TABLES OF THE LOGARITHMIC SERIES DISTRIBUTION

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1. Summary. The paper tabulates individual and cumulative terms of the logarithmic series probability distribution using the mean of the distribution as entry argument. The range of the table is $\mu = 1.1\ (0.1)\ 2.0\ (0.5)\ 5.0\ (1.0)\ 10.0$. The tables are prefaced by introductory material, outlining the history and applications of the distribution, deducing generating functions and moments, and giving a method of obtaining the terms of logarithmic series distributions outside the range of those tabled.

2. History and applications. The logarithmic series distribution was first brought to the notice of statisticians by Fisher [3], in connection with work by Corbet on the distribution of butterflies in the Malayan Peninsula, and data by Williams on the number of moths of different species caught in a light-trap in a given period.

In connection with the above data, Fisher assumed that for a given species the number of individuals caught in unit time followed a Poisson distribution with parameter λ. Because of the unequal abundance of different species, however, different varieties have unequal probabilities of being caught, this variation in risk being represented by different values of λ in the Poisson distribution $e^{-\lambda}\lambda^n/n!$ (cf. Greenwood and Yule [5]).

If λ is distributed with a gamma-type probability density function

$$(c^k/\Gamma(k))\, e^{-c\lambda}\, \lambda^{k-1}\, d\lambda,$$

the relative frequency with which exactly n individuals of a species are captured is given by the coefficient of t^n in

$$(c^k/\Gamma(k)) \int_0^\infty e^{-c\lambda}\lambda^{k-1}e^{-\lambda+\lambda t}\, d\lambda,$$

which on the substitution of $\lambda(c + 1 - t) = u$ becomes $c^k/(c + 1 - t)^k$. The probability of collecting $0, 1, 2, \cdots$ representatives of a species is therefore

$$\left(\frac{c}{c+1}\right)^k \left\{1, \frac{k}{c+1}, \frac{k(k+1)}{2!(c+1)^2}, \cdots\right\}.$$

On putting $c/(c + 1) = p$ and $1/(c + 1) = q$ the probability generating function is $p^k(1 - qt)^{-k}$.

The negative binomial distribution has often been found to provide a good fit to data where there is inequality of risk. Unfortunately, when we come to test

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the goodness of fit of this distribution to the data of Corbet and Williams, we note that only frequencies of captures greater than zero are observable, since the collection gives no indication of the number of species not found in the sample. If it is assumed that a large number of species are so rare that the probability of their capture is small, a negative binomial distribution, if it could be fitted to the data, would have a value of k approaching zero. We are thus led to consider the limiting form of the negative binomial distribution where k tends to zero and the zero class is ignored.

The negative binomial distribution is formed by the expansion of the probability generating function $p^k(1 - qt)^{-k}$. If we exclude the probability of no events the resulting distribution will have probability generating function

$$G(t) = [1/(1 - p^k)] \{p^k(1 - qt)^{-k} - p^k\}$$

which, on rearranging and substituting θ for q and $(1 - \theta)$ for p, becomes

$$G(t) = [(1 - \theta t)^{-k} - 1]/[(1 - \theta)^{-k} - 1].$$

Allowing k to tend to zero, we obtain by l'Hôpital's rule

$$\lim_{k \to 0} G(t) = \log (1 - \theta t)/\log (1 - \theta).$$

The distribution having this probability generating function is called the logarithmic series distribution, the probability of n events being

$$[-1/\log (1 - \theta)]\, \theta^n/n \qquad (n = 1, 2, 3, \cdots).$$

Biologists have found this distribution capable of describing a wide variety of natural phenomena, including the number of individuals per species, the number of species per genus, the number of genera per sub-family and even the number of research papers per biologist, in a particular year (Williams [10]). An example of the goodness of fit of the distribution to Corbet's data is given in Table 1.

Fisher [3] computed a useful table, in terms of N, the number of individuals, and S, the number of species, to enable the distribution to be fitted to data of this type. Table 3 of the present paper is a somewhat simpler table to perform the same function.

D. G. Kendall [6] has given a description of a Markov process which also leads to a logarithmic series distribution.

Consider the growth of a population whose numbers fluctuate due to reproduction (binary fission), mortality, and immigration from outside. Each of these three activities is Markovian, i.e. the probability of any particular event in time interval t is the same for all time intervals of this length and independent of the past history of the system. Then if $\beta\delta t$ is the probability that a member of the population reproduces itself during time interval δt, $\omega\delta t$ is the probability that a member of the population dies during time interval δt, and $\kappa\delta t$ is the probability that an individual not initially a member of the population joins the population during time interval δt, a set of differential equations may be set up to describe the growth of the population.

TABLE 1

Frequencies of butterfly species collected in Malaya (Corbet, 1943)

Number of individuals per species	Observed frequency	Frequencies given by the logarithmic distribution
1	118	135.1
2	74	67.3
3	44	44.8
4	24	33.5
5	29	26.7
6	22	22.2
7	20	19.0
8	19	16.5
9	20	14.7
10	15	13.1
11	12	11.9
12	14	10.9
13	6	10.0
14	12	9.3
15	6	8.6
16	9	8.1
17	9	7.6
18	6	7.1
19	10	6.7
20	10	6.4
21	11	6.1
22	5	5.8
23	3	5.5
24	3	5.3

If $P_n(t)$ is the probability that there are n individuals in the system at time t, these equations are

$$P'_n(t) = [(n-1)\beta + \kappa]P_{n-1}(t) + (n+1)\omega P_{n+1}(t) - [n(\omega+\beta)+\kappa]P_n(t) \quad (n \geqq 1)$$

$$P'_0(t) = \omega P_1(t) - \kappa P_0(t).$$

Solving these equations, given the initial conditions $P_0(0) = 1$, $P_n(0) = 0$, $(n = 1, 2, \cdots)$, we obtain for the distribution of population size, after the process has been developing for time T, a negative binomial distribution with index κ/β and mean

$$\kappa\{e^{(\beta-\omega)T} - 1\}/(\beta - \omega).$$

If $\beta \geqq \omega$, the population grows te an infinite size. If, however, $\beta < \omega$, the population size has a steady state distribution with mean $\kappa/(\omega - \beta)$. Assume, now, that κ is small compared with β. Thus, whilst immigration is necessary to generate the first individual after the population has died out, this immigration plays no effective role in perpetuating growth if there are individuals present.

In this situation, the population size, given that a population exists, follows a logarithmic series distribution.

Recently the authors have found the logarithmic series distribution relevant to problems in the field of Operational Research, particularly in the study of a certain class of inventory control problem. In many cases it is a reasonable assumption that arrivals of demands at a store follow a Poisson process. However, not all demands are for single items; the number of items requested per demand may itself be a random variable. It can be shown, e.g. Feller [2], that if $G_1(t)$ is the probability generating function of the distribution of demands per unit time, and $G_2(t)$ is the probability generating function of the distribution of items requested per demand, then the distribution of items requested per unit time has the compound generating function $G_1(G_2(t))$. Thus if the former distribution is Poisson, with parameter λ, the generating function of the compound distribution is

$$G_3(t) = \exp\{\lambda[G_2(t) - 1]\}. \tag{1}$$

Table 2 shows in the second column the frequency of the number of steel blooms requested per demand from a steel stockist. Comparison with these data shows that the logarithmic frequencies given in column 3 of the table provide a reasonably good fit.

Substituting the generating function of the logarithmic series distribution in (1), letting $k = -\lambda/\log(1 - \theta)$, $q = \theta$, and $p = 1 - \theta$, we obtain

$$G_3(t) = p^k(1 - qt)^{-k}$$

(cf Quenouille [9]), which is the probability generating function of the negative binomial distribution. Thus the withdrawals of items from the store may be represented by a tabulated function [11].

TABLE 2

Quantities of steel requested per demand from a steel merchant

Quantity requested	Observed frequency	Frequencies given by logarithmic distribution
1	77	85.6
2	35	34.4
3	29	18.4
4	16	11.0
5	3	7.1
6	4	4.7
7	4	3.3
8	1	2.3
9	0	1.6
10	1	1.2
11 and over	3	3.4
	173	173

There are thus several quite distinct situations which lead to the logarithmic series distribution, and no doubt there are others not discussed here.

3. Moments. If the probability law of the logarithmic series distribution is

$$P(n) = \alpha\theta^n/n \qquad (n = 1, 2, 3, \cdots),$$

where $\alpha = -1/\log(1 - \theta)$, the distribution has characteristic function $\varphi(s) = -\alpha \log(1 - \theta e^{is})$. On expanding and collecting coefficients of $(is)^r/r!$ we have for the moments about the origin

$$\mu_1' = \alpha\theta/(1-\theta), \quad \mu_2' = \alpha\theta/(1-\theta)^2, \quad \mu_3' = \alpha\theta(1+\theta)/(1-\theta)^3$$

$$\text{and} \quad \mu_4' = \alpha\theta(1 + 4\theta + \theta^2)/(1-\theta)^4.$$

4. Estimation of the parameter. In order to fit the logarithmic series distribution to observed data, an estimate, $\hat{\theta}$, of the population parameter is required from the sample. It can be shown that the sample mean, m, is a sufficient statistic and the maximum likelihood estimate of the population mean. Unfortunately the solution of the equation

$$m = -\hat{\theta}/(1-\hat{\theta}) \log(1-\hat{\theta})$$

for $\hat{\theta}$ is not particularly straightforward. Table 3 has been constructed using the iterative process of Newton, to determine $\hat{\theta}$ for values of the mean between 1 and 50. Combined with Table 4, which tables second and fourth central differences where necessary, the value of $\hat{\theta}$ for any mean in the above range, may be obtained by Everett's Central Difference Interpolation Formula (see Section 5).

Detailed treatments of the estimation of the population parameter of the logarithmic series distribution have been given elsewhere by Anscombe [1], Good [4] and Patil [7].

5. Description of the tables.

Computation. The probabilities of 1, 2, 3, $\cdots$ events generated by the logarithmic series distribution are

$$\alpha\theta, \quad \alpha\theta^2/2, \quad \alpha\theta^3/3, \quad \cdots.$$

The mean, μ, is the most suitable argument for entering the table and hence the parameter θ is obtained by solution of

$$\mu = -\theta/(1-\theta)\log(1-\theta).$$

As α is $-1/\log(1 - \theta)$ the probability of one event, $P(1)$, is $\mu(1 - \theta)$. The second term, $P(2)$, is obtained from the first by multiplying by the factor $\theta/2$ and in general, the $(n + 1)$th term, $P(n + 1)$, is obtained from the nth, $P(n)$, by multiplying by $n\theta/(n + 1)$.

The cumulative probability, $F(n)$, is the probability of n or fewer events, and is obtained by summing the first n individual terms of the distribution.

TABLE 3

The parameter θ for values of the mean between 1 and 50

Mean	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	.00000	.17112	.29826	.39567	.47220	.53359	.58372	.62529	.66020	.68987
2	.71533	.73738	.75663	.77355	.78853	.80186	.81379	.82452	.83421	.84300
3	.85100	.85831	.86502	.87118	.87687	.88212	.88699	.89152	.89574	.89967
4	.90335	.90680	.91003	.91307	.91593	.91863	.92118	.92358	.92587	.92803
5	.93008	.93203	.93388	.93565	.93733	.93894	.94047	.94194	.94334	.94469
6	.94597	.94721	.94839	.94953	.95063	.95168	.95269	.95366	.95460	.95551
7	.95638	.95722	.95804	.95882	.95959	.96032	.96103	.96172	.96239	.96303
8	.96366	.96427	.96486	.96543	.96599	.96653	.96705	.96756	.96806	.96854
9	.96901	.96946	.96991	.97034	.97076	.97117	.97158	.97197	.97235	.97272
	0	1	2	3	4	5	6	7	8	9
10	.97308	.97628	.97885	.98095	.98270	.98418	.98544	.98653	.98748	.98831
20	.98904	.98970	.99029	.99082	.99129	.99173	.99212	.99248	.99282	.99312
30	.99341	.99367	.99391	.99414	.99435	.99455	.99473	.99491	.99507	.99523
40	.99537	.99551	.99564	.99576	.99588	.99599	.99610	.99620	.99629	.99638
50	.99647									

TABLE 4

Second and fourth central differences necessary for interpolation in Table 3

Mean	δ^2									
	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1		.04399	.02974	.02088	.01513	.01126	.00857	.00665	.00525	.00420
2	.00341	.00280	.00233	.00195	.00165	.00140	.00120	.00104	.00090	.00079
3	.00069	.00061	.00054	.00048	.00043	.00038	.00035	.00031	.00028	.00026
4	.00023	.00021	.00019	.00018	.00016	.00015	.00014	.00013	.00012	.00011
5	.00010	.00010	.00009	.00008	.00008	.00007	.00007	.00006	.00006	.00006
6	.00005	.00005	.00005	.00004	.00004	.00004	.00004			
	0	1	2	3	4	5	6	7	8	9
10	.00088	.00063	.00046	.00035	.00027	.00022	.00017	.00014	.00012	.00010
20	.00008	.00007	.00006	.00005	.00004	.00004				

Mean	δ^4									
	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1			.00539	.00311	.00187	.00118	.00077	.00052	.00036	.00026

All differences are negative.

The argument proceeds in 0.1 steps from 1.1 to 2.0, in 0.5 steps to 5.0, and finally in 1.0 steps to 10.0, thus giving a total of 21 tables. For each distribution the computation was continued until $F(n)$ was at least 0.999. The interval of n was increased from 1 to 5 for values of n greater than 50.

Interpolation. Most of the time required to compute a table for a given value of the mean is concerned with obtaining the value of the parameter, θ. The time required to compute the terms of the distribution, once θ is known is small, and is a very straightforward procedure on a desk calculating machine. Accordingly, no differences have been tabulated for interpolating between adjacent columns of the table. Instead, Table 4 has been constructed to provide the even central differences necessary to evaluate θ for any mean in the range covered by Table 3.

Assume a given mean, m_x, lies between two tabulated values, m_0 and m_1, where

$$(m_x - m_0)/(m_1 - m_0) = x$$

and

$$(m_1 - m_x)/(m_1 - m_0) = y$$

then by Everett's Central Difference Formula

$$\theta_{m_x} = y\theta_{m_0} + x\theta_{m_1} - \tfrac{1}{6}\, xy\, \{(1 + y)\, \delta^2_{m_0} + (1 + x)\delta^2_{m_1}\}$$
$$+ (1/120)\, xy\, (1 + x)\, (1 + y)\, \{(2 + y)\, \delta^4_{m_0} + (2 + x)\delta^4_{m_1}\} - \cdots .$$

Pearson [8] has shown that it is unnecessary to use any second difference (δ^2) less than 4, and any fourth difference (δ^4) less than 20. Table 4, therefore, provides all the differences necessary to evaluate θ for any mean in the stated range.

Cumulative sums of the logarithmic series distribution calculated using a value of θ correct to five decimal places, will generally be correct to four decimal places. Using the procedure described

$$P(n) = \mu(1 - \theta)\, \theta^{n-1}/n.$$

Therefore $\delta\, P(n) = [\mu\theta^{n-2}\delta\theta/n]\, \{n(1 - \theta) - 1\}$, which tends to zero as n tends to infinity. Also $F(n) = \mu\, (1 - \theta)\, \{1 + \theta/2 + \theta^2/3 + \cdots + \theta^{n-1}/n\}$. Therefore

$$\delta\, F(n) = \mu\, \delta\theta\{(\tfrac{1}{2} - 1) + (\tfrac{2}{3} - 1)\, \theta + \cdots$$
$$+ [(n - 1)/n - 1]\, \theta^{n-2} - \theta^{n-1}\} = (\delta\theta/\theta)\, \{\mu(1 - \theta^n) - F(n)/(1 - \theta)\}.$$

The error in $F(n)$ attains a maximum as n tends to infinity, which, combined with the maximum error in θ, is

$$(5 \times 10^{-6}/\theta)\, \{\mu - 1/(1 - \theta)\}.$$

TABLE 5

The logarithmic series distribution

mean = 1.1, θ = 0.171125

n	$P(n)$	$F(n)$
1	0.91176	0.91176
2	0.07801	0.98978
3	0.00890	0.99868
4	0.00114	0.99982

mean = 1.2, θ = 0.298265

n	$P(n)$	$F(n)$
1	0.84208	0.84208
2	0.12558	0.96766
3	0.02497	0.99263
4	0.00559	0.99822
5	0.00133	0.99955

mean = 1.3, θ = 0.395670

n	$P(n)$	$F(n)$
1	0.78563	0.78563
2	0.15542	0.94105
3	0.04100	0.98205
4	0.01217	0.99422
5	0.00385	0.99807
6	0.00127	0.99934

mean = 1.4, θ = 0.472196

n	$P(n)$	$F(n)$
1	0.73893	0.73893
2	0.17446	0.91338
3	0.05492	0.96830
4	0.01945	0.98775
5	0.00735	0.99510
6	0.00289	0.99799
7	0.00117	0.99916

mean = 1.5, θ = 0.533589

n	$P(n)$	$F(n)$
1	0.69962	0.69962
2	0.18665	0.88627
3	0.06640	0.95267
4	0.02657	0.97924
5	0.01134	0.99058
6	0.00504	0.99563
7	0.00231	0.99793
8	0.00108	0.99901

mean = 1.6, θ = 0.583723

n	$P(n)$	$F(n)$
1	0.66604	0.66604
2	0.19439	0.86044
3	0.07565	0.93608
4	0.03312	0.96920
5	0.01547	0.98467
6	0.00752	0.99219
7	0.00376	0.99595
8	0.00192	0.99788
9	0.00100	0.99887
10	0.00052	0.99940

mean = 1.7, θ = 0.625288

n	$P(n)$	$F(n)$
1	0.63701	0.63701
2	0.19916	0.83617
3	0.08302	0.91919
4	0.03893	0.95812
5	0.01948	0.97760
6	0.01015	0.98775
7	0.00544	0.99319
8	0.00298	0.99616
9	0.00165	0.99782
10	0.00093	0.99875
11	0.00053	0.99928

mean = 1.8, θ = 0.660203

n	$P(n)$	$F(n)$
1	0.61164	0.61164
2	0.20190	0.81354
3	0.08886	0.90240
4	0.04400	0.94640
5	0.02324	0.96964
6	0.01279	0.98243
7	0.00724	0.98966
8	0.00418	0.99384
9	0.00245	0.99630
10	0.00146	0.99775
11	0.00087	0.99863
12	0.00053	0.99916

mean = 1.9, θ = 0.689869

n	$P(n)$	$F(n)$
1	0.58925	0.58925
2	0.20325	0.79250
3	0.09348	0.88598
4	0.04837	0.93434
5	0.02669	0.96104
6	0.01535	0.97638
7	0.00907	0.98546
8	0.00548	0.99093
9	0.00336	0.99429
10	0.00209	0.99638
11	0.00131	0.99769
12	0.00083	0.99851
13	0.00053	0.99904

mean = 2.0, θ = 0.715332

n	$P(n)$	$F(n)$
1	0.56934	0.56934
2	0.20363	0.77297
3	0.09711	0.87008
4	0.05210	0.92218
5	0.02981	0.95199
6	0.01777	0.96976
7	0.01090	0.98066
8	0.00682	0.98748
9	0.00434	0.99182
10	0.00279	0.99461
11	0.00182	0.99643
12	0.00119	0.99762
13	0.00079	0.99840
14	0.00052	0.99893
15	0.00035	0.99928

mean = 2.5, θ = 0.801861

n	$P(n)$	$F(n)$
1	0.49535	0.49535
2	0.19860	0.69395
3	0.10617	0.80011
4	0.06385	0.86396
5	0.04096	0.90492
6	0.02737	0.93229
7	0.01881	0.95110
8	0.01320	0.96430
9	0.00941	0.97370
10	0.00679	0.98049
11	0.00495	0.98544
12	0.00364	0.98908
13	0.00269	0.99177
14	0.00200	0.99378
15	0.00150	0.99528
16	0.00113	0.99640
17	0.00085	0.99726
18	0.00064	0.99790
19	0.00049	0.99839
20	0.00037	0.99876
21	0.00028	0.99905

mean = 3.0, θ = 0.851001

n	$P(n)$	$F(n)$
1	0.44700	0.44700
2	0.19020	0.63720
3	0.10791	0.74510
4	0.06887	0.81397
5	0.04689	0.86086
6	0.03325	0.89411
7	0.02425	0.91836
8	0.01806	0.93642
9	0.01366	0.95009
10	0.01046	0.96055
11	0.00809	0.96864
12	0.00631	0.97496
13	0.00496	0.97992
14	0.00392	0.98384
15	0.00311	0.98695
16	0.00248	0.98944
17	0.00199	0.99143
18	0.00160	0.99303
19	0.00129	0.99431
20	0.00104	0.99536
21	0.00084	0.99620
22	0.00069	0.99689
23	0.00056	0.99745
24	0.00046	0.99790
25	0.00037	0.99827
26	0.00030	0.99858
27	0.00025	0.99883
28	0.00020	0.99903

mean = 3.5, θ = 0.882122

n	$P(n)$	$F(n)$
1	0.41257	0.41257
2	0.18197	0.59454
3	0.10701	0.70155
4	0.07080	0.77235
5	0.04996	0.82231
6	0.03673	0.85904
7	0.02777	0.88681
8	0.02143	0.90825
9	0.01681	0.92505
10	0.01334	0.93840
11	0.01070	0.94910
12	0.00865	0.95775
13	0.00705	0.96479
14	0.00577	0.97057
15	0.00475	0.97532
16	0.00393	0.97925
17	0.00326	0.98251
18	0.00272	0.98523
19	0.00227	0.98750
20	0.00190	0.98940
21	0.00160	0.99100
22	0.00135	0.99235
23	0.00114	0.99348
24	0.00096	0.99444
25	0.00081	0.99526
26	0.00069	0.99595
27	0.00059	0.99653
28	0.00050	0.99703
29	0.00042	0.99745
30	0.00036	0.99782
31	0.00031	0.99813
32	0.00026	0.99839
33	0.00023	0.99862
34	0.00019	0.99881
35	0.00017	0.99897
36	0.00014	0.99912

mean = 4.0, θ = 0.903350

n	$P(n)$	$F(n)$
1	0.38660	0.38660
2	0.17462	0.56122
3	0.10516	0.66638
4	0.07125	0.73762
5	0.05149	0.78911
6	0.03876	0.82787
7	0.03001	0.85788
8	0.02372	0.88161
9	0.01905	0.90066
10	0.01549	0.91614
11	0.01272	0.92886
12	0.01053	0.93939
13	0.00878	0.94818
14	0.00737	0.95554
15	0.00621	0.96175
16	0.00526	0.96701
17	0.00447	0.97148
18	0.00382	0.97530
19	0.00327	0.97857
20	0.00280	0.98137
21	0.00241	0.98378
22	0.00208	0.98586
23	0.00180	0.98765
24	0.00156	0.98921
25	0.00135	0.99056
26	0.00117	0.99173
27	0.00102	0.99275
28	0.00089	0.99364
29	0.00077	0.99441
30	0.00068	0.99509
31	0.00059	0.99568
32	0.00052	0.99619
33	0.00045	0.99665
34	0.00040	0.99704
35	0.00035	0.99739
36	0.00031	0.99770
37	0.00027	0.99797
38	0.00024	0.99820
39	0.00021	0.99841
40	0.00018	0.99860
41	0.00016	0.99876
42	0.00014	0.99890
43	0.00013	0.99903

mean = 4.5, θ = 0.918628

n	$P(n)$	$F(n)$
1	0.36617	0.36617
2	0.16819	0.53436
3	0.10300	0.63736
4	0.07097	0.70833
5	0.05215	0.76048
6	0.03992	0.80041
7	0.03144	0.83184
8	0.02527	0.85711
9	0.02063	0.87774
10	0.01706	0.89480
11	0.01425	0.90905
12	0.01200	0.92104
13	0.01017	0.93122
14	0.00868	0.93989
15	0.00744	0.94733
16	0.00641	0.95374
17	0.00554	0.95928
18	0.00481	0.96408
19	0.00418	0.96827
20	0.00365	0.97192
21	0.00319	0.97511
22	0.00280	0.97791
23	0.00246	0.98037
24	0.00217	0.98254
25	0.00191	0.98445
26	0.00169	0.98614

TABLE 5—*Continued*

27	0.00149	0.98763
28	0.00132	0.98895
29	0.00117	0.99012
30	0.00104	0.99116
31	0.00093	0.99209
32	0.00082	0.99291
33	0.00073	0.99365
34	0.00065	0.99430
35	0.00058	0.99489
36	0.00052	0.99541
37	0.00047	0.99587
38	0.00042	0.99629
39	0.00037	0.99666
40	0.00033	0.99700
41	0.00030	0.99730
42	0.00027	0.99757
43	0.00024	0.99781
44	0.00022	0.99802
45	0.00019	0.99822
46	0.00017	0.99839
47	0.00016	0.99855
48	0.00014	0.99869
49	0.00013	0.99882
50	0.00011	0.99893
55	0.00007	0.99936

mean = 5.0, θ = 0.930080

n	$P(n)$	$F(n)$
1	0.34960	0.34960
2	0.16258	0.51218
3	0.10081	0.61299
4	0.07032	0.68331
5	0.05232	0.73563
6	0.04055	0.77618
7	0.03233	0.80851
8	0.02631	0.83482
9	0.02175	0.85657
10	0.01821	0.87478
11	0.01540	0.89018
12	0.01313	0.90330
13	0.01127	0.91457
14	0.00973	0.92430
15	0.00845	0.93275
16	0.00737	0.94012
17	0.00645	0.94656
18	0.00566	0.95223
19	0.00499	0.95722
20	0.00441	0.96163
21	0.00391	0.96554
22	0.00347	0.96900
23	0.00309	0.97209
24	0.00275	0.97484
25	0.00246	0.97729
26	0.00220	0.97949
27	0.00197	0.98146
28	0.00176	0.98322
29	0.00158	0.98481
30	0.00142	0.98623
31	0.00128	0.98751
32	0.00115	0.98867
33	0.00104	0.98971
34	0.00094	0.99065
35	0.00085	0.99150
36	0.00077	0.99227
37	0.00070	0.99296
38	0.00063	0.99359
39	0.00057	0.99416
40	0.00052	0.99468
41	0.00047	0.99515
42	0.00043	0.99557
43	0.00039	0.99596
44	0.00035	0.99631
45	0.00032	0.99663
46	0.00029	0.99692
47	0.00027	0.99719
48	0.00024	0.99743
49	0.00022	0.99765
50	0.00020	0.99785
55	0.00013	0.99862
60	0.00008	0.99911

mean = 6.0, θ = 0.945975

n	$P(n)$	$F(n)$
1	0.32415	0.32415
2	0.15332	0.47747
3	0.09669	0.57416
4	0.06860	0.64276
5	0.05192	0.69468
6	0.04093	0.73561
7	0.03318	0.76879
8	0.02747	0.79626
9	0.02310	0.81935
10	0.01966	0.83902
11	0.01691	0.85593
12	0.01466	0.87059
13	0.01280	0.88340
14	0.01125	0.89464
15	0.00993	0.90457
16	0.00881	0.91338
17	0.00784	0.92122
18	0.00701	0.92823
19	0.00628	0.93450
20	0.00564	0.94015
21	0.00508	0.94523
22	0.00459	0.94982
23	0.00415	0.95397
24	0.00377	0.95774
25	0.00342	0.96116
26	0.00311	0.96427
27	0.00283	0.96710
28	0.00258	0.96968
29	0.00236	0.97204
30	0.00216	0.97420
31	0.00198	0.97618
32	0.00181	0.97799
33	0.00166	0.97965
34	0.00153	0.98118
35	0.00140	0.98258
36	0.00129	0.98387
37	0.00119	0.98505
38	0.00109	0.98614
39	0.00101	0.98715
40	0.00093	0.98808
41	0.00086	0.98894
42	0.00079	0.98973
43	0.00073	0.99046
44	0.00068	0.99114
45	0.00063	0.99176
46	0.00058	0.99234
47	0.00054	0.99288
48	0.00050	0.99337
49	0.00046	0.99383
50	0.00043	0.99426
55	0.00029	0.99597
60	0.00020	0.99716
65	0.00014	0.99799
70	0.00010	0.99857
75	0.00007	0.99897
80	0.00005	0.99926

TABLE 5—*Continued*

mean = 7.0, θ = 0.956381

n	$P(n)$	$F(n)$
1	0.30533	0.30533
2	0.14601	0.45134
3	0.09309	0.54443
4	0.06677	0.61120
5	0.05109	0.66229
6	0.04072	0.70301
7	0.03338	0.73639
8	0.02793	0.76432
9	0.02375	0.78807
10	0.02044	0.80850
11	0.01777	0.82627
12	0.01558	0.84185
13	0.01375	0.85561
14	0.01221	0.86782
15	0.01090	0.87872
16	0.00978	0.88850
17	0.00880	0.89730
18	0.00795	0.90524
19	0.00720	0.91244
20	0.00654	0.91899
21	0.00596	0.92495
22	0.00544	0.93039
23	0.00498	0.93536
24	0.00456	0.93992
25	0.00419	0.94411
26	0.00385	0.94796
27	0.00355	0.95151
28	0.00327	0.95478
29	0.00302	0.95780
30	0.00279	0.96059
31	0.00258	0.96318
32	0.00239	0.96557
33	0.00222	0.96779
34	0.00206	0.96985
35	0.00191	0.97177
36	0.00178	0.97355
37	0.00166	0.97520
38	0.00154	0.97675
39	0.00144	0.97819
40	0.00134	0.97953
41	0.00125	0.98078
42	0.00117	0.98194
43	0.00109	0.98304
44	0.00102	0.98406
45	0.00095	0.98501
46	0.00089	0.98590
47	0.00084	0.98674
48	0.00078	0.98752
49	0.00073	0.98825
50	0.00069	0.98894
55	0.00050	0.99178
60	0.00037	0.99386
65	0.00027	0.99539
70	0.00020	0.99653
75	0.00015	0.99738
80	0.00011	0.99801
85	0.00008	0.99849
90	0.00006	0.99885
95	0.00005	0.99912

mean = 8.0, θ = 0.963661

n	$P(n)$	$F(n)$
1	0.29071	0.29071
2	0.14007	0.43078
3	0.08999	0.52077
4	0.06504	0.58581
5	0.05014	0.63595
6	0.04027	0.67621
7	0.03326	0.70947
8	0.02804	0.73751
9	0.02402	0.76154
10	0.02083	0.78237
11	0.01825	0.80062
12	0.01612	0.81675
13	0.01434	0.83109
14	0.01283	0.84392
15	0.01154	0.85546
16	0.01043	0.86589
17	0.00946	0.87535
18	0.00861	0.88396
19	0.00786	0.89182
20	0.00719	0.89901
21	0.00660	0.90561
22	0.00607	0.91169
23	0.00560	0.91729
24	0.00517	0.92246
25	0.00478	0.92724
26	0.00443	0.93167
27	0.00411	0.93578
28	0.00382	0.93961
29	0.00356	0.94316
30	0.00331	0.94647
31	0.00309	0.94956
32	0.00288	0.95245
33	0.00269	0.95514
34	0.00252	0.95766
35	0.00236	0.96002
36	0.00221	0.96223
37	0.00207	0.96430
38	0.00194	0.96625
39	0.00183	0.96808
40	0.00172	0.96979
41	0.00161	0.97140
42	0.00152	0.97292
43	0.00143	0.97435
44	0.00135	0.97570
45	0.00127	0.97696
46	0.00119	0.97816
47	0.00113	0.97928
48	0.00106	0.98035
49	0.00100	0.98135
50	0.00095	0.98230
55	0.00072	0.98632
60	0.00055	0.98936
65	0.00042	0.99170
70	0.00032	0.99349
75	0.00025	0.99488
80	0.00020	0.99596
85	0.00015	0.99681
90	0.00012	0.99747
95	0.00009	0.99799
100	0.00007	0.99840
105	0.00006	0.99872
110	0.00005	0.99898
115	0.00004	0.99918

mean = 9.0, θ = 0.969008

n	$P(n)$	$F(n)$
1	0.27893	0.27893
2	0.13514	0.41407
3	0.08730	0.50138
4	0.06345	0.56482
5	0.04919	0.61401
6	0.03972	0.65373
7	0.03299	0.68671
8	0.02797	0.71468
9	0.02409	0.73878
10	0.02101	0.75979
11	0.01851	0.77830

TABLE 5—*Continued*

12	0.01644	0.79474
13	0.01471	0.80944
14	0.01323	0.82267
15	0.01197	0.83464
16	0.01087	0.84551
17	0.00991	0.85543
18	0.00907	0.86450
19	0.00833	0.87283
20	0.00767	0.88050
21	0.00708	0.88757
22	0.00655	0.89412
23	0.00607	0.90019
24	0.00563	0.90582
25	0.00524	0.91106
26	0.00488	0.91594
27	0.00456	0.92050
28	0.00426	0.92476
29	0.00398	0.92874
30	0.00373	0.93247
31	0.00350	0.93597
32	0.00328	0.93926
33	0.00309	0.94234
34	0.00290	0.94525
35	0.00273	0.94798
36	0.00257	0.95055
37	0.00243	0.95298
38	0.00229	0.95527
39	0.00216	0.95743
40	0.00204	0.95948
41	0.00193	0.96141
42	0.00183	0.96323
43	0.00173	0.96496
44	0.00164	0.96660
45	0.00155	0.96815
46	0.00147	0.96962
47	0.00139	0.97102
48	0.00132	0.97234
49	0.00126	0.97359
50	0.00119	0.97479
55	0.00093	0.97992
60	0.00073	0.98393
65	0.00057	0.98708
70	0.00045	0.98957
75	0.00036	0.99156
80	0.00029	0.99315
85	0.00023	0.99442
90	0.00019	0.99545
95	0.00015	0.99628
100	0.00012	0.99695
105	0.00010	0.99750
110	0.00008	0.99794
115	0.00007	0.99830
120	0.00005	0.99860
125	0.00004	0.99885
130	0.00004	0.99905

mean = 10.00, θ = 0.973082

n	$P(n)$	$F(n)$
1	0.26918	0.26918
2	0.13097	0.40015
3	0.08496	0.48511
4	0.06201	0.54712
5	0.04827	0.59539
6	0.03914	0.63453
7	0.03265	0.66718
8	0.02780	0.69498
9	0.02404	0.71902
10	0.02106	0.74008
11	0.01863	0.75870
12	0.01662	0.77532
13	0.01492	0.79024
14	0.01349	0.80373
15	0.01225	0.81597
16	0.01117	0.82715
17	0.01023	0.83738
18	0.00940	0.84678
19	0.00867	0.85545
20	0.00801	0.86347
21	0.00743	0.87089
22	0.00690	0.87779
23	0.00642	0.88421
24	0.00599	0.89020
25	0.00559	0.89580
26	0.00523	0.90103
27	0.00490	0.90593
28	0.00460	0.91054
29	0.00432	0.91486
30	0.00407	0.91893
31	0.00383	0.92276
32	0.00361	0.92637
33	0.00341	0.92977
34	0.00322	0.93299
35	0.00304	0.93603
36	0.00288	0.93891
37	0.00272	0.94163
38	0.00258	0.94421
39	0.00245	0.94666
40	0.00232	0.94898
41	0.00220	0.95119
42	0.00209	0.95328
43	0.00199	0.95527
44	0.00189	0.95716
45	0.00180	0.95896
46	0.00171	0.96068
47	0.00163	0.96231
48	0.00156	0.96386
49	0.00148	0.96535
50	0.00141	0.96676
55	0.00112	0.97292
60	0.00090	0.97783
65	0.00072	0.98177
70	0.00059	0.98496
75	0.00048	0.98755
80	0.00039	0.98967
85	0.00032	0.99140
90	0.00026	0.99283
95	0.00022	0.99400
100	0.00018	0.99498
105	0.00015	0.99579
110	0.00013	0.99646
115	0.00010	0.99702
120	0.00009	0.99749
125	0.00007	0.99788
130	0.00006	0.99821
135	0.00005	0.99849
140	0.00004	0.99872
145	0.00004	0.99892
150	0.00003	0.99908

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