx, Gx	0	0	0
Hx	$12p^6$	$2d^6$	$-224a^6$
Ix	$4p^6$	0	$-32a^6$

of each of these arrangements in order that they will form second order rotatable designs.

The complete design has parameters

$$\lambda_2 N = 12(7^{\frac{1}{3}} + 3)a^2, \quad \lambda_4 N = 4(7^{\frac{2}{3}} + 5)a^4, \quad \lambda_6 N = 16a^6.$$

3. The second design. Consider the following point sets in four dimensions: $S(p, p, 0, 0)$, $S(d, 0, 0, 0)$, $S(a, a, a, a)$, $S(b, b, b, b)$ and $S(c, 0, 0, 0)$. The excess functions for each of these sets can be obtained from those in Table 1. These sets together will form a third order rotatable arrangement if $2d^4 + 2c^4 = 32a^4 + 32b^4$, $12p^6 + 2d^6 + 2c^6 = 224a^6 + 224b^6$ and $4p^6 = 32a^6 + 32b^6$. In order to obtain a solution to these equations, we let $b = 2^{\frac{1}{2}}a$. After eliminating p and d, we find that a solution of the resulting equation is $c = 2(2)^{\frac{1}{2}}a$. Therefore, $d = 2a$ and $p = 2^{\frac{1}{2}}9^{\frac{1}{6}}a$. Since at least two of the point sets have different radii, S forms a third order rotatable design of 72 points.

If we let $S_1 = (2^{\frac{1}{2}}9^{\frac{1}{6}}a, 2^{\frac{1}{2}}9^{\frac{1}{6}}a, 0, 0)$, $S_2 = S(2a, 0, 0, 0) + S(a, a, a, a) + S(2^{\frac{1}{2}}a, 2^{\frac{1}{2}}a, 2^{\frac{1}{2}}a, 2^{\frac{1}{2}}a) + S(2(2)^{\frac{1}{2}}a, 0, 0, 0)$, $S_3 = S(2a, 0, 0, 0) + S(a, a, a, a)$ and $S_4 = S(2(2)^{\frac{1}{2}}a, 0, 0, 0) + S(2^{\frac{1}{2}}a, 2^{\frac{1}{2}}a, 2^{\frac{1}{2}}a, 2^{\frac{1}{2}}a)$, then S_1, S_3 and S_4 will form second order rotatable arrangements of 24 points in four dimensions. Points must be added at the centre of each of these arrangements in order that they will form second order rotatable designs. S_2 forms a second order rotatable design with 48 points.

The complete design has parameters

$$\lambda_2 N = 24(9^{\frac{1}{3}} + 3)a^2, \quad \lambda_4 N = 16(9^{\frac{2}{3}} + 5)a^4, \quad \lambda_6 N = 144a^6.$$

4. Acknowledgment. The author is grateful to Professor N. Shklov for his help and guidance and to the referee for his comments.

REFERENCES

Das, M. N. and Narasimham, V. L. (1962). Construction of rotatable designs through balanced incomplete block designs. *Ann. Math. Statist.* **33** 1421–1439.

Draper, Norman R. (1960a). Third order rotatable designs in three dimensions. *Ann. Math. Statist.* **31** 865–874.

Draper, Norman R. (1960b). A third order rotatable design in four dimensions. *Ann. Math. Statist.* **31** 875–877.

Gardiner, D. A., Grandage, A. H. E. and Hader, R. J. (1959). Third order rotatable designs for exploring response surfaces. *Ann. Math. Statist.* **30** 1082–1096.