A TEST FOR REALITY OF A COVARIANCE MATRIX IN A CERTAIN COMPLEX GAUSSIAN DISTRIBUTION

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1. Problem. Let $\mathbf{X}_1 : p \times n$ and $\mathbf{X}_2 : p \times n$ be real random variables having the joint density function

$$(1) \qquad (2\pi)^{-pn}|\mathbf{\Sigma}_0|^{-\frac{1}{2}n} \exp\{-\tfrac{1}{2} \operatorname{tr} \mathbf{\Sigma}_0^{-1}(\mathbf{X} - \boldsymbol{\nu})(\mathbf{X} - \boldsymbol{\nu})'\}, \qquad (-\infty \leqq \mathbf{X} \leqq \infty),$$

where $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}$, $\quad \mathbf{\Sigma}_0 = \begin{pmatrix} \mathbf{\Sigma}_1 & -\mathbf{\Sigma}_2 \\ \mathbf{\Sigma}_2 & \mathbf{\Sigma}_1 \end{pmatrix}$, $\quad \boldsymbol{\nu} = \begin{pmatrix} \boldsymbol{\mu}_1 & -\boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_2 & \boldsymbol{\mu}_1 \end{pmatrix} \begin{pmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{pmatrix}$,

$\mathbf{\Sigma}_1 : p \times p$ is a real symmetric positive definite (p.d.) matrix, $\mathbf{\Sigma}_2 : p \times p$ is a real skew-symmetric matrix, $\boldsymbol{\mu}_j : p \times q$ and $\mathbf{M}_j : q \times n (j = 1, 2)$. $\mathbf{M}_j : q \times n (j = 1, 2)$ are given matrices or their joint distribution does not contain $\mathbf{\Sigma}_1$, $\mathbf{\Sigma}_2$, $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ as parameters. Then it has been shown by Goodman [2] that the distribution of the complex matrix $\mathbf{Z} = \mathbf{X}_1 + i\mathbf{X}_2$, $(i = (-1)^{\frac{1}{2}})$, is complex Gaussian and its density function is given by

$$(2) \qquad L = \pi^{-pn}|\mathbf{\Sigma}|^{-n} \exp[-\operatorname{tr} \mathbf{\Sigma}^{-1}(\mathbf{Z} - \boldsymbol{\mu}\mathbf{M})(\overline{\mathbf{Z} - \boldsymbol{\mu}\mathbf{M}})']$$

where $\mathbf{\Sigma} = \mathbf{\Sigma}_1 + i\mathbf{\Sigma}_2$ is hermitian p.d., $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + i\boldsymbol{\mu}_2$ and $\mathbf{M} = \mathbf{M}_1 + i\mathbf{M}_2$. In this paper, we consider the problem of testing the independence of $\mathbf{X}_1$ and $\mathbf{X}_2$ sets of variates, i.e. testing the hypothesis of the reality of $\mathbf{\Sigma}$ as

$$(3) \qquad H_0(\mathbf{\Sigma}_2 = \mathbf{0}),$$

against the alternative that $H(\mathbf{\Sigma}_2 \neq \mathbf{0})$.

2. Solution. In this section, we shall derive the likelihood ratio criterion and propose two other test procedures.

Under the alternative hypothesis H, it has been shown [3] that

$$(4) \qquad (\operatorname{Max}_H L) = \pi^{-pn}|\boldsymbol{\psi}|^{-n} \exp(-np),$$

where if $\boldsymbol{\beta} = \mathbf{Z}\overline{\mathbf{M}}'(\mathbf{M}\overline{\mathbf{M}}')^{-1}$

$$(5) \qquad \boldsymbol{\psi} = n^{-1}(\mathbf{Z} - \boldsymbol{\beta})(\overline{\mathbf{Z} - \boldsymbol{\beta}})' = n^{-1}\mathbf{Z}[\mathbf{I} - \overline{\mathbf{M}}'(\mathbf{M}\overline{\mathbf{M}}')^{-1}\mathbf{M}]\overline{\mathbf{Z}}'$$
$$= n^{-1}(\mathbf{S}_1 + i\mathbf{S}_2), \quad \text{say}.$$

Then $\mathbf{S}_1 : p \times p$ is real symmetric p.d. and $\mathbf{S}_2 : p \times p$ is real skew-symmetric.

The maximum likelihood estimates of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ under H_0 are obtained by the same technique given in [3] and they are

$$\text{estimate of } \boldsymbol{\mu} \text{ under } H_0 = \hat{\boldsymbol{\mu}},$$

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and

$$\text{estimate of } \mathbf{\Sigma} \text{ under } H_0 = \text{Real part of } \boldsymbol{\psi} = n^{-1}\mathbf{S}_1 .$$

Hence, we get

$$(\text{Max}_{H_0} L) = \pi^{-pn}|n^{-1}\mathbf{S}_1|^{-n} \exp(-np). \tag{6}$$

Using (4) and (6), we get the likelihood ratio statistic Λ given by

$$\Lambda = [(\text{Max}_{H_0}L)/(\text{Max}_H L)]^{1/n} = |\mathbf{S}_1 + i\mathbf{S}_2|/|\mathbf{S}_1|. \tag{7}$$

Noting the following result given by Goodman [2],

$$\left|\begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}\right| = |\mathbf{A} + i\mathbf{B}|^2, \tag{8}$$

we can write Λ as

$$\Lambda = |\mathbf{I} + i\mathbf{Q}| = |\mathbf{I} + \mathbf{Q}^2|^{\frac{1}{2}} = |I - \mathbf{Q}\mathbf{Q}'|^{\frac{1}{2}} \tag{9}$$

where

(10) $\mathbf{Q} = \mathbf{S}_1^{-\frac{1}{2}}\mathbf{S}_2\mathbf{S}_1^{-\frac{1}{2}}$ is a real skew-symmetric matrix and $\mathbf{S}_1 = (\mathbf{S}_1^{\frac{1}{2}})^2$.

Then, the likelihood ratio criterion is

$$\text{reject } H_0 \text{ if } \Lambda < \lambda_1 \tag{11}$$

where λ_1 satisfies $\Pr(\Lambda < \lambda_1 \mid H_0) = \alpha$.

We propose here two other test procedures whose critical regions are as follows

$$\text{tr} = \text{tr}\,[\mathbf{S}_2\mathbf{S}_1^{-1}\mathbf{S}_2'(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_1^{-1}\mathbf{S}_2')^{-1}] > \lambda_2 \tag{12}$$

and

$$C_{\max} = \text{max. ch. root of } [\mathbf{S}_2\mathbf{S}_1^{-1}\mathbf{S}_2'(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_1^{-1}\mathbf{S}_2')^{-1}] > \lambda_3 \tag{13}$$

where $S_2' = -S_2$, and

$$\Pr(\text{tr} > \lambda_2 \mid H_0) = \Pr(C_{\max} > \lambda_3 \mid H_0) = \alpha. \tag{14}$$

3. Distribution of Λ under H_0. It has been shown by Khatri [3] that the distribution of $\mathbf{S} = \mathbf{S}_1 + i\mathbf{S}_2$ is complex Wishart $(\mathbf{S}; p, n - q, \mathbf{\Sigma})$. Hence under H_0, we write the joint density function of $\mathbf{S}_1$ and $\mathbf{S}_2$ as

$$\{\Gamma_p(n - q)\}^{-1}|\mathbf{\Sigma}_1|^{-(n-q)}|\mathbf{S}_1 + i\mathbf{S}_2|^{n-q-p} \exp[-\text{tr}\,\mathbf{\Sigma}_1^{-1}\mathbf{S}_1], \tag{15}$$

where $\Gamma_p(n - q) = \pi^{\frac{1}{2}p(p-1)}[\prod_{j=1}^{p} \Gamma(n - q - j + 1)]$ and $\mathbf{S}_1 + i\mathbf{S}_2$ is hermitian p.d.

Now, in (15), we use the transformation $\mathbf{Q} = \mathbf{S}_1^{-\frac{1}{2}}\mathbf{S}_2\mathbf{S}_1^{-\frac{1}{2}}$. The Jacobian of the transformation is $J(\mathbf{S}_2 ; \mathbf{Q}) = |\mathbf{S}_1|^{\frac{1}{2}(p-1)}$, $(p > 1)$. Then, it is easy to verify that $\mathbf{S}_1$ and $\mathbf{Q}$ are independently distributed under H_0, the distribution of $\mathbf{S}_1$ is real Wishart $(\mathbf{S}_1 ; 2(n - q), p, \mathbf{\Sigma}_1)$ and the density function of $\mathbf{Q}$ is

$$c|\mathbf{I} + i\mathbf{Q}|^{n-p-q} \quad \text{or} \quad c|\mathbf{I} - \mathbf{Q}\mathbf{Q}'|^{\frac{1}{2}(n-p-q)}, \qquad (p > 1), \tag{16}$$

where $\mathbf{Q}$ is real skew-symmetric, $\mathbf{I} - \mathbf{Q}\mathbf{Q}'$ is p.d. and

(17) $$c = \pi^{-\frac{1}{4}p(p-1)}[\prod_{j=1}^{p} \{\Gamma(n - q - \tfrac{1}{2}j + \tfrac{1}{2})/\Gamma(n - q - j + 1)\}].$$

From (16) and (17), it is easy to see that

(18) $$E(\Lambda^r) = \prod_{j=1}^{p} \left[\frac{\Gamma(n - q - \frac{1}{2}j + \frac{1}{2})\Gamma(n - q + r - j + 1)}{\Gamma(n - q - j + 1)\,\Gamma(n - q + r - \frac{1}{2}j + \frac{1}{2})} \right],$$

or (18) can be rewritten as

(19) $$E(\Lambda^r) = \prod_{j=1}^{t'} \left[\frac{\Gamma(n - q - j + \frac{1}{2})\Gamma(n - q - t + r - j + 1)}{\Gamma(n - q - t - j + 1)\Gamma(n - q + r - j + \frac{1}{2})} \right]$$

where

(20) $$t' = t = \tfrac{1}{2}p \text{ if } p \text{ is even, and } t' = t - 1 \text{ and } t = \tfrac{1}{2}(p + 1) \text{ if } p \text{ is odd } (>1).$$

Now let us consider t' independent real Beta variables $\omega_j (j = 1, 2, \cdots, t')$ given by

(21) $$[\Gamma(n - q - j + \tfrac{1}{2})/\{\Gamma(t - \tfrac{1}{2})\Gamma(n - q - t - j + 1)\}] \cdot \omega_j^{n-q-t-j}(1 - \omega_j)^{t-\frac{3}{2}}, \ (0 \leqq \omega_j \leqq 1).$$

Then it is easy to verify that

(22) $$E(\Lambda^r) = \prod_{j=1}^{t'} E(\omega_j^r) = E(\prod_{j=1}^{t'} \omega_j)^r.$$

Since the range of Λ is finite and it satisfies the moment relation (22), the distribution of Λ is the same as that of the product of t' independent real Beta variates ω_j given by (21).

Using the results of (5.3) of Khatri [3], it can be easily verified for large values of $2(n - q) - p - \frac{1}{2} = m$ that

(23) $$\Pr(-m \log \Lambda \leqq \xi) = \Pr(\chi_f^2 \leqq \xi) + \gamma_2 m^{-2}[\Pr(\chi_{f+4}^2 \leqq \xi) - \Pr(\chi_f^2 \leqq \xi)] + O(m^{-3})$$

where

(24) $$m = 2(n - q) - p - \tfrac{1}{2}, f = \tfrac{1}{2}p(p - 1) \text{ and } \gamma_2 = p(p - 1)\{p^2 + (p - 1)^2 - 8\}/48.$$

4. Distribution of ch. roots of QQ' or $S_1^{-1}S_2'S_1^{-1}S_2$ under H_0.

LEMMA 1. *Let a random matrix* $\mathbf{Q}$: $p \times p$ *be skew-symmetric. Then there exists a real orthogonal matrix* $\mathbf{\Delta}$: $p \times p = (\delta_{jj'})$ *such that*

(25) $$\mathbf{Q} = \mathbf{\Delta W \Delta'}$$

where $$\mathbf{W} = \text{diag.}\left[\begin{pmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega_2 \\ -\omega_2 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & \omega_t \\ -\omega_t & 0 \end{pmatrix}\right] \quad \textit{if } p = 2t$$

$$= \text{diag.}\left[\begin{pmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega_2 \\ -\omega_2 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & \omega_t \\ -\omega_t & 0 \end{pmatrix}, 0\right] \textit{if } p = 2t + 1$$

with $\omega_1^2 > \omega_2^2 > \cdots > \omega_t^2$. *The nonzero ch. roots of* Q *are* $i\omega_j, -i\omega_j (j = 1, 2, \cdots, t)$ *and so* $\omega_j^2 (j = 1, 2, \cdots, t)$ *(each repeated twice are the nonzero ch. roots of* $\mathbf{QQ}'$ *or* $\mathbf{Q}'\mathbf{Q}$*).*

For proof, we may refer to [1] p. 3.

We note that in $\mathbf{Q}$ there are $\frac{1}{2}p(p-1)$ random variables, while the apparent number of variables on the right of (25) are $\frac{1}{2}p(p-1)$ in $\mathbf{\Delta}$ and t in $\mathbf{W}$. We may note that when $p = 2$, $\mathbf{\Delta}$ can be any real orthogonal matrix and so it can be taken as $\mathbf{I}$. This shows that there must be $\{\frac{1}{2}p(p-1) - t\}$ only random variables in $\mathbf{\Delta}$. Hence in order to establish one to one relation between random variables of (25), we shall assume that $\{\frac{1}{2}p(p+1) + t\}$ elements in $\mathbf{\Delta}$ will satisfy $\mathbf{\Delta}'\mathbf{\Delta} = \mathbf{I}$ and t elements will not be random. We may take these fixed t elements as zero. With these remarks, we shall prove the following lemma.

LEMMA 2. *The Jacobian of the transformation given in* (25) *is*

$$(26)\qquad \begin{aligned} J(\mathbf{Q}; \mathbf{\Delta}, \mathbf{W}) &= [\textstyle\prod_{k=1}^{t-1} \prod_{j=k+1}^{t} (\omega_j^2 - \omega_k^2)^2] J(\mathbf{\Delta}) \text{ if } p = 2t \\ &= (\textstyle\prod_{j=1}^{t} \omega_j^2)[\prod_{k=1}^{t-1} \prod_{j=t+1}^{t} (\omega_j^2 - \omega_k^2)^2] J(\mathbf{\Delta}) \\ &\qquad\qquad \text{if } p = 2t+1 \end{aligned}$$

where $J(\mathbf{\Delta})$ *is a function of the elements of* $\mathbf{\Delta}$.

PROOF. Taking the differential of $\mathbf{Q} = \mathbf{\Delta W \Delta}'$ and writing $\mathbf{R}^*$ the differential of $\mathbf{R}$, we get

$$(27)\qquad \mathbf{\Delta}'\mathbf{Q}^*\mathbf{\Delta} = \mathbf{\Delta}'\mathbf{\Delta}^*\mathbf{W} + \mathbf{W}\mathbf{\Delta}'^*\mathbf{\Delta} + \mathbf{W}^*.$$

Let $\mathbf{B} = \mathbf{\Delta}'\mathbf{Q}^*\mathbf{\Delta}$ and $\mathbf{A} = \mathbf{\Delta}'\mathbf{\Delta}^*$. Then $\mathbf{A}$ is a real skew-symmetric matrix in $[\frac{1}{2}p(p-1) - t]$ random elements, on account of $\mathbf{\Delta}^*$ having $[\frac{1}{2}p(p-1) - t]$ random elements. Hence, we shall assume that $a_{2j-1,2j} (j = 1, 2, \cdots, t)$ can be determined in terms of the remaining elements of $\mathbf{A}$. Then (27) can be written as

$$(28)\qquad \mathbf{B} = \mathbf{AW} - \mathbf{WA} + \mathbf{W}^*.$$

That is, we have

$$(28')\qquad \begin{aligned} b_{2j-1,2j} &= \omega_j^*, \\ b_{2k-1,2j-1} &= -\omega_j a_{2k-1,2j} - \omega_k a_{2k,2j-1}, \\ b_{2k,2j-1} &= \omega_j a_{2k,2j} + \omega_k a_{2k-1,2j-1}, \\ b_{2k-1,2j} &= \omega_j a_{2k-1,2j-1} - \omega_k a_{2k,2j}, \\ \text{and}\qquad b_{2k,2j} &= \omega_j a_{2k,2j-1} + \omega_k a_{2k-1,2j}, \end{aligned}$$

for $j > k$; $j = 1, 2, \cdots, t$; $k = 1, 2, \cdots, t-1$ if $p = 2t$, and if $p = 2t+1$, $j > k$; $j = 1, 2, \cdots, t+1$; $k = 1, 2, \cdots, t$ and $\omega_{t+1} = 0$. Note that (28′) is free from $a_{2j-1,2j} (j = 1, 2, \cdots, t)$ elements and so justifies the remarks made before Lemma 2.

Now using the Jacobian theorem on conditional transformation, we get

$$(29)\qquad \begin{aligned} J(\mathbf{Q}; \mathbf{\Delta}, \mathbf{W}) &= J(\mathbf{Q}^*; \mathbf{\Delta}^*, \mathbf{W}^*) = J(\mathbf{Q}^*; \mathbf{B}) J(\mathbf{B}; \mathbf{A}, \mathbf{W}^*) J(\mathbf{A}; \mathbf{\Delta}^*) \\ &= J(\mathbf{B}; \mathbf{A}, \mathbf{W}^*) J(\mathbf{A}; \mathbf{\Delta}^*) \end{aligned}$$

for $J(\mathbf{Q}^*; \mathbf{B}) = |\mathbf{\Delta}|^{p-1} = 1$. Let $J(\mathbf{A}; \mathbf{\Delta}^*) = J(\mathbf{\Delta})$, and from (28′), we can show that

$$(30)\quad \begin{aligned} J(\mathbf{B}; \mathbf{A}, \mathbf{W}^*) &= [\textstyle\prod_{k<j=1}^{t} (\omega_j^2 - \omega_k^2)^2] && \text{if } p = 2t \\ &= (\textstyle\prod_{k=1}^{t} \omega_k^2)[\prod_{k<j=1}^{t} (\omega_j^2 - \omega_k^2)^2] && \text{if } p = 2t + 1. \end{aligned}$$

Using (30) in (29), we get Lemma 2.

Now, we shall derive the joint distribution of the ch. roots $\mathbf{S}_1^{-1}\mathbf{S}_2'\mathbf{S}_1^{-1}\mathbf{S}_2$ or $\mathbf{QQ}'$. The density function of $\mathbf{Q}$, (a skew-symmetric matrix) is given by (16). We apply the transformation given in Lemma 1 and its Jacobian is given by (26). Let $\lambda_j = \omega_j^2 (j = 1, 2, \cdots, t)$, which are the nonzero ch. roots (each repeated twice) of $\mathbf{QQ}'$. Then integrating over $\mathbf{\Delta}$, we get the joint density function of $1 \geqq \lambda_1 \geqq \cdots \geqq \lambda_t > 0$ as

$$(31)\quad c_1[\textstyle\prod_{j=1}^{t} \{\lambda_j^{-\frac{1}{2}}(1 - \lambda_j)^{n-p-q}\}][\prod_{k=1}^{t-1} \prod_{j=k+1}^{t} (\lambda_j - \lambda_k)^2] \qquad \text{if } p = 2t,$$

and

$$(32)\quad c_2[\textstyle\prod_{j=1}^{t} \{\lambda_j^{\frac{1}{2}}(1 - \lambda_j)^{n-p-q}\}][\prod_{k=1}^{t-1} \prod_{j=k+1}^{t} (\lambda_j - \lambda_k)^2] \qquad \text{if } p = 2t + 1,$$

where c_1 and c_2 are constants. Comparing (31) and (32) with the distribution of the ch. roots of a hermitian p.d. given in Section 7 of [3], we find the values of c_1 and c_2 as

$$(33)\quad \begin{aligned} c_1 = \textstyle\prod_{j=1}^{t} [\Gamma(n - q - j + \tfrac{1}{2})/\{\Gamma(n - q - t - j + 1) \\ \cdot\Gamma(t - j + \tfrac{1}{2})\Gamma(t - j + 1)\}] \end{aligned}$$

and

$$(34)\quad \begin{aligned} c_2 = \textstyle\prod_{j=1}^{t} [\Gamma(n - q - j + \tfrac{1}{2})/\{\Gamma(n - q - t - j) \\ \cdot\Gamma(t - j + \tfrac{1}{2})\Gamma(t - j + 1)\}]. \end{aligned}$$

The distribution of tr $\mathbf{QQ}'(\mathbf{I} - \mathbf{QQ}')^{-1} = \sum_{j=1}^{t} [\lambda_j(1 - \lambda_j)]$ and λ_1 can also be obtained under H_0 with the help of (31) and (32).

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