MONOTONICITY OF THE POWER FUNCTIONS OF SOME TESTS OF THE MULTIVARIATE LINEAR HYPOTHESIS

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1. Summary. The test procedures, invariant under certain groups of transformations [4], for testing a set of multivariate linear hypotheses in the linear normal model depend on the characteristic roots of a random matrix. The power function of such a test depends on the characteristic roots of a corresponding population matrix as parameters; these roots may be regarded as measures of deviation from the hypothesis tested. In this paper sufficient conditions on the procedure for the power function to be a monotonically increasing function of each of the parameters are obtained. The likelihood-ratio test [1], Lawley-Hotelling trace test [1], and Roy's maximum root test [6] satisfy these conditions. The monotonicity of the power function of Roy's test has been shown by Roy and Mikhail [5] using a geometrical method.

2. Introduction. Consider a $p \times n$ random matrix $\mathbf{X}$ whose columns are independently distributed according to p-variate normal distributions with the common covariance matrix $\mathbf{\Sigma}$ and expectations given by

$$\mathcal{E}\mathbf{X} = \mathbf{\Theta A},$$

where $\mathbf{A}$ is a known $m \times n$ matrix of rank r and $\mathbf{\Theta}$ is a $p \times m$ matrix of unknown parameters. It is assumed that $r \leqq \min(m, n - p)$. In this model consider the problem of testing the hypothesis

$$\mathcal{H}_0 : \mathbf{\Theta C} = \mathbf{0}(p \times s),$$

where $\mathbf{C}$ is a known $m \times s$ matrix of rank s $(\leqq r)$ such that $\mathbf{\Theta C}$ is estimable, against all alternatives. This problem can be transformed to the following canonical form [1], [6]: Let $\mathbf{X}^* = [\mathbf{X}_1^*(p \times s),\ \mathbf{X}_2^*(p \times (n - r)),\ \mathbf{X}_3^*(p \times (r - s))]$ be a random matrix whose column vectors are independently distributed according to p-variate normal distributions with the common covariance matrix $\mathbf{\Sigma}$ and expectations given by

$$\mathcal{E}\mathbf{X}_1^* = \mathbf{\Lambda}(p \times s), \qquad \mathcal{E}\mathbf{X}_2^* = \mathbf{0}(p \times (n - r)), \qquad \mathcal{E}\mathbf{X}_3^* = \mathbf{\Gamma}(p \times (r - s)).$$

The hypothesis $\mathcal{H}_0$ is equivalent to the hypothesis: $\mathbf{\Lambda} = \mathbf{0}(p \times s)$. The matrices of sums of products due to the hypothesis and due to error are given by

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$\mathbf{S}_h = \mathbf{X}_1^*\mathbf{X}_1^{*\prime}$ and $\mathbf{S}_e = \mathbf{X}_2^*\mathbf{X}_2^{*\prime}$, respectively. The problem is invariant under transformations $\mathbf{B}\mathbf{X}_1^*\mathbf{F}_1$, $\mathbf{B}\mathbf{X}_2^*\mathbf{F}_2$, and $\mathbf{B}\mathbf{X}_3^*\mathbf{F}_3 + \mathbf{G}$, where $\mathbf{B}$ is nonsingular and $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ are orthogonal matrices. In this paper we shall consider the test procedures which are invariant under these transformations. These invariant test procedures depend only on $c_1 \geqq c_2 \geqq \cdots \geqq c_p$, the characteristic roots of $\mathbf{S}_h\mathbf{S}_e^{-1}$, and the power function of any such test depends only on the parameters $\theta_1, \cdots, \theta_t$, where $\theta_1 \geqq \cdots \geqq \theta_t$ are the possibly nonzero characteristic roots of $\mathbf{\Lambda}\mathbf{\Lambda}'\mathbf{\Sigma}^{-1}$, and $t = \min(p, s)$.

Some invariant procedures are:

(i) The likelihood-ratio test [1] whose acceptance region is $\prod_{i=1}^{p} (1 + c_i) \leqq \lambda_1$, a constant.

(ii) Lawley-Hotelling trace test [1] whose acceptance region is $\sum_{i=1}^{p} c_i \leqq \lambda_2$, a constant.

(iii) Roy's maximum root test [6] whose acceptance region is $c_1 \leqq \lambda_3$, a constant.

3. Tests of multivariate linear hypothesis. The following theorem is the basic result of this paper.

THEOREM 1. *Let the random vectors* $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_s$ *and the matrix* $\mathbf{Y}$ *be mutually independent, the distribution of* $\mathbf{x}_i$ *being* $N(k_i\boldsymbol{\mu}_i, \mathbf{\Sigma}_i)$, $i = 1, \cdots, s$. *If a set* ω *in the sample space is convex and symmetric in each* $\mathbf{x}_i$ *given the other* $\mathbf{x}_j$*'s and* $\mathbf{Y}$, *then* Prob (ω) *decreases with respect to each* $k_i(\geqq 0)$.

This theorem is proved using the following result due to Anderson [2]:

THEOREM 2. *Let E be a convex set in the n-dimensional Euclidean space, symmetric about the origin. For* $\mathbf{x}(n \times 1)$, *let* $f(\mathbf{x}) \geqq 0$ *be a function such that* (i) $f(\mathbf{x}) = f(-\mathbf{x})$, (ii) $\{\mathbf{x} \mid f(\mathbf{x}) \geqq u\} = K_u$ *is convex for every* $u(0 < u < \infty)$, *and* (iii) $\int_E f(\mathbf{x})\, d\mathbf{x} < \infty$. *Then*

$$\int_E f(\mathbf{x} + k\mathbf{y})\, d\mathbf{x} \geqq \int_E f(\mathbf{x} + \mathbf{y})\, d\mathbf{x} \tag{3.1}$$

for every vector $\mathbf{y}$ *and for* $0 \leqq k \leqq 1$.

PROOF OF THEOREM 1. Define $f_i(\mathbf{x}_i)$ to be the density of $N(\mathbf{0}, \mathbf{\Sigma}_i)$ at $\mathbf{x}_i$. Then f_i satisfies the conditions (i) to (iii) of Theorem 2. From Theorem 2

$$\int_R f_i(\mathbf{x}_i + k_i\, \boldsymbol{\mu}_i)\, d\mathbf{x}_i \geqq \int_R f_i(\mathbf{x}_i + k_i^*\, \boldsymbol{\mu}_i)\, d\mathbf{x}_i, \tag{3.2}$$

where $k_i \leqq k_i^*$, and $R = \omega(\mathbf{x}_i \mid \mathbf{x}_1, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_s, \mathbf{Y})$ is the set of vectors $\mathbf{x}_i$ that belong to ω with given values of $\mathbf{x}_1, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_s, \mathbf{Y}$. Multiplying both the sides of the inequality (3.2) by the joint density of the temporarily fixed variables and integrating with respect to them, we obtain

$$\begin{aligned} &\text{Prob}\,(\omega \mid k_1, \cdots, k_{i-1}, k_i, k_{i+1}, \cdots, k_s) \\ &\qquad \geqq \text{Prob}\,(\omega \mid k_1, \cdots, k_{i-1}, k_i^*, k_{i+1}, \cdots, k_s), \end{aligned} \tag{3.3}$$

for $k_i \leqq k_i^*$, and any k_j's $(j \neq i)$.

Now we shall apply this theorem to the problem of testing multivariate linear hypotheses. The roots of $\mathbf{S}_h\mathbf{S}_e^{-1}$ are the roots of $(\mathbf{UU}')(\mathbf{VV}')^{-1}$, if $\mathbf{U} = \mathbf{BX}_1^*\mathbf{F}_1$, and $\mathbf{V} = \mathbf{BX}_2^*\mathbf{F}_2$; hence invariant tests depend only on the roots of the matrix $(\mathbf{UU}')(\mathbf{VV}')^{-1}$. The matrices $\mathbf{B}$, $\mathbf{F}_1$ and $\mathbf{F}_2$ can be found (see [6], p. 86) so that the density of $\mathbf{U} = [u_{ij}]: p \times s$ and $\mathbf{V} = [v_{ij}]: p \times (n-r)$ is

$$(2\pi)^{-\frac{1}{2}(s+m)p}\cdot\exp\left[-\frac{1}{2}\left\{\operatorname{tr}(\mathbf{VV}') + \sum_{i=1}^{t}(u_{ii}-\theta_i)^2 + \sum_{i=t+1}^{p} u_{ii}^2 + \sum_{\substack{i=1\\ j\neq i}}^{p}\sum_{j=1}^{s} u_{ij}^2\right\}\right], \tag{3.4}$$

where $m = n - r$, and the hypothesis $\mathcal{H}_0$ holds if and only if $\theta_1 = \theta_2 = \cdots = \theta_t = 0$.

THEOREM 3. *If the acceptance region of an invariant test is convex in the space of each column vector of* $\mathbf{U}$ *for each set of fixed values of* $\mathbf{V}$ *and of the other column vectors of* $\mathbf{U}$, *then the power of the test increases monotonically in each* θ_i.

PROOF. Let the jth column vector of $\mathbf{U}$ be denoted by $\mathbf{u}_j (j = 1, \cdots, s)$. Note that the vectors $\mathbf{u}_1, \cdots, \mathbf{u}_s$ and $\mathbf{V}$ are mutually independent, the distribution of $\mathbf{u}_j (t < j \leqq s)$ is $N(\mathbf{0}, \mathbf{I})$ and the distribution of $\mathbf{u}_j (1 \leqq j \leqq t)$ is $N(\theta_j \mathbf{e}_j, \mathbf{I})$, where the ith element of $\mathbf{e}_j$ is δ_{ij}, the Kronecker's delta. Since the test is invariant, the acceptance region is symmetric in each of the column vectors of $\mathbf{U}$. The result now follows from Theorem 1.

The following corollaries are implied by the above theorem:

COROLLARY 3.1. *If the acceptance region of an invariant test is convex in* $\mathbf{U}$ *for each fixed* $\mathbf{V}$, *then the power of the test increases monotonically in each* θ_i.

COROLLARY 3.2. *If the acceptance region of an invariant test is the interior and the boundary of an ellipsoid in the space of each column vector of* $\mathbf{U}$ *for each set of fixed values of* $\mathbf{V}$ *and of the other column vectors of* $\mathbf{U}$, *then the power of the test increases monotonically in each* θ_i.

COROLLARY 3.3. *The maximum root test of Roy, the acceptance region of which is given by*

$$\operatorname{ch}_1[(\mathbf{UU}')(\mathbf{VV}')^{-1}] \leqq \mu,$$

has a power function which is monotonically increasing in each θ_i. ($\operatorname{ch}_1(\mathbf{A})$ denotes the maximum characteristic root of $\mathbf{A}$.)

The above corollary follows from Corollary 3.1 and the following lemma:

LEMMA 1. *For any symmetric matrix* $\mathbf{B}: n \times n$ *the region*

$$E = [\mathbf{A}: n \times m \mid \operatorname{ch}_1(\mathbf{AA}'\mathbf{B}) \leqq \mu]$$

is convex in $\mathbf{A}$.

PROOF. Let $\mathbf{B} = \mathbf{T}'\mathbf{T}$, where $\mathbf{T}$ is an $n \times n$ matrix. Then

$$\operatorname{ch}_1(\mathbf{AA}'\mathbf{B}) = \operatorname{ch}_1[(\mathbf{TA})(\mathbf{TA})'].$$

Let $\mathbf{A}_1 \varepsilon E$, $\mathbf{A}_2 \varepsilon E$, and $\mathbf{A} = \lambda\mathbf{A}_1 + (1-\lambda)\mathbf{A}_2$ for $0 \leqq \lambda \leqq 1$. It follows from the Cauchy-Schwarz inequality that for any vector $\mathbf{x}$

$$\begin{aligned}\mathbf{x}'\mathbf{TAA}'\mathbf{T}'\mathbf{x} &= \lambda^2\mathbf{x}'\mathbf{TA}_1\mathbf{A}_1'\mathbf{T}'\mathbf{x} + (1-\lambda)^2\mathbf{xTA}_2\mathbf{A}_2'\mathbf{T}'\mathbf{x} \\ &\quad + \lambda(1-\lambda)[\mathbf{x}'\mathbf{TA}_1\mathbf{A}_2'\mathbf{T}'\mathbf{x} + \mathbf{x}'\mathbf{TA}_2\mathbf{A}_1'\mathbf{T}'\mathbf{x}] \leqq [\lambda(\mathbf{x}'\mathbf{TA}_1\mathbf{A}_1'\mathbf{T}'\mathbf{x})^{\frac{1}{2}} \\ &\quad + (1-\lambda)(\mathbf{x}'\mathbf{TA}_2\mathbf{A}_2'\mathbf{T}'\mathbf{x}')^{\frac{1}{2}}]^2 \leqq \mu\mathbf{x}'\mathbf{x}.\end{aligned}$$

Thus $\mathbf{A} \varepsilon E$.

Let $c_1 \geqq \cdots \geqq c_p$ be the roots of $(\mathbf{UU}')(\mathbf{VV}')^{-1}$, and $d_i = 1 + c_i$ $(i = 1, \cdots, p)$. Let W_k be the sum of all different products of $d_1, \cdots, d_p$ taken k at a time $(k = 1, \cdots, p)$.

THEOREM 4. *An invariant test having acceptance region* $\sum_{k=1}^{p} a_k W_k \leqq \mu(a_k\text{'s} \geqq 0)$ *has a power function which is monotonically increasing in each* θ_i.

This theorem is proved using the following Lemma 2. Consider a matrix $\mathbf{A} = [\mathbf{a}_1, \cdots, \mathbf{a}_n]: m \times n$, where $\mathbf{a}_j$'s are the column vectors of $\mathbf{A}$. Define $W_k(\mathbf{A})$ as the sum of all k-rowed principal minors of $\mathbf{AA}' + \mathbf{I}_m$, or equivalently as the sum of all different products of the roots of $\mathbf{AA}' + \mathbf{I}_m$ taken k at a time.

LEMMA 2. *For any j and $k(j = 1, \cdots, n;\ k = 1, \cdots, m)$ and for $\mathbf{a}_i$'s fixed, $i \neq j$, $W_k(\mathbf{A})$ is a positive definite quadratic form in $\mathbf{a}_j$ plus a constant.*

PROOF. For simplicity we prove this for $j = 1$. Consider a k-rowed principal minor of $\mathbf{AA}' + \mathbf{I}_m$, say the one with rows and columns numbered $1, 2, \cdots, k$; it is

$$\tag{3.5} \begin{aligned} &\left| \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{kn} \end{bmatrix} + \mathbf{I}_k \right| \\ &\quad = [(a_{11}, a_{21}, \cdots, a_{k1})\mathbf{B}^{-1}(1; 1, 2, \cdots, k)(a_{11}, a_{21}, \cdots, a_{k1})' + 1] \\ &\qquad \cdot |\mathbf{B}(1; 1, 2, \cdots, k)|, \end{aligned}$$

where

$$\mathbf{B}(1; 1, 2, \cdots, k) = \begin{bmatrix} a_{12} & \cdots & a_{1n} \\ a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{k2} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{kn} \end{bmatrix} + \mathbf{I}_k .$$

Thus the determinant (3.5) is a constant plus a positive definite quadratic form in $(a_{11}, \cdots, a_{k1})$ with the $\mathbf{a}_j$'s $(j \neq 1)$ fixed. Since the sum of positive definite quadratic forms is a positive quadratic form, we see that $W_k(\mathbf{A})$ is a positive definite quadratic form in $\mathbf{a}_1$ plus a constant when the $\mathbf{a}_j$'s $(j \neq 1)$ are fixed.

PROOF OF THEOREM 4. Let $(\mathbf{VV}')^{-1} = \mathbf{T}'\mathbf{T}$, where $\mathbf{T}$ is a $p \times p$ matrix, and $\mathbf{U}^* = \mathbf{TU}$. It follows from Lemma 2 that $\sum_{k=1}^{p} a_k W_k(a_k\text{'s} \geqq 0)$ is a positive definite quadratic form in each column vector of $\mathbf{U}^*$ plus a constant when the other column vectors of $\mathbf{U}^*$ are fixed. Thus the region $E_j^* = [\mathbf{u}_j^* \mid \sum_{k=1}^{p} a_k W_k \leqq \mu, \mathbf{u}_i^*\text{'s fixed}, i \neq j]$ is convex and symmetric about the origin in the space of the

jth column vector $\mathbf{u}_j^*$ of $\mathbf{U}^*$. Since $\mathbf{u}_j^* = \mathbf{T}\mathbf{u}_j$, the region $E_j = [\mathbf{u}_j \mid \sum_{k=1}^p a_k W_k \leqq \mu$, $\mathbf{V}$, $\mathbf{u}_i$'s fixed, $i \neq j]$ is convex and symmetric about the origin in the space of $\mathbf{u}_j (j = 1, \cdots, s)$. The theorem now follows from Theorem 3.

The following corollaries are special cases of Theorem 4.

COROLLARY 4.1. *The likelihood-ratio test having the acceptance region of the form $W_p \leqq \mu$ has a power function which is monotonically increasing in each θ_i.*

COROLLARY 4.2. *The Lawley-Hotelling trace test with the acceptance region of the form $W_1 \leqq \mu$ has a power function which is monotonically increasing in each θ_i.*

In order that the power function of an invariant test should increase monotonically in each θ_i it is not necessary that the acceptance region of the test be convex in each $\mathbf{u}_i$ given the other $\mathbf{u}_j$'s and $\mathbf{V}$. The following theorem gives another sufficient condition on the acceptance region. There are some tests for which both the conditions are satisfied; neither of these two conditions implies the other. Let ω denote the acceptance region of an invariant test, and let the region $\omega(\mathbf{u}_i \mid \mathbf{u}_j\text{'s}, j \neq i; \mathbf{V})$ be the section of ω in the space of $\mathbf{u}_i$ for a set of fixed values of $\mathbf{u}_j$'s $(j \neq i)$ and $\mathbf{V}$. For simplicity, we shall denote this section by $\omega_i(\mathbf{u}_i)$.

THEOREM 5. *For each $i (i = 1, \cdots, s)$ and for each set of fixed values of $\mathbf{u}_j$'s $(j \neq i)$ and $\mathbf{V}$, suppose there exists an orthogonal transformation: $\mathbf{u}_i \rightarrow \mathbf{M}\mathbf{u}_i = \mathbf{u}_i^* = (u_{1i}^*, \cdots, u_{pi}^*)'$ such that the region $\omega_i(\mathbf{u}_i)$ is transformed into the region $\omega_i^*(\mathbf{u}_i^*)$ which has the following property: Any section of $\omega_i^*(\mathbf{u}_i^*)$ for fixed values of $u_{ki}^* (k \neq j)$ is an interval, symmetric about $u_{ji}^* = 0$. Then the power function of the test, having the acceptance region ω, monotonically increases in each θ_i.*

PROOF. Let $\mathbf{M} = [m_{jk}]$ be the orthogonal matrix satisfying the condition of the theorem. For simplicity, we take $i = 1$. We shall indicate the proof by treating the case $p = 2$ in detail. Let the section of $\omega_1^*(\mathbf{u}_1^*)$ in the coordinate u_{k1}^* for fixed u_{j1}^* be the interval $[-\omega_{1k}^*(u_{j1}^*), \omega_{1k}^*(u_{j1}^*)], j \neq k$. Since $\mathbf{M}$ is orthogonal we have

$$(2\pi)^{-1} \int_{\omega_1(\mathbf{u}_1)} \exp\left[-\tfrac{1}{2}(u_{11} - \theta_1)^2 - \tfrac{1}{2}u_{21}^2\right] du_{11}\, du_{21}$$
$$= (2\pi)^{-1} \int_{\omega_1^*(\mathbf{u}_1^*)} \exp\left[-\tfrac{1}{2}(u_{11}^* - \theta_1 m_{11})^2 - \tfrac{1}{2}(u_{21}^* - \theta_1 m_{21})^2\right] du_{11}^*\, du_{21}^*.$$

For $0 \leqq \lambda \leqq 1$, we have from Theorem 2

$$\int_{\omega_1^*(\mathbf{u}_1^*)} \exp\left[-\tfrac{1}{2}(u_{11}^* - \theta_1 m_{11})^2 - \tfrac{1}{2}(u_{21}^* - \theta_1 m_{21})^2\right] du_{11}^*\, du_{21}^*$$
$$= \int \left\{\exp\left[-\tfrac{1}{2}(u_{11}^* - \theta_1 m_{11})^2\right] \int_{-\omega_{12}^*(u_{11}^*)}^{\omega_{12}^*(u_{11}^*)} \exp\left[-\tfrac{1}{2}(u_{21}^* - \theta_1 m_{21})^2\right] du_{21}^*\right\} du_{11}^*$$
$$\leqq \int \left\{\exp\left[-\tfrac{1}{2}(u_{11}^* - \theta_1 m_{11})^2\right] \int_{-\omega_{12}^*(u_{11}^*)}^{\omega_{12}^*(u_{11}^*)} \exp\left[-\tfrac{1}{2}(u_{21} - \lambda\theta_1 m_{21})^2\right] du_{21}^*\right\} du_{11}^*$$
$$= \int_{\omega_1^*(\mathbf{u}_1^*)} \exp\left[-\tfrac{1}{2}(u_{11}^* - \theta_1 m_{11})^2 - \tfrac{1}{2}(u_{21}^* - \lambda\theta_1 m_{21})^2\right] du_{11}^*\, du_{21}^*$$

$$= \int \left\{ \exp\left[-\tfrac{1}{2}(u_{21}^* - \lambda\theta_1 m_{21})^2\right] \int_{-\omega_{11}^*(u_{21}^*)}^{\omega_{11}^*(u_{21}^*)} \exp\left[-\tfrac{1}{2}(u_{11}^* - \theta_1 m_{11})^2\right] du_{11}^* \right\} du_{21}^*$$

$$\leqq \int \left\{ \exp\left[-\tfrac{1}{2}(u_{21}^* - \lambda\theta_1 m_{21})^2\right] \int_{-\omega_{11}^*(u_{21}^*)}^{\omega_{11}^*(u_{21}^*)} \exp\left[-\tfrac{1}{2}(u_{11}^* - \lambda\theta_1 m_{11})^2\right] du_{11}^* \right\} du_{21}^*$$

$$= \int_{\omega_1^*(\mathbf{u}_1^*)} \exp\left[-\tfrac{1}{2}(u_{11}^* - \lambda\theta_1 m_{11})^2 - \tfrac{1}{2}(u_{21}^* - \lambda\theta_1 m_{21})^2\right] du_{11}^* \, du_{21}^*$$

$$= \int_{\omega_1(\mathbf{u}_1)} \exp\left[-\tfrac{1}{2}(u_{11} - \lambda\theta_1)^2 - \tfrac{1}{2}u_{21}^2\right] du_{11} \, du_{21} .$$

The rest of the proof is similar to the proof of Theorem 1.

It may be noted that if the acceptance region satisfies the condition of Corollary 3.2, then it will also satisfy the condition of Theorem 5.

REMARK. It can be seen ([3], p. 33) that any root of $\mathbf{\Lambda\Lambda}'\mathbf{\Sigma}^{-1}$ increases monotonically in each of the diagonal elements of $\mathbf{\Lambda\Lambda}'$. Hence, if the power function of a test increases monotonically in each root of $\mathbf{\Lambda\Lambda}'\mathbf{\Sigma}^{-1}$, then it increases monotonically in each of the diagonal elements of $\mathbf{\Lambda\Lambda}'$ for each set of fixed $\mathbf{\Sigma}$ and the nondiagonal elements of $\mathbf{\Lambda\Lambda}'$.

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