THE ENUMERATION OF ELECTION RETURNS BY NUMBER OF LEAD POSITIONS

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0. Summary. In an election return with two candidates A and B, if α_r is the number of votes for A in the first r counted, β_r the similar number for B, then r is a c-lead position for A if $\alpha_r > \beta_r + c - 1$. With final vote (n, m) (n for A, m for B), what is the number $l_j(n, m; c)$ of returns with j c-lead positions? Or, what is the enumerator $l_{nm}(x; c) = \sum l_j(n, m; c)x^j$ of election returns by number of lead positions?

For $c = 0, \pm 1, \pm 2, \cdots$ it is shown that all enumerators may be expressed in terms of $l_{nm}(x; 0)$ and $l_{nm}(x; 1)$, which are given explicit expression.

1. Introduction. In an election return with candidates A and B, if α_r is the number of votes for A among the first r votes, β_r the similar number for B, then r is a c-lead position for A if $\alpha_r > \beta_r + c - 1$. For given c, every return with final vote n for A, m for B, has a count of c-lead positions. What is the number $l_k(n, m; c)$ of returns with k c-leads? Or, what is the enumerator

$$l_{nm}(x; c) = \sum l_k(n, m; c)x^k \tag{1}$$

of returns by number of lead positions? Partial answers appear in Lajos Takács [3].

The first election return problem, posed by J. Bertrand [1] in 1887, asked only for the number $l_{n+m}(n, m; 0)$; Bertrand's answer was

$$a_{nm} = \binom{n+m}{m} - \binom{n+m}{m-1} = \frac{n+1-m}{n+1}\binom{n+m}{m}, \qquad n \geqq m. \tag{2}$$

Note that $a_{nm} = a_{n,m-1} + a_{n-1,m}$.

Here it is shown that

$$\begin{aligned} l_{nm}(x; 0) &= a_{m-1,n} + \sum_{j=0}^{n-1} b_j(n, m)x^{2j}(x + x^2), & n < m, \\ &= \sum_{j=0}^{m-2} b_{m-2-j}(m-1, n+1)x^{n-m+2j}(x + x^2) \\ &\quad + a_{n,m-1}x^{n+m-1} + a_{nm}x^{n+m}, & n \geqq m, \end{aligned} \tag{3}$$

with $b_j(n, m) = \sum_{k=0}^{j} a_{jk}a_{m-1-k,n-1-j}$ and

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$$
(4)\qquad
\begin{aligned}
l_{nm}(x;1) &= a_{mn} + a_{m,n-1}x + \sum_{j=0}^{n-2} b_j(m+1, n-1)x^{2j+2}(1+x), && n \leqq m,\\
&= \sum_{j=0}^{m-1} b_{m-1-j}(m,n)x^{n-m+2j}(1+x) + a_{n-1,m}x^{n+m}, && n > m.
\end{aligned}
$$

These two enumerators serve as a basis for all others, as will be shown.

2. Recurrence relations. For given c, recurrence relations for the enumerators are found by considering the effect of the last vote cast. The final vote is a lead position only if $n > m + c - 1$; hence

$$
(5)\qquad
\begin{aligned}
l_{nm}(x;c) &= l_{n,m-1}(x;c) + l_{n-1,m}(x,c), && n < m + c - 1,\\
&= xl_{n,m-1}(x;c) + xl_{n-1,m}(x,c), && n > m + c - 1.
\end{aligned}
$$

Boundary conditions are:

$$
(6)\qquad
\begin{aligned}
l_{0m}(x;c) &= 1, && m = 0, 1, \cdots,\\
l_{n0}(x;c) &= 1, && n = 0, 1, \cdots, c-1,\\
&= x^{n+1-c}, && n = c, c+1, \cdots.
\end{aligned}
$$

Recurrences (5) and the boundary conditions (6) determine the enumerators completely.

Recurrences for changes in c are found by considering the effect of the first vote cast. If this vote is for A, the first position is a lead position only if $2 > c$, and the remaining votes are enumerated by $l_{n-1,m}(x; c-1)$. If the first vote is for B the first position is a lead position only if $0 < c$, and the remaining votes are enumerated by $l_{n,m-1}(x; c+1)$. Hence

$$
(7)\qquad
\begin{aligned}
l_{nm}(x;c) &= l_{n-1,m}(x;c-1) + l_{n,m-1}(x, c+1), && c = 2, 3, \cdots\\
l_{nm}(x;1) &= xl_{n-1,m}(x;0) + l_{n,m-1}(x;2)\\
l_{nm}(x;0) &= xl_{n-1,m}(x;-1) + l_{n,m-1}(x;1)\\
l_{nm}(x;-c) &= xl_{n-1,m}(x;-c-1) + xl_{n,m-1}(x;-c+1), && c = 1, 2, \cdots.
\end{aligned}
$$

It is clear that all enumerators may be expressed in terms of any consecutive pair. Thus

$$
\begin{aligned}
l_{nm}(x;2) &= l_{n,m+1}(x;1) - xl_{n-1,m+1}(x;0)\\
l_{nm}(x;3) &= l_{n,m+1}(x;2) - l_{n-1,m+1}(x;1)\\
&= l_{n,m+2}(x;1) - l_{n-1,m+1}(x;1) - xl_{n-1,m+2}(x;0)
\end{aligned}
$$

and in general

$$
(8)\qquad l_{nm}(x; 2+c) = L_c(-f)l_{n,m+c+1}(x;1) - xL_{c-1}(-f)l_{n-1,m+c+1}(x;0)
$$

with $fl_{n,m}(x;c) = l_{n-1,m+1}(x;c)$ and

$$L_n(x) = \sum_{k=0}^{N} \binom{n-k+1}{k} x^k, \qquad N = [(n+1)/2].$$

$L_n(x)$ is the "largest rook polynomial" (cf., [2], p. 183).

For negative c, first

$$\begin{aligned} x l_{nm}(x; -1) &= l_{n+1,m}(x; 0) - l_{n+1,m-1}(x; 1) \\ x^2 l_{nm}(x; -2) &= x l_{n+1,m}(x; -1) - x^2 l_{n+1,m-1}(x; 0) \\ &= l_{n+2,m}(x; 0) - x^2 l_{n+1,m-1}(x; 0) - l_{n+2,m-1}(x; 1). \end{aligned}$$

Further iterations lead to

$$x^{c+1} l_{nm}(x; -c-1) = L_c(-x^2F) l_{n+c,m}(x; 0) - L_{c-1}(-x^2F) l_{n+c,m-1}(x; 1) \tag{9}$$

with $F l_{nm}(x; c) = l_{n-1,m-1}(x; c)$.

3. Weak leads. The case $c = 0$ ($\alpha_r \geqq \beta_r$) is often referred to as that of weak leads, a convenient designation for this section. The enumerators are completely determined by recurrences (5) and boundary conditions (6) for $c = 0$, but their structure is found most readily by beginning with the tie returns: $n = m$. These may be classified according to first occurrences of ties. If the first tie is (k, k), $k = 1, 2, \cdots, n$, the remaining votes are enumerated by $l_{n-k,n-k}(x; 0)$; if the first vote is for A, the votes up to the tie are enumerated by x^{2k} (every position is a lead position), while if it is for B, the enumerator is x. The total number of returns with first tie at (k, k), and first vote for A is $a_{k-1,k-1} = c_{k-1}$, with c_{k-1} a Catalan number (this is a well known result which follows from the fact that the last vote must be for B, and from Equation (2)). By symmetry there are an equal number of returns with first vote for B. Hence, if $l_{00}(x; 0) = 1$

$$l_{nn}(x; 0) = \sum_{k=1}^{n} c_{k-1}(x + x^{2k}) l_{n-k,n-k}(x; 0), \qquad n = 1, 2, \cdots. \tag{10}$$

In particular, omitting arguments

$$\begin{aligned} l_{11} &= c_0(x + x^2) l_{00} = x + x^2 \\ l_{22} &= c_0(x + x^2) l_{11} + c_1(x + x^4) l_{00} = x + x^2 + 2x^3 + 2x^4 \\ l_{33} &= c_0(x + x^2) l_{22} + c_1(x + x^4) l_{11} + c_2(x + x^6) l_{00} = 2x \\ &\quad + 2x^2 + 3x^3 + 3x^4 + 5x^5 + 5x^6. \end{aligned}$$

These suggest writing

$$l_{nn}(x; 0) = (x + x^2) g_n(x^2), \qquad n = 1, 2, \cdots \tag{11}$$

with $g_n(x)$ a polynomial in x of degree $n - 1$: $g_1(x) = 1$, $g_2(x) = 1 + 2x$, $g_3(x) = 2 + 3x + 5x^2$.

Substituting (11) in (10) shows that

$$\begin{aligned} g_n(x^2) &= \sum_{k=1}^{n-1} c_{k-1}(x + x^{2k}) g_{n-k}(x^2) \\ &\quad + c_{n-1}(x + x^{2n})(x + x^2)^{-1}, \quad n = 2, 3, \cdots. \end{aligned} \tag{12}$$

In particular, $g_1(x) = 1$, and

$$g_2(x^2) = (x + x^2)g_1(x^2) + 1 - x + x^2 = x^2 g_1(x^2) + 1 + x^2 + x[g_1(x^2) - 1]$$

or

$$g_2(x) = xg_1(x) + 1 + x = 1 + 2x$$

and

$$g_2(x) + g_1(x) = 2(1 + x) = c_2(1 + x).$$

Next

$$g_3(x^2) = (x + x^2)g_2(x^2) + (x + x^4)g_1(x^2) + c_2(1 - x + x^2 - x^3 + x^4)$$

or, separating even and odd powers of x,

$$g_3(x^2) = x^2 g_2(x^2) + x^4 g_1(x^2) + c_2(1 + x^2 + x^4) + x[g_2(x^2) + g_1(x^2) - c_2(1 + x^2)]$$

so that

$$g_3(x) = xg_2(x) + x^2 g_1(x) + c_2(1 + x + x^2)$$

and

$$g_3(x) + g_2(x) + c_2 g_1(x) = c_3(1 + x + x^2).$$

It is clear that the equation

$$c_0 g_n(x) + c_1 g_{n-1}(x) + \cdots + c_{n-1} g_1(x) = c_n(1 + \cdots + x^{n-1}) \tag{13}$$

implies, separating even and odd powers of x in Equation (12),

$$\begin{aligned} g_{n+1}(x) = {} & c_0 x g_n(x) + c_1 x^2 g_{n-1}(x) \\ & + \cdots + c_{n-1} x^n g_1(x) + c_n(1 + \cdots + x^n). \end{aligned} \tag{14}$$

Hence both equations are true if the fact that (14) holds up to $n + 1$ and (13) up to n implies that (13) holds up to $n + 1$. This implication is perhaps clearest in the next instance of the development above: $n = 4$; thus, omitting arguments

$$g_4 = xg_3 + x^2 g_2 + c_2 x^3 g_1 + c_3(1 + x + x^2 + x^3)$$

$$g_3 = xg_2 + x^2 g_1 + c_2(1 + x + x^2)$$

$$c_2 g_2 = c_2 x g_1 + c_2(1 + x), \qquad c_3 g_1 = c_3 .$$

Then, using (13) for $n = 3, 2, 1$, and summing columns as arranged above

$$\begin{aligned} & g_4 + g_3 + c_2 g_2 + c_3 g_1 \\ & \quad = (c_3 + c_2 + c_2 + c_3)(1 + x + x^2 + x^3) = c_4(1 + x + x^2 + x^3) \end{aligned}$$

since $c_n = \sum_{j=0}^{n-1} c_j c_{n-1-j}$. The argument is general and (13) holds in general.

Now introduce the generating functions

$$(15)\qquad g(x, y) = \sum_{n=1}^{\infty} x^n g_n(y)$$

$$(16)\qquad c(x) = \sum_{n=0}^{\infty} c_n x^n = [1 - (1 - 4x)^{\frac{1}{2}}]/2x = [1 - xc(x)]^{-1}.$$

Then by (13)

$$(17)\qquad (1 - y)c(x)g(x, y) = c(x) - c(xy)$$

while by (14)

$$(18)\qquad (1 - y)[1 - xyc(xy)]g(x, y) = xc(x) - xyc(xy).$$

From either of (17) or (18), it follows, using $c(x)(1 - xc(x)) = 1$, that

$$(19)\qquad (1 - y)g(x, y) = 1 - c(xy) + xc(x)c(xy).$$

Equating coefficients of x^n in (19) gives

$$(20)\qquad \begin{aligned}&(1 - y)g_n(y)\\ &\quad = -c_n y^n + c_0 c_{n-1} + c_1 c_{n-2} y + \cdots + c_k c_{n-1-k} k + \cdots + c_{n-1} c_0 y^{n-1}.\end{aligned}$$

Then if $g_n(y) = \sum_{k=0}^{n-1} g_{nk} y^k$ it follows from (20) that

$$(21)\qquad \begin{aligned} g_{n,n-1} &= c_n \\ g_{n,k} - g_{n,k-1} &= c_k c_{n-1-k}, \qquad k = 0, 1, \cdots, n - 2.\end{aligned}$$

Iterations of the second of Equations (21) lead to

$$(22)\qquad g_{nk} = \sum_{j=0}^{k} c_j c_{n-1-j}, \qquad k = 0, 1, \cdots, n - 1.$$

Note that (22) holds for $k = n - 1$ by the relation (used above): $c_n = \sum_{j=0}^{n-1} c_j c_{n-1-j}$. Note also that

$$(23)\qquad g_{nk} + g_{n,n-2-k} = \sum_{j=0}^{k} c_j c_{n-1-j} + \sum_{j=k+1}^{n-1} c_{n-1-j} c \;= c_n$$

or

$$(23a)\qquad xg_n(x) + x^{n-1} g_n(x^{-1}) = c_n(1 + x + \cdots + x^n).$$

Another form for g_{nk} has been found by my friend and colleague, S. O. Rice; it is

$$(24)\qquad 2g_{nk} = c_n + [(2k + 2 - n)(n - k)(k + 2)/n(n + 1)]c_{k+1}c_{n-1-k}.$$

It may be proved by (22), but proof is omitted.

Turn now to the enumerator $l_{n-1,n}(x; 0)$; from

$$l_{nn}(x; 0) = xl_{n-1,n}(x; 0) + xl_{n+1,n}(x; 0)$$

and

$$l_{nm}(x; 0) = l_{n-1,m}(x, 0) + l_{n,m-1}(x; 0), \qquad n < m,$$

it is clear that Equation (10) implies

$$(25) \qquad l_{n-1,n}(x; 0) = \sum_{k=1}^{n} c_{k-1} l_{n-k,n-k}(x; 0), \qquad n = 1, 2, \cdots$$

$$(26) \qquad x l_{n+1,n}(x; 0) = \sum_{k=1}^{n} c_{k-1} x^{2k} l_{n-k,n-k}(x; 0), \qquad n = 1, 2, \cdots$$

with $l_{00}(x; 0) = 1$. Then, by (25), (11) and (13)

$$(27) \qquad \begin{aligned} &l_{n-1,n}(x; 0) \\ &= \sum_{k=1}^{n-1} c_{k-1} g_{n-k}(x^2)(x + x^2) + c_{n-1} = c_{n-1}(1 + x + \cdots + x^{2n-2}) \end{aligned}$$

while by (26) and (14)

$$(28) \qquad \begin{aligned} x l_{n,n-1}(x, 0) &= \sum_{k=1}^{n-1} c_{k-1} x^{2k} g_{n-k}(x^2)(x + x^2) + c_{n-1} x^{2n} \\ &= (x + x^2) g_n(x^2) - c_{n-1}(x + \cdots + x^{2n-1}) \\ &= l_{nn}(x; 0) - x l_{n-1,n}(x; 0) \end{aligned}$$

with the last line a verification.

With these results, the way is open to the determination of the enumerators for all n and m. Note first that the recurrence $a_{nm} = a_{n,m-1} + a_{n-1,m}$ and Equation (2) imply

$$\begin{aligned} c_n &= a_{n,n} = a_{n,n-1} \\ &= a_{n,n-2} + a_{n-1,n-1} = a_{n,n-2} + a_{n-1,n-2}\,, \\ &= a_{n,n-3} + 2a_{n-1,n-2} = a_{n,n-3} + 2a_{n-1,n-3} + 2a_{n-2,n-3}\,, \end{aligned}$$

leading to the conjecture, easily proved by induction,

$$(29) \qquad c_n = \sum_{j=0}^{k} a_{kj} a_{n-j,n-1-k}\,, \qquad k = 1, 2, \cdots .$$

It is interesting to notice that this is an inverse relation to

$$a_{n,n-k} = \sum_{j=0}^{K} (-1)^j \binom{k-j}{j} c_{n-j}\,, \quad K = [k/2], \quad k = 1,2, \cdots .$$

For $n < m$, first notice that a_{nm} and $l_{nm}(x; 0)$ have recurrences of the same form namely

$$\begin{aligned} a_{nm} &= a_{n-1,m} + a_{n,m-1} \\ l_{nm}(x; 0) &= l_{n-1,m}(x; 0) + l_{n,m-1}(x; 0) \end{aligned}$$

the latter by (5). Hence it may be expected that $l_{nm}(x; 0)$ may be expressed in terms of the numbers a_{nm}, the form being fixed by the boundary conditions, supplied by $l_{m-1,m}(x; 0)$ as given by (27), modified appropriately by (28). The modification needed develops naturally by examination of the first few cases.

Thus as already noticed in (6), $l_{0m}(x; 0) = 1$, and it is easy to see that, omitting arguments

$$l_{1m} = m - 1 + x + x^2 = a_{m-1,1} + a_{m-1,0}(x + x^2), \qquad m > 1.$$

Then, using (27) and (29),

$$\begin{aligned} l_{23} &= c_2(1 + x + x^2 + x^3 + x^4) \\ &= a_{22} + a_{21}(x + x^2) + (a_{20} + a_{10})(x^3 + x^4) \\ l_{24} &= l_{23} + l_{14} \\ &= a_{22} + a_{31} + (a_{21} + a_{30})(x + x^2) + (a_{20} + a_{10})(x^3 + x^4) \\ &= a_{32} + a_{31}(x + x^2) + (a_{30} + a_{20})(x^3 + x^4) \end{aligned}$$

and if

$$\begin{aligned} l_{2,m-1} &= a_{m-2,2} + a_{m-2,1}(x + x^2) + (a_{m-2,0} + a_{m-3,0})(x^3 + x^4) \\ l_{2,m} &= l_{2,m-1} + l_{1m} \\ &= a_{m-1,2} + a_{m-1,1}(x + x^2) + (a_{m-1,0} + a_{m-2,0})(x^3 + x^4) \\ &= a_{m-1,2} + b_0(2, m)(x + x^2) + b_1(2, m)(x^3 + x^4) \end{aligned}$$

where $b_j(n, m)$ is the expression defined following Equation (3).

The general result, given by Equation (3), now follows by induction, writing

$$\begin{aligned} l_{m-1,m} &= a_{m-1,m-1} + (x + x^2) \sum_{j=0}^{m-2} x^{2j} \sum_{k=0}^{j} a_{jk}a_{m-1-k,m-2-j} \\ &= a_{m-1,m-1} + (x + x^2) \sum_{j=0}^{m-2} x^{2j} b_j(m - 1, m). \end{aligned}$$

Now, since

$$\begin{aligned} l_{nn} = l_{n,n+1} - l_{n-1,n+1} = a_{nn} - a_{n,n-1} + \sum_{j=0}^{n-2} [b_j(n, n + 1) \\ - b_j(n - 1, n + 1)]x^{2j}(x + x^2) + b_{n-1}(n, n + 1)x^{2n-2}(x + x^2) \end{aligned}$$

and

$$\begin{aligned} b_j(n, n + 1) - b_j(n - 1, n + 1) &= b_j(n, n), \qquad j = 0, 1, \cdots, n - 2 \\ b_{n-1}(n, n + 1) &= c_n \end{aligned}$$

a new expression for l_{nn} appears, namely

$$l_{nn}(x, 0) = \sum_{j=0}^{n-2} b_j(n, n)x^{2j}(x + x^2) + c_n x^{2n-2}(x + x^2). \tag{11a}$$

This implies $g_{nj} = b_j(n, n)$, $j = 0, 1, \cdots, n-2$, $g_{n,n-1} = c_n$ and using the first of these and (23)

$$\begin{aligned} b_j(n, n) &= c_n - b_{n-2-j}(n, n) \\ &= \sum_{k=0} a_{n-2-j,k}(a_{n-k,j+1} - a_{n-1-k,j+1}) \qquad (30) \\ &= \sum a_{n-2-j,k}a_{n-k,j} = b_{n-2-j}(n-1, n+1). \end{aligned}$$

Thus a third expression for l_{nn} is

$$(11b) \quad l_{nn}(x; 0) = \sum_{j=0}^{n-2} b_{n-2-j}(n-1, n+1)x^{2j}(x + x^2) + c_n x^{2n-2}(x + x^2).$$

For concreteness, the first two interesting values of these expressions are, omitting arguments,

$$\begin{aligned} l_{33} &= (x + x^2)[a_{22} + (a_{21} + a_{11})x^2 + c_3x^4] \\ &= (x + x^2)[a_{30} + a_{20} + a_{31}x^2 + c_3x^4] \\ l_{44} &= (x + x^2)[a_{33} + (a_{32} + a_{22})x^2 + (a_{31} + 2a_{21} + 2a_{11})x^4 + c_4x^6] \\ &= (x + x^2)[(a_{40} + 2a_{30} + 2a_{10}) + (a_{41} + a_{31})x^2 + a_{42}x^4 + c_4x^6]. \end{aligned}$$

Equation (11b) is particularly apt in determining $l_{nm}(x; 0)$ for $n \geqq m$. As already noted in Equation (6), $l_{n0}(x; 0) = x^n$ and it is easy to see that $l_{n1} = x^n + nx^{n+1} = x^n + a_{n1}x^{n+1}$. Then, using (11b) and $c_2 = a_{22} = a_{21}$

$$\begin{aligned} l_{22} &= a_{20}(x + x^2) + a_{21}x^3 + a_{22}x^4 \\ l_{32} &= xl_{22} + xl_{31} \\ &= a_{20}(x^2 + x^3) + (a_{21} + 1)x^4 + (a_{22} + a_{31})x^5 \\ &= a_{30}(x^2 + x^3) + a_{31}x^4 + a_{32}x^5 \end{aligned}$$

and by induction

$$l_{n2} = a_{n0}(x^{n-1} + x^n) + a_{n1}x^{n+1} + a_{n2}x^{n+2}.$$

In the same way

$$l_{n3} = (a_{n0} + a_{n-1,0})(x^{n-2} + x^{n-1}) + a_{n1}(x^n + x^{n+1}) + a_{n2}x^{n+3} + a_{n3}x^{n+4}.$$

Note that $a_{n0} + a_{n-1,0} = b_1(2, n+1)$, $a_{n1} = b_0(2, n+1)$. The general result given in the second of Equations (3) is obtained by induction, using (11b) in the form

$$l_{nn}(x; 0) = \sum_{j=0}^{n-2} b_{n-2-j}(n-1, n+1)x^{2j}(: + x^2) + a_{n,n-1}x^{2n-1} + a_{nn}x^{2n}.$$

Another form for $l_{nm}(x; 0)$, $n \geqq m$ has been found independently by Ora Engelberg (private communication) and in present terms is as follows

$$l_{nm}(x;0) = \sum_{j=0}^{m-2} c_j(n,m)x^{n-m+2j}(x+x^2) + c_{m-1}(n,m)x^{n+m-1} + a_{nm}x^{n+m}$$

with $c_j(n,m) = \sum_{k=0}^{j} c_{m-1-k}a_{n-m+k,k}$. The identities appearing on comparison of the two forms are

$$c_j(n,m) = b_{m-2-j}(m-1,n+1), \qquad j = 0,1,\cdots,m-2$$

$$c_{m-1}(n,m) = a_{n,m-1}.$$

They are readily proved, but proof is omitted.

4. Strict leads. This is the case $c = 1$. There are many similarities to weak leads, which permit greater brevity in the development. First, by an argument similar to that for Equation (10),

$$l_{nn}(x;1) = \sum_{k=1}^{n} c_{k-1}(1+x^{2k-1})l_{n-k,n-k}(x;1) \tag{31}$$

with $l_{00}(x;1) = 1$. The first few values are

$$l_{11}(x;1) = 1+x$$

$$l_{22}(x;1) = (1+x)(2+x^2)$$

$$l_{33}(x;1) = (1+x)(5+3x^2+2x^4).$$

The numbers in these are familiar; for any n, they are the g_{nk} of the preceding section written in reverse order. If

$$l_{nn}(x;1) = (1+x)g_n(x^2;1) \tag{32}$$

then

$$g_n(x;1) = x^{n-1}g_n(x^{-1}). \tag{33}$$

Indeed substitution of (32) in (31) gives

$$g_n(x^2;1) = \sum c_{k-1}(1+x^{2k-1})g_{n-k}(x^2;1) + c_{n-1}(1+x^{2n-1})(1+x)^{-1}$$

and use of (33) in this gives (12).

Next (31), like (10), may be partitioned into

$$\begin{aligned} l_{n-1,n}(x;1) &= \sum_{k=1}^{n} c_{k-1}l_{n-k,n-k}(x;1) \\ l_{n,n-1}(x;1) &= \sum_{k=1}^{n} c_{k-1}x^{2k-1}l_{n-k,n-k}(x;1) \end{aligned} \tag{34}$$

but now $l_{n,n-1}(x;1)$ has a common numerical factor:

$$l_{n,n-1}(x;1) = c_{n-1}(x+x^2+\cdots+x^{2n-1}). \tag{35}$$

Proof is by the first of the following two recurrences (derived by using (33) in (13) and (14)):

$$\sum_{k=0}^{n-1} c_k x^k g_{n-k}(x, 1) = c_n(1 + \cdots + x^{n-1})$$

$$g_{n+1}(x; 1) = \sum_{k=0}^{n-1} c_k g_{n-k}(x; 1) + c_n(1 + \cdots + x^n).$$

For $n \leqq m$, first $l_{0m}(x; 1) = 1$ and $l_{1m}(x; 1) = m + x = a_{m1} + x$. Then

$$l_{22}(x; 1) = a_{22} + a_{21}x + a_{20}(x^2 + x^3)$$

$$l_{23}(x; 1) = l_{22}(x; 1) + l_{13}(x; 1)$$

$$= a_{32} + a_{31}x + a_{30}(x^2 + x^3)$$

and if

$$l_{2m}(x; 1) = a_{m2} + a_{m1}x + a_{m0}(x^2 + x^3)$$

$$l_{2,m+1}(x; 1) = l_{2m}(x; 1) + l_{1,m+1}(x; 1)$$

$$= a_{m+1,2} + a_{m+1,1}x + a_{m+1,0}(x^2 + x^3).$$

In the same way

$$l_{3m}(x; 1) = a_{m3} + a_{m2}x + a_{m1}(x^2 + x^3) + (a_{m0} + a_{m-1,0})(x^4 + x^5)$$

$$l_{4m}(x; 1) = a_{m4} + a_{m3}x + a_{m2}(x^2 + x^3) + (a_{m1} + a_{m-1,1})(x^4 + x^5)$$
$$+ (a_{m0} + 2a_{m-1,0} + 2a_{m-2,0})(x^6 + x^7)$$

$$= a_{m4} + a_{m3} + \sum_{j=0}^{2} b_j(m + 1, 3)x^{2j}(x^2 + x^3).$$

Mathematical induction and

$$l_{mm}(x; 1) = a_{mm} + a_{m,m-1}x + \sum_{j=0}^{m-2} b_j(m + 1, m - 1)x^{2j}(x^2 + x^3)$$

proves the first half of (4).

For $n > m$, first $l_{n0}(x; 1) = x^n$ and

$$l_{n1}(x; 1) = x^{n-1} + x^n + (n - 1)x^{n+1}$$

$$= x^{n-1} + a_{n-1,0}x^n + a_{n-1,1}x^{n+1}, \qquad n > 1.$$

Next

$$l_{32}(x; 1) = (a_{20} + a_{10})(x + x^2) + a_{21}(x^3 + x^4) + a_{22}x^5$$

$$l_{42}(x; 1) = xl_{32}(x; 1) + xl_{41}(x; 1)$$

$$= (a_{30} + a_{20})x^2(1 + x) + a_{31}x^4(1 + x) + a_{32}x^6$$

and if

$$l_{n2}(x;\,1) = (a_{n-1,0} + a_{n-2,0})x^{n-2}(1 + x) + a_{n-1,1}x^{n}(1 + x) + a_{n-1,2}x^{n+2}$$

then

$$\begin{aligned} l_{n+1,2}(x;\,2) &= (a_{n0} + a_{n-1,0})x^{n-1}(1 + x) + a_{n1}x^{n+1}(1 + x) + a_{n2}x^{n+3} \\ &= b_1(2,\,n + 1)x^{n-1}(1 + x) + b_0(2,\,n + 1)x^{n+1}(1 + x) + a_{n2}x^{n+3}. \end{aligned}$$

Writing

$$l_{n+1,n}(x;\,1) = \sum_{j=0}^{n-1} b_{n-1-j}(n,\,n + 1)x^{1+2j}(1 + x) + a_{nn}x^{2n+1}$$

the second half of (4) follows by mathematical induction.

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