SOME THEOREMS ON FUNCTIONALS OF MARKOV CHAINS

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1. Introduction. In this paper we shall investigate various phenomena associated with a Markov process in discrete time, extending results found in [3], [6], [7], and [13]. The paper is divided into three parts. In part one we focus our attention on recurrent events (i.e., to successive entrances into some fixed state of a Markov chain with the positive integers as states) and show that the waiting time distribution is completely determined by the sequence $\{EY_n\}$, where Y_n is the time as observed from n that the event last took place. Moreover, we show that criteria for the event to be persistent, transient, positive, etc., may be given directly in terms of the EY_n. In part two we examine a particular class of null events called β-regular (see Section 2 for the definition), where we find various joint limit distributions for some of the functionals usually associated with these events. In part three we extend these limit laws to situations more general than recurrent events, and these extended results are then applied to several concrete situations.

2. Criteria for recurrent events. Let e be a recurrent event on the positive integers with waiting times W_k, these being independent, identically distributed, positive integer-valued random variables which may also assume the value ∞. We recall that e is called transient or persistent according as to whether or not $\rho = P(W_1 < \infty) < 1$. A persistent event is positive if $EW_1 < \infty$ and is null if $EW_1 = \infty$. A recurrent event is periodic of period λ if e may only occur at times $n\lambda$ for $n = 0, 1, 2, \cdots$. If $\lambda = 1$ we say that e is aperiodic. From now on we shall always assume that the recurrent event is aperiodic. The methods needed to extend results to the case of periodic events are both simple and standard [8]. We introduce the following functions. For $n > 0$ let

$N_n = \sup\{k \leqq n: W_1 + \cdots + W_k \leqq n\}$ (number of occurrences by time n)

$Y_n = W_1 + \cdots + W_{N_n}$ (time of last occurrence),

$Y_n' = n - Y_n$ (time elapsed since last occurrence),

$Z_n = W_1 + \cdots + W_{N_n+1} - n$ (time to elapse until next occurrence). For $n = 0$ let $N_0 = Y_0 = 0$.

Let $u_n = P(Y_n = n)$, $q_n = P(Y_n = 0)$, and for $|t| < 1$ let

$$F(t) = E(t^{W_1}; W_1 < \infty) = \sum_{k=1}^{\infty} t^k P(W_1 = k),$$

$$U(t) = 1 + \sum_{n=1}^{\infty} u_n t^n,$$

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and

$$T(t) = \sum_{n=1}^{\infty} P(Y_n = 0)t^n = \sum_{n=0}^{\infty} q_n t^n.$$

Recall (see [8]) the well-known relations:

$$[1 - F(t)]^{-1} = U(t), \qquad (1 - t)^{-1}[1 - F(t)] = T(t).$$

We are now in a position to show that the various criteria associated with e may be given directly in terms of the sequence EY_n. This result was motivated by the results of Andersen [1] and Spitzer [17] which implicitly contain these criteria for the special recurrent event, "Ladder Point." (See the example at the end of this section.)

THEOREM 2.1. *Suppose e is an aperiodic recurrent event. Let $\Delta_n = E(Y_n - Y_{n-1})$ if $n > 0$ and let $\Delta_0 = 0$. Then e is transient if and only if $\sum_{k=1}^{\infty} \Delta_k k^{-1} < \infty$, and*

$$P(W_1 = \infty) = 1 - \rho = \exp\left\{-\sum_{k=1}^{\infty} \Delta_k k^{-1}\right\}. \tag{2.1}$$

Moreover, we have

$$EW_1 = \exp\left\{\sum_{k=1}^{\infty} (1 - \Delta_k)k^{-1}\right\} \tag{2.2}$$

in the sense that both sides of (2.2) are finite or infinite together and always equal.

PROOF. In [14] we showed that for any recurrent event

$$U(t) = \exp\left\{\sum_{k=1}^{\infty} t^k \Delta_k k^{-1}\right\}. \tag{2.3}$$

Since $\Delta_k \geqq 0$, (2.1) follows from (2.3) by an extended version of Abel's theorem. From (2.3) we also have $T(t) = \exp \sum_{k=1}^{\infty} t^k(1 - \Delta_k)k^{-1}$. Now, suppose

$$\sum_{k=1}^{\infty}(1 - \Delta_k)k^{-1} \tag{2.4}$$

converges. Then, as e^x is continuous, from Abel's theorem we have

$$EW_1 = \lim_{t\to 1^-} T(t) = \exp \sum_{k=1}^{\infty}(1 - \Delta_k)k^{-1}.$$

Conversely, suppose $EW_1 < \infty$. Then we know at once that

$$\lim_{t\to 1^-} \sum_{k=1}^{\infty} t^k(1 - \Delta_k)k^{-1} \tag{2.5}$$

exists, and to conclude the proof we must show that the series (2.4) converges. If we knew that

$$\lim_{n\to\infty} \Delta_n = 1, \tag{2.6}$$

then this result would follow at once from (2.5) and Tauber's theorem. To verify

(2.6) we may proceed as follows. Observe that

$$(2.7) \qquad (1-t)\sum_{k=1}^{\infty} \Delta_k t^k = tF'(t)U(t)(1-t).$$

At $t = 1$, the series $U(t)(1-t)$ becomes the series $u_0 + (u_1 - u_0) + (u_2 - u_1) + \cdots = \lim_{n\to\infty} u_n$. By the fundamental limit theorem on recurrent events [8], the above limit is $(EW_1)^{-1}$. Moreover, $F'(1) = EW_1$. Thus by Merten's theorem ([11]; Theorem 161, p. 228) we have from (2.7),

$$(2.8) \qquad \lim_{n\to\infty} \Delta_n = EW_1(EW_1)^{-1} = 1.$$

As a corollary of the proof we have the following.

COROLLARY 2.2. *If e is a positive, aperiodic event, then* (2.8) *holds and the series* (2.4) *converges to a finite positive value.*

We introduce next the following definition.

DEFINITION. *A recurrent event is β-regular if*

$$(2.9) \qquad \lim_{n\to\infty}(1/n)\sum_{k=1}^{n} \Delta_k = \lim_{n\to\infty} E(Y_n/n) = \beta.$$

The following result is essentially due to Lamperti [13].

THEOREM 2.3. *e is β-regular if and only if*

$$(2.10) \qquad 1 - F(t) \sim (1-t)^{\beta} h(1/(1-t))$$

where $h(x)$ is a slowly varying function. Moreover,

$$(2.11) \qquad h(1/(1-t)) \sim L(1/(1-t)) = \exp\sum_{k=1}^{\infty} t^k(\beta - \Delta_k)k^{-1}.$$

PROOF. Lamperti in [13] showed that (2.10) and (2.9) were equivalent and now (2.11) easily follows from (2.10) and (2.3).

We conclude this section with an illustration of the preceding results. Let S_n be the nth partial sum of independent random variables with a common distribution. Set $S_0 = 0$, and define e' to be the event $S_n > M_{n-1}$ where $M_n = \max(S_0, \cdots, S_n)$. It is well known that e' is an aperiodic recurrent event, and it is easily verified that Y_n for the event e' is the random variable $L_n = \inf\{k \leqq n : S_k = M_n\}$. If $Z_1, Z_2, \cdots$, are the successive, "ladder random variables," (i.e., if W_k are the waiting times for e', then $Z_k = S_{W_1+\cdots+W_k} - (Z_1 + \cdots + Z_{k-1})$, where $Z_1 = S_{W_1}$) then $M_n = Z_1 + \cdots + Z_{N_n}$. The equivalence principle of Andersen-Feller asserts that L_n and $Q_n = \sum_{k=1}^{n}(1 + \operatorname{sgn} S_k)/2$ have the same distribution. From this, it follows at once that for the event e' we have $\Delta_n = P(S_n > 0)$. If M and N denote the limits as $n \to \infty$ of M_n and N_n respectively then it is clear that M is finite with probability one if and only if N is. Consequently, Theorem 2.1 gives the following results of Spitzer. If $\sum_{n=1}^{\infty} P(S_n > 0)n^{-1} < \infty$ then $P(M < \infty) = 1$ while $P(M = \infty) = 1$ if the above series diverges. Moreover, for the event e' we have

$$P(W_1 < \infty) = 1 - \exp\left[-\sum_{k=1}^{\infty} P(S_k > 0)k^{-1}\right]$$

and

$$EW_1 = \exp \sum_{k=1}^{\infty} P(S_k \leqq 0)k^{-1}.$$

3. Limit laws for recurrent events. Feller in [7], and independently Darling and Kac in [3], showed that if there are constants C_n such that N_n/C_n has a nondegenerate limit distribution, then

$$1 - F(t) \sim (1 - t)^{\beta} h(1/(1 - t)) \tag{3.1}$$

for some β, $0 \leqq \beta < 1$ and some slowly varying function h. Conversely, if (3.1) is satisfied then one can take $C_n = n^{\beta}h(n)^{-1}$ and N_n/C_n will have a nondegenerate limit distribution. Similarly Dynkin [6], and independently Lamperti [13] showed that when e was a β-regular event for some β, $0 < \beta < 1$ then Y_n/n had a nondegenerate limit law, and conversely if Y_n/n had a nondegenerate limit law, then e was β-regular for some β, $0 < \beta < 1$.

Now by Theorem 2.3, (3.1) and β-regularity are equivalent, and moreover,

$$h(n) \sim L(n) = \exp \sum_{k=1}^{\infty}((n - 1)/n)^k[\beta - \Delta_k]k^{-1}. \tag{3.2}$$

From now on we shall always take the normalizing constants on N_n to be

$$b_n = n^{\beta}L(n)^{-1}. \tag{3.3}$$

The above discussion makes it plausible that the following result holds.

THEOREM 3.1. *In order that $(N_n/b_n, Y_n/n)$ should converge in law to (N, Y) having a nondegenerate distribution, it is both necessary and sufficient that e be β-regular for some β, $0 < \beta < 1$. The distribution of (N, Y) is then uniquely determined by its moments:*

$$E(N^m Y^k) = \frac{(-1)^k k!\, m!}{\Gamma(m\beta + k + 1)} \binom{-\beta(m + 1)}{k}. \tag{3.4}$$

In the proof given below, we shall follow the method used by Lamperti to establish the corresponding result for Y_n/n. It seems to have first been used in problems of this type by Spitzer [17].

PROOF. If $(N_n/b_n, Y_n/n)$ is to have a nondegenerate limit distribution, then Y_n/n must have one. But as was mentioned above, this is only possible when e is β-regular for some β, $0 < \beta < 1$. If then, e is such an event, we obtain from Theorem 2.3 that (3.1) and (3.2) hold, and we shall now show that this implies $(N_n/b_n, Y_n/n)$ has the nondegenerate limit law with moments (3.4).

We have $P(N_n = k, Y_n = m) = P(W_1 + \cdots + W_k = m)q_{n-m}$ and thus for $|x| \leqq 1$, $|y| \leqq 1$, $|t| < 1$,

$$\sum_{n=0}^{\infty} t^n E[x^{Y_n} y^{N_n}] = T(t)[1 - yF(xt)]^{-1}, \tag{3.5}$$

which is an analytic function in (x, y, t) for $|x| \leqq 1$, $|y| \leqq 1$, $|t| < 1$. Taking the mth derivative of (3.5) with respect to y at 1 results in

$$\sum_{n=0}^{\infty} t^n E(x^{Y_n} N_n^{(m)}) = m!\, F(tx)^m [1 - F(t)](1 - t)^{-1} [1 - F(xt)]^{-(m+1)}, \tag{3.6}$$

where $N_n^{(m)} = N_n(N_n - 1) \cdots (N_n - m + 1)$. Set $x = e^{-\lambda(1-t)}$ in (3.6) and expand in powers of λ to obtain

$$\begin{aligned}\sum_{k=0}^{\infty} [(-1)^k/k!]\lambda^k (1 - t)^k \sum_{n=0}^{\infty} t^n E(Y_n^k N_n^{(m)}) \\ = m![1 - F(t)](1 - t)^{-1} F(te^{-\lambda(1-t)})^m [1 - F(te^{-\lambda(1-t)})]^{-(m+1)}.\end{aligned} \tag{3.7}$$

As $t \to 1^-$,

$$1 - te^{-\lambda(1-t)} \sim (1 + \lambda)(1 - t),$$

and so, taking account of the slowly varying nature of L, we obtain from (3.7) (after a slight rearrangement),

$$\begin{aligned}\lim_{t\to 1^-} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lambda^k (1 - t)^{k+1} (1 - t)^{\beta m} L\left(\frac{1}{1 - t}\right)^m \\ \cdot \sum_{n=0}^{\infty} t^n E(Y_n^k\, N_n^{(m)}) = m!(1 + \lambda)^{-\beta(m+1)}.\end{aligned} \tag{3.8}$$

Since the quantity in (3.7) is analytic in λ, t for $|t| < 1$ and $\lambda \geqq 0$, and the right-hand side of (3.8) is analytic in λ at $\lambda = 0$, we obtain from (3.8), by an appeal to the Weierstrass theorem,

$$\begin{aligned}\lim_{t\to 1^-} (1 - t)^{k+\beta m} L\left(\frac{1}{1 - t}\right)^m \sum_{n=1}^{\infty} t^n E\,(Y_n^k\, N_n^{(m)} - Y_{n-1}^k\, N_{n-1}^{(m)}) \\ = (-1)^k k!\, m! \binom{-\beta(m + 1)}{k}.\end{aligned} \tag{3.9}$$

Now, $E(Y_n^k N_n^{(m)})$ is, for each fixed k and m, a monotone increasing function in n. Hence we may apply Karamata's Tauberian theorem (see [4], p. 507) to (3.9) and conclude that

$$E(N_n^{(m)} Y_n^k) \sim n^{\beta m} n^k L(n)^{-m} \frac{(-1)^k k!\, m!}{\Gamma(m\beta + k + 1)} \binom{-\beta(m + 1)}{k}. \tag{3.10}$$

Finally, as $E(N_n^{(m)} Y_n^k) \sim E(N_n^m Y_n^k)$, $n \to \infty$, we have (3.4). To complete the proof, we must show that these moments uniquely determine a distribution. By Theorem 1.12 of [15] this will be true provided

$$\sum_{n=0}^{\infty} E(N^n + Y^n)^{-1/2n} = \infty. \tag{3.11}$$

A simple computation shows that

$$E(N^n + Y^n)^{-1/2n} \geqq (1 + EN^n)^{-1/2n} \geqq \text{const.}/(n + 1)^{\frac{1}{2}},$$

and so (3.11) certainly holds. This completes the proof.

It is possible to give an interesting characterization of the distribution of (N, Y) and even to write down its density function.

COROLLARY 3.2. (This was brought to our attention by M. Dwass.) *(N, Y) is distributed like $(\tilde{N}Y^{\beta}, Y)$ where $(\tilde{N}, Y)$ are independent. Y has a density on $[0, 1]$ which is a generalized* arc-sin *law density,*

$$dP(Y \leqq x) = f_{\beta}(x) = (\sin \pi\beta/\pi)x^{\beta-1}(1 - x)^{-\beta} \tag{3.12}$$

while $\tilde{N}$ is a positive random variable with a density

$$h_{\beta}(x) = \Gamma(\beta + 1)xg_{\beta}(x) \tag{3.13}$$

on $[0, \infty)$ where $g_{\beta}(x)$ is the density of the Mittag-Leffler distribution of index β. (See [3] *for a discussion of the Mittag-Leffler distribution.)*

PROOF. A simple computation shows that

$$\begin{aligned} &\frac{(-1)^k k!\, m!}{\Gamma(m\beta + k + 1)} \binom{-\beta(m+1)}{k} \\ &\qquad = \frac{(m+1)!\,\Gamma(\beta+1)}{\Gamma(1 + \beta(m+1))} \frac{\Gamma(k + \beta(m+1))}{\Gamma(\beta)\Gamma(\beta m + k + 1)} \end{aligned} \tag{3.14}$$

and that

$$\int_0^1 x^{\beta m + k} f_{\beta}(x)\, dx = \frac{\Gamma(k + \beta(m+1))}{\Gamma(\beta)\Gamma(\beta m + k + 1)}.$$

Finally, as the mth moment of $\int_0^x g_{\beta}(x)\, dx$ is $m![\Gamma(1 + m\beta)]^{-1}$ it is easily seen that the first quantity on the right in (3.14) is the mth moment of the distribution on $(0, \infty)$ with density given in (3.13).

COROLLARY 3.3. *Under the same conditions as Theorem 3.1 we have that $(N_n/b_n, Y'_n/n)$ converges to (N, Y') in distribution. Moreover (N, Y') has the same distribution as $(\tilde{N}Y^{\beta}, (1 - Y))$, and*

$$E(N^m Y'^k) = m!\, k!(-1)^k \binom{\beta - 1}{k} \Big/ \Gamma(1 + k + \beta m).$$

PROOF.

$$\lim_{n\to\infty} E[(N_n/b_n)^m (Y'_n/n)^k] = \lim_{n\to\infty} E[(N_n/b_n)^m (1 - Y_n/n)^k].$$

A simple computation and Corollary 3.2 now completes the proof.

From the joint distribution of (N, Y) we may obtain the joint limit distribution of $(N_n/b_n, Z_n/n)$ and $(N_n/b_n, [N_{n+[rn]} - N_n]b_n^{-1})$. Since the proofs of these results follow closely those used by Lamperti in [13] to obtain the limit distribution of Z_n/n from that of Y'_n/n, and by Dynkin in [6] to obtain the limit distribution of $[N_{n+[rn]} - N_n]b_n^{-1}$ from that of Z_n/n, we shall merely state the results with no proofs.

THEOREM 3.4. *β-regularity $(0 < \beta < 1)$ is both necessary and sufficient condition for $(N_n/b_n, Z_n/n)$ to have a nondegenerate limit law. The density of the limit law is*

$$h_{\beta}(x(1 + u)^{\beta})(1 + u)^{\beta} f_{\beta}(1/(1 + u))(1 + u)^{-2} = dP(N \leqq x, Z \leqq u). \tag{3.15}$$

THEOREM 3.5. *Under the same conditions as Theorem 3.4 we have*

$$\lim_{n\to\infty} P(N_n \leqq xb_n, N_{n+[rn]} - N_n > ub_n) \tag{3.16}$$
$$= \int_0^r G_\beta\left(\frac{r-s}{u^{1/\beta}}\right) P(\tilde{N} \leqq x(1+s)^\beta)(1+s)^{-2} f_\beta\left(\frac{1}{1+s}\right) ds$$

where $G_\beta(x)$ is the stable law on $(0, \infty)$ with Laplace transform $e^{-\lambda^\beta}$.

At times it is possible to assert that a stronger version of the limit laws for Y_n and Y_n' hold.

THEOREM 3.6. *Suppose $0 < s < 1$ and $k \sim sn$. Then for $0 < \beta < 1$*

$$\lim_{n\to\infty} nP(Y_n = k) = \lim_{n\to\infty} nP(Y_n' = n-k) = f_\beta(s), \tag{3.17}$$

and uniformly so for s bounded away from 0 and 1, if and only if

$$u_n \sim n^{\beta-1}L(n)^{-1}\Gamma(\beta)^{-1}, \tag{3.18}$$

where $L(n)$ and $f_\beta(s)$ are as before.

PROOF. If (3.18) holds a well-known Abelian theorem ([4], p. 460) gives us that

$$1 - F(t) \sim (1-t)^\beta L(1/1-t), \tag{3.19}$$

and thus by Karamata's theorem ([4], p. 508) we have

$$q_0 + \cdots + q_n \sim n^{1-\beta}L(n)\Gamma(2-\beta)^{-1}.$$

As q_n is monotone, an application of another Tauberian Theorem ([4] p. 517)[1] allows us to conclude that

$$q_n \sim n^{-\beta}L(n)\Gamma(1-\beta)^{-1}. \tag{3.20}$$

Equation (3.17) now follows at once from the relation

$$P(Y_n = k) = P(Y_n' = n-k) = u_k q_{n-k}. \tag{3.21}$$

Conversely, suppose we know that (3.17) holds for all s, $0 < s < 1$. A simple argument then shows that for $0 < a < b < 1$,

$$P(an < Y_n < bn) \sim \int_a^b f_\beta(s)\, ds,$$

and thus Y_n/n converges in distribution to the nondegenerate law on $(0, 1)$ with density $f_\beta(s)$. Theorems (3.1) and (2.3) then tell us that (3.19) holds, and thus (3.20) holds. Setting $k = [sn]$ in (3.21) we obtain, by a simple computation, that $u_{[sn]} \sim [sn]^{\beta-1}\Gamma(\beta)^{-1}L([sn])^{-1}$.

REMARK. In the course of the above proof we showed that β-regularity and (3.20) were equivalent and that (3.18) always implied (3.20). Recently the problem of when (3.20) implied (3.18) was attacked by A. Garsia and J. Lamperti in [9]. Their results show that for $\frac{1}{2} < \beta < 1$ this is always the case, while for $0 < \beta \leqq \frac{1}{2}$ this need not, in general, be true. However, it is worth pointing out that if u_n is monotone, then (3.20) implies (3.18) for all values of β.

[1] The extension of this result to include a slowly varying function causes no difficulty.

The results of Theorem 3.1 and Corollary 3.2 make it seem very plausible that

(3.22) $\lim_{n\to\infty} P(N_n \leqq xEN_n \mid Y_n = [sn]) = P(\tilde{N} \leqq xs^{-\beta}), \quad 0 < s \leqq 1,$

for every β-regular event. Under the more stringent condition (3.18), Dwass and Karlin [5] have shown that (3.22) is true. By the above remark then, (3.22) is true for all β-regular events with $\frac{1}{2} < \beta < 1$, but whether this can be extended to $0 < \beta \leqq \frac{1}{2}$ remains open. It might be well to point out that if (3.22) holds, then the event is β-regular. To see this, observe that when (3.22) holds we have,

$$\lim_{n\to\infty} E(N_n/EN_n \mid Y_n = [sn]) = s^{\beta} \lim_{n\to\infty} E(N_n/EN_n \mid Y_n = n)$$

and thus, for all s, $0 < s \leqq 1$, we have $\lim_{n\to\infty} EN_{[sn]}/EN_n = s^{\beta}$, from which it is easy to deduce that $EN_n \sim n^{\beta}h(n)$ where $h(x)$ is a slowly varying function. Application of a familiar Abelian theorem ([4], p. 460) and Theorem 2.3 then show that e is β-regular.

Let us conclude this section with the following applications to the recurrent event e' introduced at the close of Section 2. Assume that $ES_1 = 0$ and that $ES_1^2 = \sigma^2 < \infty$. In [16] Spitzer showed that under these conditions the series

$$\gamma = \sum_{k=1}^{\infty} [\tfrac{1}{2} - P(S_k > 0)]k^{-1}$$

was convergent and that $EZ_1 = \sigma e^{\gamma}/2^{\frac{1}{2}}$. By the strong law of large numbers, we then have $\lim_{n\to\infty} M_n/N_n = EZ_1$ with probability one. Moreover, in this case we may choose the b_n of Theorem 3.1 to be $n^{\frac{1}{2}}e^{-\gamma}$. Consequently, from Theorem 3.1 and Corollary 3.2 we have, by the well-known fact ([3]) that $g_{\frac{1}{2}}(x) = \pi^{-\frac{1}{2}}e^{-x^2/4}$,

$$\lim_{n\to\infty} P(M_n \leqq x\sigma n^{\frac{1}{2}}, N_n \leqq yb_n 2^{\frac{1}{2}}, Y_n \leqq tn)$$
$$= \pi^{-1} \int_0^{\min(x,y)} \int_0^t uv^{-1}(1 - v)^{-\frac{1}{2}}e^{-u^2/2v}\, du\, dv.$$

Let us further observe that for the event e' we have $u_n = P(S_i > 0, 1 \leqq i \leqq n)$, which is obviously a monotone function of n, and thus by the remark following Theorem 3.6, if e' is β-regular then (3.18) holds. But e' is β-regular if and only if

$$\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} P(S_k > 0) = \beta,$$

and since $Y_n = L_n$, and Q_n have the same distribution, we have (by Theorem 3.6) the following stronger version of the generalized arc-sin laws for L_n and Q_n:

THEOREM 3.7. *If $0 < s < 1$, and if $k \sim sn$, then for $0 < \beta < 1$*

$$\lim_{n\to\infty} nP(L_n = k) = \lim_{n\to\infty} nP(Q_n = k) = f_{\beta}(s),$$

and uniformly so for s bounded away from 0 *and* 1, *if and only if*

$$\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} P(S_k > 0) = \beta.$$

4. Extensions. In this section X_n will denote a Markov chain with states in a measurable space $(E, \mathfrak{F})$. Its nth step transition function will be denoted by $P^n(x; A) = P(X_n \varepsilon A \mid X_0 = x)$, where, of course, $x \varepsilon E$ and $A \varepsilon \mathfrak{F}$. For $A \varepsilon \mathfrak{F}$, define (in analogy to the recurrent event case)

$$\begin{aligned} N_n(A) &= \sum_{j=1}^{n} \delta_A(X_j), && n > 0, \\ &= 0, && n = 0, \end{aligned}$$

where $\delta_A(x)$ is the characteristic function of A, and

$$\begin{aligned} Y_n(A) &= 0, && n = 0. \\ &= \sup\{r \leqq n : X_r \varepsilon A\}, && n > 0 \text{ and } N_n(A) > 0. \\ &= 0, && n > 0 \text{ and } N_n(A) = 0. \end{aligned}$$

For $|t| < 1$, $x \varepsilon E$, $A \varepsilon \mathfrak{F}$ let $U_t(x, A) = \sum_{n=0}^{\infty} t^n P^n(x; A)$, and for a given set $A \varepsilon \mathfrak{F}$, let

$$\begin{aligned} {}_AU_t(x, B) &= U_t(x, B) && \text{if } x \varepsilon E, \text{ and } B \varepsilon \mathfrak{F},\ B \subset A, \\ &= 0 && \text{otherwise,} \end{aligned}$$

be the restriction of U_t to the set A. Let

$$\begin{aligned} V_A &= \inf\{r > 0 : X_r \varepsilon A\}, && \text{if } X_r \varepsilon A \text{ for some } r > 0, \\ &= \infty, && \text{if there is no such } r, \end{aligned}$$

be the time of first return to A, and for $B \varepsilon \mathfrak{F}$, $B \subset A$, let

$${}_AH_t(x, B) = \sum_{n=1}^{\infty} P(X_n \varepsilon B,\ V_A = n \mid X_0 = x) t^n.$$

Finally, let ${}_AT_t(x) = \sum_{n=0}^{\infty} P(V_A > n \mid X_0 = x) t^n$.

From now on we shall focus our attention on a fixed set A, and it will be our purpose in this section to show under appropriate regularity conditions on ${}_AU_t$ and A, that it is possible to extend Theorem 3.1 to $(N_n(A), Y_n(A))$. The regularity condition we shall take is as follows.

ASSUMPTION 4.1. *There exists a finite measure $\pi(\cdot)$, with $\pi(A) > 0$, on the measurable subsets of A, and a function $\varphi(t)$ such that $\varphi(t) \to \infty$ as $t \to 1^-$ and such that for every bounded measurable function $f(y)$ we have*

$$(4.1) \qquad \int_A {}_AU_t(x, dy) f(y) = \varphi(t) \int_A f(x)\pi(dx) + \int_A \psi_1(x, y, t) f(y) \pi(dy),$$

where for each fixed x,

$$\lim_{t \to 1^-} \sup_{y \varepsilon A} |\psi_1(x, y, t)| / \varphi(t) = 0,$$

and moreover,

$$(4.2) \qquad \lim_{t \to 1^-} \sup_{x, y \varepsilon A} |\psi_1(x, y, t)| / \varphi(t) = 0.$$

In addition to Assumption 4.1 we shall also assume that the following recurrence condition holds:

$$P(V_A < \infty \mid X_0 = x) = 1 \qquad \text{for all } x \in E. \tag{4.3}$$

The Condition (4.1) above is similar to the conditions employed by Darling and Kac [3] and Lamperti [13] to establish corresponding results for $N_n(A)$ and $Y_n(A)$, respectively. Probably the result we establish here is true under less drastic conditions than the above, but this condition suffices for the applications we have in mind.

THEOREM 4.2. *If for the set A we have that* (4.3) *and Assumption* 4.1 *holds, and if for some β, $0 < \beta < 1$, and some slowly varying function $h(x)$ we have*

$$\varphi(t) \sim (1 - t)^{-\beta} h(1/(1 - t))^{-1}, \tag{4.4}$$

then if we choose

$$B_n = n^\beta h(n)^{-1} \tag{4.5}$$

we have, for any initial point x,

$$\lim_{n\to\infty} P(N_n(A)/\pi(A)B_n \leqq u,\ Y_n(A)/n \leqq t \mid X_0 = x) = P(N \leqq u,\ Y \leqq t),$$

where (N, Y) is as in Theorem 3.1.

PROOF. The same argument used to establish Theorem 3.1 will show that the desired result follows from (4.4), provided we can show

$$\sum_{n=0}^{\infty} t^n E(N_n^{(m)}(A) s^{Y_n(A)} \mid X_0 = x) \sim \pi(A)^m \varphi(st)^{m+1} (1 - t)^{-1} \varphi(t)^{-1} m!, \tag{4.6}$$

when $s = \exp[-\lambda(1 - t)]$. This will be accomplished by use of the following lemmas.

LEMMA 4.3. *Let ${}_AU_t^1 = {}_AU_t$, and for $m > 1$ define*

$${}_AU_t^{m+1}(x, B) = \int_A {}_AU_t^m(x, dy)\ {}_AU(y, B). \tag{4.7}$$

Then under Assumption 4.1 *for every bounded measurable function $f(x)$, we have*

$$\begin{aligned} \int_A {}_AU_t^m(x, dy) f(y) = \varphi(t)^m \pi(A)^{m-1} \int_A f(y)\pi(dy) \\ + \int_A \psi_m(x, y, t) f(y)\pi(dy) \end{aligned} \tag{4.8}$$

where

$$\lim_{t\to 1^-} \sup_{x,y\in A} |\psi_m(x, y, t)|/\varphi(t)^m = 0. \tag{4.9}$$

PROOF. We proceed by induction on m. For $m = 1$, this is just part of Assumption 4.1. Suppose we have already established (4.8) and (4.9) for all $m \leqq m_0$.

Let

$$\begin{aligned}(4.10)\qquad \psi_{m_0+1}(x, y, t) &= \varphi(t) \int_A \psi_{m_0}(z, y, t)\pi(dz) \\ &+ \varphi(t)^{m_0}\pi(A)^{m_0-1} \int_A \psi_1(x, z, t)\pi(dz) + \int_A \psi_1(x, z, t)\psi_{m_0}(z, y, t)\pi(dz).\end{aligned}$$

A simple computation shows that for any bounded measurable function f, we have

$$\int_A {}_AU_t^{m_0+1}(x, dy)f(y) - \varphi(t)^{m_0+1}\pi(A)^{m_0} \int_A f(y)\pi(dy) = \int_A \psi_{m_0+1}(x, y, t)f(y)\pi(dy).$$

But from (4.10) we have

$$\sup_{x,y\varepsilon A} \frac{|\psi_{m_0+1}(x, y,t)|}{\varphi(t)^{m_0+1}} \leqq \frac{\pi(A) \sup |\psi_{m_0}|}{\varphi(t)^{m_0}} + \frac{\pi(A)^{m_0} \sup |\psi_1|}{\varphi(t)} + \frac{\pi(A) \sup |\psi_1| \sup |\psi_{m_0}|}{\varphi(t)^{m_0+1}}$$

and thus, by the induction assumption, the limit as $t \to 1^-$ in the above expression is 0. This establishes the lemma.

LEMMA 4.4. *Let $\mathcal{B}(E)$ be the Banach space of bounded measurable functions. On $\mathcal{B}(E)$ define the operator*

$${}_AH_t f(x) = \int_A {}_AH_t(x, dy)f(y).$$

Then, for $|t| < 1$, and $|u| \leqq 1$ we have that

$$(4.11)\qquad (I - u\, {}_AH_t)^{-1} = I + \sum_{n=1}^{\infty} u^n\, {}_AH_t^n ,$$

and the expression in (4.11) *defines for $|u| \leqq 1$ an analytic function of u. Moreover, its* mth *derivative at $u = 1$ is given by the formula*

$$(4.12)\qquad \frac{d^m}{du^m} (I - u\, {}_AH_t)^{-1} |_{u=1} = m!\, {}_AH_t^m\, {}_AU_t^{m+1} .$$

PROOF. Let $V_A^1, V_A^2, \cdots$, be the times between the successive returns to A, that is $V_A^1 = V_A$, and for $n > 0$,

$$V_A^{n+1} = \inf\{r > V_A^1 + \cdots + V_A^n\} - (V_A^1 + \cdots + V_A^n),$$

and note that (4.3) assures us that these are defined with probability one. A simple renewal type argument shows

$$(4.13)\quad {}_AH_t^n(x, B) = \sum_{r=1}^{\infty} t^r P(V_A^1 + \cdots + V_A^n = r, X_r \varepsilon B \mid X_0 = x).$$

(See Chung [2] where the details are carried out for the case of a denumerable state space E. An examination of his proof, however, reveals that it may be carried over to the case of an arbitrary state space.)

If $f \varepsilon \mathcal{B}(E)$ then for $|t| < 1$

$$\|{}_AH_t f\| = \sup_x \left| \int_A {}_AH_t(x, dy) f(y) \right|$$
$$\leqq \sup_x |f(x)|\, |t| \sum_{n=1}^{\infty} P(V_A^1 = n \mid X_0 = x) \leqq \|f\|\, |t|,$$

and thus $\|{}_AH_t\| \leqq |t| < 1$. Consequently by a fundamental theorem in the theory of linear operators (see e.g. Section 5.1 of [18]) we have for all u, $|u| \leqq 1$, that the expression in (4.11) holds, defines an analytic function of u, and has at $u = 1$ the derivative $m!\, {}_AH_t^m(I - {}_AH_t)^{-(m+1)}$. However, the relation

$$P(X_n \varepsilon B \mid X_0 = x) = \sum_{r=1}^{n} P(V_A^1 + \cdots + V_A^r = n, X_n \varepsilon B \mid X_0 = x), \quad n > 0,$$
$$= \delta_B(x), \qquad n = 0,$$

shows at once that

$$(4.14)\qquad {}_AU_t = (I - {}_AH_t)^{-1}.$$

We may now quickly establish that (4.6) holds. From the obvious relation

$$P(N_n(A) = k, Y_n(A) = r \mid X_0 = x)$$
$$= \int_A P(V_A^1 + \cdots + V_A^k = r,$$
$$X_r \varepsilon dy \mid X_0 = x) P(V_A > n - r \mid X_0 = y), \quad k > 0$$
$$= P(V_A > n \mid X_0 = x), \qquad k = 0$$

we obtain, upon taking generating functions, that

$$(4.15)\quad \sum_{n=0}^{\infty} E(s^{Y_n(A)} u^{N_n(A)} \mid X_0 = x) t^n = (I - u\, {}_AH_{st})^{-1}\, {}_AT_t(x).$$

Use of Lemma 4.4 then gives us that

$$\sum_{n=0}^{\infty} E(s^{Y_n(A)} N_n^{(m)}(A) \mid X_0 = x) t^n = m!\, {}_AH_{st}^m\, {}_AU_{st}^{m+1}\, {}_AT_t(x).$$

From Lemma 4.3 and Assumption 4.1 we then have, when $s = \exp[-\lambda(1 - t)]$,

$$\lim_{t \to 1^-} \left| \frac{{}_AH_{st}^m\, {}_AU_{st}^{m+1}\, {}_AT_t(x)}{\varphi(st)^{m+1} \pi(A)^m \int_A {}_AT_t(x) \pi(dx)} - 1 \right| = 0.$$

Consequently we have, for all $m \geqq 0$

$$(4.16) \qquad m!\, {}_AH_{st}^m\, {}_AU_{st}^{m+1}\, {}_AT_t(x) \sim m!\, \varphi(st)^{m+1}\pi(A)^m \int_A {}_AT_t(x)\pi(dx), \qquad t \to 1.$$

If we set $s = u = 1$ in (4.15) we obtain $(1 - t)^{-1} = {}_AU_t\, {}_AT_t(x)$, and if we set $s = 1$ (i.e. $\lambda = 0$), $m = 0$ in (4.16) we obtain ${}_AU_t\, {}_AT_t(x) \sim \varphi(t) \int_A {}_AT_t(x)\pi(dx)$. Thus, $(1 - t)^{-1}\, \varphi(t)^{-1} \sim \int_A {}_AT_t(x)\pi(dx)$. Substitution of this result into (4.16) yields (4.6).

Remark. The converse of Theorem 4.2 is also true. Namely, if (4.3) and Assumption 4.1 hold and if for some constants D_n we have that $(N_n(A)/D_n,\ Y_n(A)/n)$ has a nondegenerate limit law, then we must have that $\varphi(t)$ is of the form given in (4.4). This follows at once from the fact, that under a condition which includes Condition 4.1, Darling and Kac [3] demonstrate that N_n/D_n has a nondegenerate limit law only if $\varphi(t)$ is of the form given in (4.4).

If we define

$$Z_n(A) = \inf\{r > n : X_r \in A\} - n,$$

then by the use of Theorem 4.2 we may also extend Theorems 3.4 and 3.5 to the same situations in which Theorem 4.2 holds.

We shall conclude this section with several examples.

Example 1. Let E be the integers and let X_n be an irreducible, recurrent chain with states in E. Then, the Doeblin ratio theorem [2] asserts,

$$(4.17) \qquad \lim_{n\to\infty} \sum_{r=0}^{n} P^r(i, \{j\}) \Big/ \sum_{r=0}^{n} P^r(0, \{0\}) = \mu(\{j\})$$

where $\mu(\cdot)$ is the invariant measure of the chain normalized so that $\mu(\{0\}) = 1$. A simple Abelian argument then shows that if A is any finite set of states we have

$$\lim_{t\to 1^-} \sum_{n=0}^{\infty} t^n P^n(i, A) \Big/ \sum_{n=0}^{\infty} t^n P^n(0, \{0\}) = \mu(A).$$

If we choose the $\pi(\cdot)$ of Assumption 4.1 to be the above $\mu(\cdot)$ and if we choose

$$(4.18) \qquad \varphi(t) = \sum_{n=0}^{\infty} t^n P^n(0, \{0\})$$

we then have that Assumption 4.1 holds for any finite set A (the required uniformity is trivial in this case since A is finite).

From the above we see that Theorem 4.2 contains Theorem 3.1 as a special case, for it is a well-known fact [8] that the successive returns to a fixed state in a Markov chain with integer states constitute a recurrent event, and conversely, that every recurrent event may be viewed as the successive returns to 0 in some irreducible chain. (In fact Y_n' is just such a Markov chain.) This leads us to the following result.

Theorem 4.3. *Let X_n be an irreducible Markov chain with the integers as states.*

Suppose the successive returns to 0 *are a* β*-regular recurrent event for some* β, $0 < \beta < 1$. *Then, for any* r *functions* $h_i(\cdot)$ *defined on the integers such that*

$$(4.19) \qquad \sum_k |h_i(k)|\mu(\{k\}) < \infty \quad \text{and} \quad \sum_k h_i(k)\mu(\{k\}) = h_i \neq 0,$$

for any finite non-empty subset of states A, *and for any initial point* l, *we have*

$$(4.20) \qquad \begin{aligned} \lim_{n\to\infty} P\left(\sum_{j=1}^{n} h_i(X_j) \leqq x_i h_i b_n, 1 \leqq i \leqq r, Y_n(A) \leqq tn \mid X_0 = l\right) \\ = P(N \leqq \min(x_1, \cdots, x_r), Y \leqq t) \end{aligned}$$

where $b_n = \varphi(1 - 1/n)$, *and* $\varphi(t)$ *is given in* (4.18).

Proof. The above results follow at once from Theorems 2.3 and 4.2 and the familiar ergodic theorem [2] that for functions satisfying (4.19) we have, for any finite non-empty set A,

$$\lim_{n\to\infty} \sum_{l=1}^{n} h_i(X_l) \Big/ N_n(A) = h_i/\mu(A)$$

with probability one.

In particular, if X_n is the nth partial sum of independent, identically distributed, integer-valued random variables such that $EX_1 = 0$, $|E(e^{i\theta X_i})| = 1$ if and only if $\theta = 2n\pi$, and such that the X_n lie in the domain of attraction of a stable law with exponent α, for $1 < \alpha \leqq 2$ we have, by the local limit law for lattice distributions [10], that the successive returns to 0 are $1 - 1/\alpha$ regular. Thus Theorem 4.3 applies.

Another example of when Theorem 4.3 holds is the following. Let V_n be independent, identically distributed, integer-valued random variables with partial sums S_n. Define

$$T_n = (V_n + T_{n-1})^+$$

where $x^+ = \max(x, 0)$. It is easily seen that T_n is an irreducible Markov chain with the nonnegative integers as states. It is a well-known fact [16] that

$$P(T_n \leqq x \mid T_0 = 0) = P(\max(0, S_1, \cdots, S_n) \leqq x).$$

Let $Y_n(\{0\}) = Y_n$. Then, it is readily seen ([14]) that Y_n has the same distribution as the position of the last minimum amongst the sums $(S_1, S_2, \cdots, S_n)$, and thus the equivalence principle shows that Y_n has the same distribution as the number of nonpositive sums amongst the first n partial sums. We therefor have $E(Y_n - Y_{n-1}) = P(S_k \leqq 0)$, and thus the event $T_n = 0$ is β-regular if and only if

$$(4.21) \qquad \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} P(S_k \leqq 0) = \beta.$$

In particular, if the sums S_n are attracted to some stable law, or have a symmetric distribution, then (4.21) holds. Consequently whenever (4.21) holds for $\beta \neq 0, 1$ we have that Theorem 4.3 is applicable to the T_n chain.

Example 2. Let E be the real line, $\mathfrak{F}$ be the Borel subsets of E, and X_n be the nth partial sum of independent random variables each with density $p_1(x)$. Assume further that

(a) X_n are attracted to a stable law with exponent α, $1 < \alpha, \leqq 2$,

(b) for some m, we have $E|X_m|^r < \infty$ for $1 < r \leqq 2$,

(c) $EX_1 = 0$.

Then by the local limit theorem for densities [10] we have, for $p_n(x)$ the density X_n and $g(x)$ the density of the limit law, that

$$B_n p_n(x) - g(B_n^{-1}x) \to 0 \quad \text{uniformly in} \quad x, \quad -\infty < x < \infty, \tag{4.22}$$

where, for some slowly varying function $h(x)$, $B_n = n^{1/\alpha}h(n)$. Now let A be a bounded Borel set. Then since $g(x)$ is continuous [10], we have

$$g(B_n^{-1}(y - x)) - g(0) \to 0 \tag{4.23}$$

uniformly in x, y for $x, y \,\varepsilon\, A$.

In Theorem 4.2 let

$$\varphi(t) = g(0) \sum_{n=1}^{\infty} t^n B_n^{-1} \tag{4.24}$$

and choose $\pi(\cdot)$ to be Lebesgue measure. Then,

$$\begin{aligned} |{}_AU_t(x, dy) - \varphi(t)\, dy| \leqq \sum_{n=1}^{\infty} |B_n p_n(y - x) - g(B_n^{-1}(y - x))| t^n B_n^{-1} \\ + \sum_{n=1}^{\infty} t^n B_n^{-1} |g(B_n^{-1}(y - x)) - g(0)|. \end{aligned}$$

Since $\varphi(t) \to \infty$ as $t \to 1^-$, we have from (4.22) and (4.23) that,

$$|{}_AU_t(x, dy) - \varphi(t)\, dy| / \varphi(t) \to 0,$$

and uniformly so for $x, y \,\varepsilon\, A$.

Finally, a familiar Abelian theorem ([4], p. 460) shows that

$$\varphi(t) \sim g(0)\Gamma(1 - 1/\alpha)h(1/1 - t))^{-1}(1 - t)^{(1/\alpha)-1}, \qquad t \to 1^-,$$

and thus we have that Theorem 4.2 is applicable in this case. We state this fact a bit more generally as follows:

THEOREM 4.4. *Let X_n be the* nth *partial sum of independent random variables with a common density $p_1(x)$, and suppose Conditions* (a)–(c) *hold. Then for any r functions $h_i(x)$ in $L_i(-\infty, \infty)$ such that $\int h_i(x)\, dx = h_i \neq 0$, we have, for any bounded Borel set A with non-zero Lebesgue measure,*

$$\lim_{n\to\infty} P\left(\sum_{k=1}^{n} h_i(X_k) \leqq x_i\, h_i B_n\, , Y_n(A)/n \leqq t\right)$$
$$= P(N \leqq \min(x_1, x_2, \cdots, x_r), Y \leqq t).$$

PROOF. This follows from Theorem 4.2 and the ergodic theorem of Harris and Robbins [12] which asserts that

$$\lim_{n\to\infty} \sum_{k=1}^{n} h_i(X_k) \Big/ N_n(A) = h_i/\pi(A)$$

with probability one.

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