ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Central Regional meeting, Lincoln, Nebraska, April 1–3, 1965. Additional abstracts will appear in future issues.)

1. Tables of Percentage Points of the Multivariate t Distribution. J. V. Armitage and P. R. Krishnaiah, Wright-Patterson Air Force Base, Ohio. (By title)

Let $x_1, x_2, \cdots, x_p$ be distributed jointly as a p-variate normal with mean vector $\mathbf{\mu}$ and covariance matrix $\Sigma = \sigma^2(\rho_{ij})$ where $\rho_{ii} = 1$ for $i = 1, 2, \cdots, p$. Also let s^2/σ^2, where $E(s^2) = \sigma^2$, be a chi-square variate with m degrees of freedom distributed independently of $x_1, x_2, \cdots, x_p$. Then the joint distribution of $t_1, t_2, \cdots, t_p$, where $t_i = x_i m^{\frac{1}{2}}/s$ for $i = 1, 2, \cdots, p$, is known to be a central or non-central p-variate t distribution with m degrees of freedom according as $\mathbf{\mu} = \mathbf{0}$ or $\mathbf{\mu} \neq \mathbf{0}$. Dunnett and Sobel (*Biometrika*, **41,** 153–169) tabulated the upper percentage points and the probability integral of the bivariate t distribution when $\rho_{12} = \pm\frac{1}{2}$. Dunnett (*J. Amer. Statist. Assoc.*, **50,** 1091–1121) and Gupta and Sobel (Ann. Math. Statist., **28** 957–967) constructed tables for the upper percentage points of the multivariate t distribution for certain values of p and m when $\rho = \frac{1}{2}$ and $\rho_{ij} = \rho$ for $i \neq j = 1, 2, \cdots, p$. In the present paper, upper 10%, 5%, 2.5% and 1% points of the multivariate t distribution are tabulated for $p = 1(1)10$, $m = 5(1)40$ and $\rho = 0.0(0.05)0.9$. Various applications of this distribution are also discussed.

2. The Compound Weibull Distribution. Satya D. Dubey, Procter and Gamble Co.

In this paper a compound Weibull distribution has been derived by compounding a 2-parameter Weibull distribution with a gamma distribution. The resulting compound distribution includes the Burr's distribution [*Ann. Math. Statist.* (1942) **13** 215–232; *Ind. Quality Control* (1963) **1** 18–26] and the Lomax's distribution [*J. Amer. Statist. Assoc.* (1954) 847–852] as special cases. The characteristic function and the expression for the kth moment of the compound distribution have been derived. The paper contains expressions for conditional expectation, conditional variance and the covariance between conditional Weibull and the gamma random variables. The expression for its conditional expectation is more general than the corresponding expression for the compound gamma case, considered by the present author elsewhere. Furthermore, the parameters of the compound Weibull distribution have been expressed in terms of the functions of the relevant moments. The compound 2-parameter Weibull distribution has been extended to the compound 3-parameter Weibull distribution. Here the shape parameter of the Weibull distribution has been expressed as functions of the moments of its location and scale parameters.

3. Compound Gamma, Beta, and F Distribution. Satya D. Dubey, Procter and Gamble Co. (By title)

In this paper a compound gamma distribution has been derived by compounding a gamma distribution with another gamma distribution. The resulting compound gamma distribution has been reduced to the Beta distributions of the first kind and the second kind and to the F distribution by suitable transformations. The above compound gamma distribution includes the Lomax's distribution (*J. Amer. Statist. Assoc.* (1954) 847–852) as a special case. The expressions for its kth moment and its characteristic function have been derived. The moment estimators of two of its parameters are explicitly given which

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tend to a bivariate normal distribution. The paper contains expressions for a probability density function of a bivariate gamma distribution, its conditional expectation, conditional variance and the correlation coefficient between two random variables. Finally, all the parameters of the compound gamma distribution have been explicitly expressed in terms of the functions of the moments of the functions of the random variables in two different ways. The compound gamma distribution includes several other compound distributions as special cases.

4. Characterization of Normal and Generalized Truncated Normal Populations Using Order Statistics. ZAKKULA GOVINDARAJULU, Case Institute of Technology. (By title)

Many contributions have been made to the problem of characterizing the normal distribution, using the property of independence of sample mean and the sample variance, maximum likelihood, etc. In this paper, using certain identities among the product (linear) moments of order statistics in a random sample, the generalized truncated (both from below and above) normal distributions, the negative normal and the positive normal distributions are characterized in the class of absolutely continuous distributions. In Theorem 3.5, the normal distribution is characterized in the class of absolutely continuous distributions having mean zero. Bennett's {1} characterization of the normal distribution (Asymptotic properties of ideal linear estimators. Unpublished Ph.D. thesis, Univ. Michigan) is a special case of our Theorem 3.5, namely Corollary 3.5.2.

5. An Optimum Solution of a Three Decision Problem Involving Classification. AKIO KUDÔ and KOJI SAKAGUCHI, Kyushu University and Iowa State University, and Kyushu University.

In this paper, we are concerned with a three decision problem. We have two normal populations with a common variance, from which independent samples are drawn. We have another observation, which is assumed to have come from one of these two populations. We want to decide among— H_0:the means of the two populations are identical, or they are different but H_1:the observation is from the first population or H_2:from the second. Under the conditions (1) the probability of choosing H_0 when H_0 is correct is $1 - \alpha$, (2) the decision is invariant under a certain transformation group, and (3) certain symmetricity holds true among the operating characteristic functions, the solution has been found.

6. A Group-Testing Problem (Preliminary Report). S. KUMAR, University of Minnesota.

The problem is to classify each of the N given units into one of the three disjoint categories by means of group testing. We shall call the three categories good, mediocre and defective. In group testing, a set of x units is tested simultaneously as a group with one of the three possible outcomes: (i) all the x units are good, (ii) at least one of the x units is mediocre and none is defective, (iii) at least one of the x units is defective. It is assumed that the N units can be represented by independent trinomial random variables with probability q_1, q_2 and $q_3 = 1 - q_1 - q_2$ of being good, mediocre and defective, respectively. The problem is to devise a procedure, for known values of q_1, q_2 and q_3, which minimizes the expected number of tests $E(T)$ required to classify all the N units as good, mediocre or defective. A procedure, which is optimal in a certain class of procedures, is proposed. Furthermore, we prove the following THEOREM: *A necessary and sufficient condition that the optimal group test plan among all procedures is unique and tests one unit at a time is that*

$$q_1 < \tfrac{1}{2}\{[5q_2^2 - 6q_2 + 5]^{\frac{1}{2}} - q_2 + 1\}. \quad (1)$$

If the inequality is reversed in (1), then $ET < N$ for the optimal group test plan and if $\leqq$ holds then $ET = N$. Some further generalizations are under consideration.

7. A Class of Ranking and Selection Procedures (Preliminary Report). DESU M. MAHAMUNULU, University of Minnesota.

Consider $k \geqq 2$ populations π_i on the real line with c d f $F(x; \theta_i)$, $(i = 1, \cdots, k)$, where $\theta_i \varepsilon \Theta$. Let $\theta_{[1]} \leqq \theta_{[2]} \leqq \cdots \leqq \theta_{[k]}$ be the unknown ordered values and suppose $\theta_{[k-t]} < \theta_{[k<t+1]}$. An experimenter's goal is to select a subset of $s(<k)$ populations (s fixed) which contains the populations with the t largest θ-values where $t \leqq s$. A random sample of size n is available from each one of the k populations. The experimenter specifies two positive constants δ^* and P^* where $\binom{k-t}{k-s}/\binom{k}{s} < P^* < 1$ and he desires to have a procedure R such that $P(CS \mid R) \geqq P^*$ whenever $d(\theta_{[k-t+1]}, \theta_{[k-t]}) \geqq \delta^*$; here CS denotes correct selection and $d(\theta_1, \theta_2)$ is an appropriate measure of distance in Θ. A procedure R based on real-valued statistics T_i, $(i = 1, \cdots, k)$, is used which states that the selected subset consists of the populations which give the s largest T-values. Assuming that the resulting distributions $G(\cdot, \theta)$ of T is a stochastically increasing family we prove the THEOREM: *$P(CS \mid R)$ is a nonincreasing function of $\theta_{[\alpha]}$, $(\alpha = 1, \cdots, k - t)$, and a nondecreasing function of* $\theta_{[\beta]}$, $(\beta = k - t + 1, \cdots, k)$. This is used to determine n for which the above $P(CS \mid R)$ requirement is met. Explicit equations for n are obtained when θ is (i) a location parameter (ii) a scale parameter for the distributions of T. These results are extended to the goal of selecting a subset of size $s(<t)$ which contains any s of the t populations with the t largest θ values (Bechhofer, *Ann. Math. Statist.* **25** (1954) 16–39), and also to the goals where small values of θ are of interest. Applications to different distributions $F(x, \theta)$ are considered.

8. Multivariate Continuous Parameter N-ple Markov Processes (Preliminary Report). V. S. MANDREKAR, University of Minnesota. (Introduced by G. B. Kallianpur.)

A definition of q-variate continuous parameter N-ple Markov processes (not necessarily stationary) is given as an extension of the definition of T. Hida [*Mem. Coll. Sci. Kyoto Univ.* **A33** (1960)] in the univariate case. For such purely non-deterministic process $\mathbf{x}_t$ where multiplicity M is less than the dimension we have the representation $\mathbf{x}_t = \sum_{i=1}^{N} \int_{-\infty}^{t} F_i(t)G_i(u)d\xi(u)$ where the $qN \times qN$ matrix $\{F_i(t_j)\}$ $i, j = 1, 2, \cdots, N$ is nonsingular $G_i(u)$, $(i = 1, 2, \cdots, N)$, is a "square integrable" $q \times M$-matrix-valued function with respect to the covariance matrix measure corresponding to the orthogonally scattered vector random measure ξ, and the integral is in the sense of M. Rosenberg (*Duke Math. J.* **31**). The kernel of the representation, $\sum_{i=1}^{N} F_i(t)G_i(u)$ is an extension of the Goursat kernel. Finally, it is observed that in stationary case these processes are a subclass of the processes with rational spectral density matrix.

9. A Test for Monotonicity of the Dose-Response Curve in Bioassays (Preliminary Report). PAUL E. LEAVERTON, University of Iowa.

In a quantal response bioassay situation, let the observations at each dose z_i $(i = 1, \cdots, k)$ be independent and binomially distributed with parameters P_i and n. The probabilities of a response lie on the dose-response curve $P = F(z)$, where $F(z)$ is assumed to be a continuous non-decreasing function. A test of this assumption is derived where no parametric form for the relationship $P = F(z)$ is specified. To do this, the hypothesis $H_o: P_1 \leqq P_2 \leqq \cdots \leqq P_k$ is tested against the alternative hypothesis $H_a: P_i > P_{i+m}$ for at least one i and m $(i = 1, 2, \cdots, k; m = 1, 2, \cdots, k - i)$.

10. Some Results on Tests for Poisson Processes. PETER A. W. LEWIS, IBM San Jose Research Laboratories.

It is shown that a test proposed by Barnard for Poisson processes, using certain distribution free statistics, is not consistent against renewal alternatives. However, empirical evidence is given which suggests that a modification of this test due to Durbin results in relatively powerful tests of the Poisson hypothesis. A simple test based on Durbin's modification is described, and its asymptotic relative efficiency with respect to the asymptotically most powerful test against the alternative of a renewal process with Gamma-distributed intervals is given.

11. On Orthogonal Arrays of Strength Four (Preliminary Report). ESTHER SEIDEN and RITA ZEMACH, Michigan State University.

A $k \times \lambda s^t$ matrix, k, λ, s, t positive integers, is called an orthogonal array of strength t size λs^t, k constraints s, levels if each $t \times \lambda s^t$ submatrix contains all possible $t \times 1$ column vectors with the same frequency λ. Methods of construction of such arrays are studied for $t = 4$. In the cases $s = 2$, λs^t equal 16, 32, 48, 64, 80 arrays with maximum number of constraints are obtained. For some values of $\lambda s^t > 80$ arrays are constructed but it isn't known yet whether the obtained number of constraints is the best possible. Attempts are being made to obtain bounds for k, sharper than the one known thus far.

12. On Some Multisample Permutation Tests Based on a Class of U-Statistics. PRANAB KUMAR SEN, University of California, Berkeley. (By Title)

The two-sample as well as single sample permutation tests developed so far, are mostly based on linear permutation statistics. They also relate only to the particular problems of tests of independence and the identity of locations. Here a class of U-statistics (which are MVU estimators) are employed for constructing various types of permutation tests that arise in the general case of c independent samples ($c \geqq 2$), and the properties of these tests are studied. The scope of applicability of the permutation tests has thus been extended.

13. The Estimation of the Parameter in the Stochastic Model for Phage Attachment to Bacteria. RAMESH SRIVASTAVA, Michigan State University.

Recently Gani [to be published in *Biometrics* (1965)] has considered a stochastic model for phage attachment to bacteria. Let n_{00} and ν_{00} be the number of bacteria and phages respectively at time $t = 0$. Also let $n_i(t)$ be the number of bacteria with i phages attached to them ($i = 0, \cdots, r$) and $\nu_0(t)$ be the number of free phages at time $t \geqq 0$. Then, it is shown by Gani that $P(n_0, \cdots, n_r; t) = [n_{00}!/(n_0(t))! \cdots (n_r(t))!] \prod_{j=0}^{r} (a_{0j}(t))^{n_j(t)}$ where $a_{0j}(t) = \sum_{i=0}^{j} (-1)^{j-1} \binom{r}{i} \binom{r-i}{j-i} \exp[-(r - i)\alpha\rho(t)]$ where $\rho(t) = \int_0^t \nu(t)\, dt$. The stochastic process $\mathbf{n}(t) = (n_0(t), \cdots, n_r(t))'$ is a Markov chain and depends on an unknown parameter α. In this paper, we describe a method, of the type originated by Ruben (*Amer. Math. Statist.* **34**), for estimating the parameter α and studying the asymptotic properties of the estimate. We show that this method yields a consistent estimate. Then we find a lower limit to the variance of a consistent estimate satisfying certain conditions and use our result to obtain the asymptotic efficiency of the estimate.

14. Repetitive Play in Finite Statistical Games with Unknown Distributions. J. VAN RYZIN, Argonne National Laboratory.

Consider a sequence of finite statistical games in which the parameter space $\Omega = \{1, \cdots, m\}$, action space $A = \{1, \cdots, n\}$ and loss matrix $\|M_{ij}\|$ are identical in each game. For the kth game, let $\theta_k \varepsilon \Omega$ denote player I's move and t_k be player II's move, which can depend on I's previous $k - 1$ moves and the random variable $\mathbf{X}_k = (X_1, \cdots, X_k)$, where X_i is distributed as P_{θ_i} and the X_i's are an independent sequence. However, the class $\mathcal{P} = \{P_i, i \varepsilon \Omega\}$ is unknown to II. Assume that each member of $\mathcal{P}$: (i) is a discrete distribution on the denumerable set U and $\sum_{u \varepsilon U} P_i^{\frac{1}{2}}(u) < \infty$, $i = 1, \cdots, m$; or (ii) has an r-dimensional Lebesgue density. For sequences $\mathbf{t} = \{t_1, t_2, \cdots\}$ and $\boldsymbol{\theta} = \{\theta_1, \theta_2, \cdots\}$, let $M_N = M_N(\mathbf{t}, \boldsymbol{\theta})$ be the average payoff in the first N games and $R_N = E_N M_N$ (E_N denoting expection on $\mathbf{X}_N$). Let $\beta(\xi_N)$ be the Bayes risk in the component game of the N-stage empirical distribution of I's moves on Ω. Then, for (i) a class of sequences $\mathbf{t}$ is given for which $N^{\frac{1}{2}}\{R_N - \beta(\xi_N)\} = O(1)$ and $N^{\frac{1}{2}}(\log N)^{-1}\{M_N - \beta(\xi_N)\} \to 0$ in probability, uniformly in $\boldsymbol{\theta}$. For (ii), a class of sequences $\mathbf{t}$ is given for which $R_N - \beta(\xi_N) \to 0$ and $M_N - \beta(\xi_N) \to 0$ in probability for each $\boldsymbol{\theta}$. Uniform (in $\boldsymbol{\theta}$) results for (ii) are given under additional restrictions on $\mathcal{P}$.

(Abstracts of papers to be presented at the Eastern Regional meeting, Tallahassee, Florida, April 29–May 1, 1965. Additional Abstracts will appear in future issues.)

1. Classification of Binary Random Patterns. KENNETH ABEND and LAVEEN KANAL, Philco Applied Research Laboratory and University of Pennsylvania.

In various pattern recognition problems such as classification of photographic data' preprocessing operations result in a two-dimensional array of binary random variables. An optimal recipe for classifying such patterns is described. It combines the generation of a "best' approximating distribution based on lower order marginals to approximate a joint probability function, with the use of an orthogonal expansion. The unwieldy nature of the optimal recipe leads to an assumption of Markovian dependence. It is shown how this assumption can be used to account for dependence of a variable on a set of K spatially nearest neighbors. The result is a classification function with a small number of parameters which nevertheless takes care of a type of dependence often associated with the problems considered. Together with the method described for estimating missing data, the result represents a feasible approach.

2. An Asymptotic Expansion for the Distribution of the Latent Roots of the Estimated Covariance Matrix. GEORGE A. ANDERSON, Trinity College.

The distribution of the latent roots depends on a definite integral over the group of orthogonal matrices. This integral defines a function of the latent roots of both the covariance matrix and the estimated covariance matrix. With an integration procedure involving first a substitution and then an expansion of the resulting integrand the first three terms of an expansion for the integral are found. This expansion is given in increasing powers of n^{-1}, where n is the sample number less one. A numerical example is given for the distribution of the latent roots using the expansion for the definite integral given in this paper. Improved maximum likelihood estimates for the latent roots are found and the likelihood function is considered in detail.

3. A Randomization Test for Equality of Means of Two Multivariate Populations with Common Covariance Matrix (Preliminary Report). RALPH BRADLEY and KANTI M. PATEL, Florida State University.

Consider p-variate populations Π_i having continuous cumulative distribution functions $F_i(x^{(1)}, \cdots, x^{(p)}) = F_i(x)$. Let there be n_i independent vector observations $X_{i\alpha}$ from Π_i,

$i = 1, 2$. We consider a test of the null hypothesis: $F_1 \equiv F_2$ against alternatives of the form $F_2(x) \equiv F_1(x - \theta)$ where not all elements of the vector θ are zero. The following test is proposed and discussed: Compute $B^2 = (n_1 n_2/N)(\bar{X}_{1\cdot} - \bar{X}_{2\cdot})'S^{-1}(\bar{X}_1 - \bar{X}_2)$ where $\bar{X}_i$ is the average vector $\sum_{\alpha=1}^{n_i} X_{i\alpha}/n_i$, $i = 1, 2$; $(N - 1)S = \sum_{i=1}^{2} \sum_{\alpha=1}^{n_i} (X_{i\alpha} - \bar{X}_{\cdot\cdot})(X_{i\alpha} - \bar{X}_{\cdot\cdot})'$ with $\bar{X}_{\cdot\cdot} = \sum_{i=1}^{2} \sum_{\alpha=1}^{n_i} X_{i\alpha}/N$ and $N = n_1 + n_2$. Reject the null hypothesis if B^2 is too large. Under the null hypothesis we use a vector randomization procedure to develop the null distribution of B^2 that assumes that $\binom{N}{n_1}$ partitions of the combined sample of N vectors are equally likely. It has been shown that use of B^2 is equivalent to use of the standard Hotelling T^2 for the two-sample problem. The first four moments about the origin are determined for B^2 under the vector randomization when the null hypothesis is true. The null distribution of B^2 is approximated by a beta distribution by equating the corresponding mean and variance. If we replace each $X_{i\alpha}$ by its corresponding vector of ranks, observations on each variate having been ranked separately in the combined sample, the B^2-test becomes a multivariate generalization of the symmetric two-sided Wilcoxon-Mann-Whitney test. Also, a multivariate generalization of the Wilcoxon paired-sample rank test and a nonparametric test for a problem of symmetry are discussed. The test procedures are illustrated by numerical examples.

4. Solutions to Some Two-Sided Boundary Problems for Sums of Variables with Alternating Distributions. J. CHOVER and G. YEO, University of Wisconsin and Yale University.

Let X_i, Y_i, $(i = 1, 2, \cdots)$, be mutually independent and identically distributed random variables and let $-\infty < a < c < b < \infty$ and $S_0 = c$, $S_n = S_{n-1} + X_n + Y_n$, $T_n = S_n + X_{n+1}$. An integral equation is found for the probability of the number of steps to the first passage beyond one of the boundaries a or b, e.g.

$$p_n = \Pr\{S_n > b; a \leqq S_i, T_i \leqq b, 1 \leqq i \leqq n - 1\}.$$

A solution is obtained in some special cases such as when the $\{X_i\}$ have a one-sided negative exponential distribution and the $\{Y_i\}$ take mass only in that part of the opposite half line at least $d(> 0)$ away from zero. The generating function $p(s)$ of $\{p_n\}$ is found as the ratio of two polynomials of degree at most $1 + [b/d]$. The method is applied to some waiting time and busy period problems for finite waiting room queues and finite dams; the above case may be made to correspond to a single server queue with Poisson arrivals and service times having a minimum greater than zero.

5. Some Remarks on the Lomax's Paper. SATYA D. DUBEY, Procter and Gamble Co.

It has been pointed out in this paper that the hyperbolic model of Lomax (*J. Amer. Statist. Assoc.* (1954) 847–852) is a special case of the compound gamma distribution and the compound Weibull distribution. And the exponential model of Lomax (*J. Amer. Statist. Assoc.* (1954) 847–852) fails to yield a proper probability distribution function. A theorem has been proved which provides the necessary and sufficient condition on the intensity function for the existence of the proper distribution function. In the light of the results of this theorem the exponential model of Lomax has been modified. A compound distribution for the modified Lomax's distribution has been derived. It has been shown that a natural complement of the exponential model of Lomax does not suffer from this drawback and a compound distribution for it is reduced to a logistic distribution. When the natural complement of the exponential model of Lomax is extended over the whole real line the resulting distribution is related to the extreme value distribution which when compounded with a gamma distribution results in a generalized logistic distribution.

6. A Bayes Approach for Combining Correlated Estimates. SEYMOUR GEISSER, National Institute of Arthritis and Metabolic Diseases.

A Bayes solution is supplied for an estimation problem involving a sample from a multivariate normal population having an arbitrary unknown covariance matrix, but a vector mean whose components are all equal. Assuming that a particular unnormed prior density is a convenient expression for displaying prior ignorance, it is then demonstrated that a posterior interval for this common mean can be based on Student's t distribution. If prior information can be conveniently represented by a natural conjugate prior density, the posterior interval will also depend on Student's t. An extension is made to the case of estimating the constant difference between two parallel profiles.

7. A Characterization of the Negative Exponential Distribution. ZAKKULA GOVINDARAJULU, Case Institute of Technology. (By title)

Let $X_{1,N} < X_{2,N} < \cdots < X_{N,N}$ denote a set of order statistics in a random sample of size N drawn from a population having $F(x - \theta)$ (for $x \geqq \theta$ and zero for $x < \theta$), for its continuous cumulative distribution function. Define the random variables $U_1 = X_{1,N}$, $U_2 = X_{2,N} - X_{1,N}, \cdots, U_N = X_{N,N} - X_{1,N}$. Then the following characterization theorem is proved. THEOREM. *The conditional distribution of* $U_2, \cdots, U_N$ *given* $U_1 = u_1$ *is independent of* u_1 *and* θ *if and only if* $F(x) = 1 - e^{-ax}$, *for some* $a > 0$.

8. Approximations to the Maximum Likelihood Estimator Using Grouped Data. B. K. KALE, Iowa State University.

The grouping of observations on a random variable (rv) X corresponds to a partition of real line into a finite number of disjoint intervals and every such grouping g defines a rv X_g which has a multinomial distribution. Let G be the class of all such groupings. Under some regularity conditions on the density function $f(x, \theta)$, $\theta \varepsilon \Omega$ of the rv X, Kale [*Biometrika* **51** pts. 3 and 4] proved that $I(\theta_0, \theta) = \text{lub}_{g \varepsilon G} \{I_g(\theta_0, \theta)\}$ where $I(\theta_0, \theta)$ and $I_g(\theta_0, \theta)$ denote the amount of discriminatory information (Kullback) supplied by the rv X and X_g respectively. Using this result, it is proved that there exists a sequence of groupings $\{g_K\} \varepsilon G$ such that (1) $\lim_{k\to\infty} I_{g_K}(\theta_0) = I(\theta_0)$ and (2) for each K and any $\delta > 0$, the infrimum of $I_g(\theta_0, \theta)$ over $|\theta_0 - \theta| > \delta$ is strictly positive whenever the infrimum of $I(\theta_0, \theta)$ over $|\theta_0 - \theta| > \delta$ is bounded away from zero. This implies that the sequence of multinomial distributions $\{X_{g_K}\}$ satisfy the regularity conditions of Rao [*Sankhyā* **18** 139–148] and determines a sequence of maximum likelihood estimators (mle) $\{\hat{\theta}_{g_K}\}$ of θ which converges in distribution, (for large samples at least) to $\hat{\theta}$ the mle of θ based on observations on X.

9. On Sequential Multinomial Estimation. A. KUDÔ, A. OKUMA, H. YAMATO and T. YANAGAWA, Kyushu University and Iowa State University Kyushu University and Kyushu University.

Consider the sequential multinomial estimation problem with k categories with probabilities $(p_1, p_2, \cdots, p_k)$. If the sampling scheme is closed (the sampling terminates with probability one) and multiply simple (among k point one step behind any boundary point this condition requires that only one is an accessible point), then there exists a unique unbiased estimator for the probabilities. This is a generalization of the result by Wolfowitz in binomial case. (*Ann. Math. Statist.* **17** 489–493.)

10. A Class of Wide-Sense Markov Processes. C. B. MEHR, International Business Machines Corporation.

In linear minimum-mean square-error prediction theory, when the processes under consideration are not Gaussian, the class of stochastic processes which are of practical importance are wide-sense Markov processes [Beutler, F. J., *Ann. Math. Statist.* **34** (1963)]. In general, Markov processes (with the exception of Gauss-Markov processes) are not Markov in the wide sense. Here we show a class of Markov processes (not necessarily Gaussian) which are also Markov in the wide sense. These processes have other nice properties in problems involving noise in non-linear devices.

11. On the Bivariate Moments of Order Statistics from a Logistic Distribution and Applications. B. K. SHAH, Yale University. (By title)

Let $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ be an ordered sample of n independent and identically distributed random variables from a logistic distribution whose cumulative distribution function is $[1 + \exp(-x)]^{-1}$. An easily computed formula for the expected value of the product of the lth and mth order statistics is obtained in terms of digamma and trigamma functions. In particular for $l = 1$, $m = 2$ and for any n, $E(x_{1,n}x_{2\,n}) = E(x_{2,n}^2) + n\{[\psi^{(0)}(n - 2) - \psi^{(0)}(n - 1)][\psi^{(0)}(0) - \psi^{(0)}(n - 2)] - \psi^{(1)}(n - 2)\}$, where $\psi^{(r)}(x) = [(d/dt)^{r+1} \log \Gamma(t + x + 1)]_{t=0}$ and $E(x_{i,n}^r)$ is the rth moment of the ith order statistic from a sample of size n. Various recurrence relations are obtained. Using the above results, the expected values of products of all pairs of order statistics are obtained and tabulated for sample sizes up to 20. The problem of estimating the location and the scale parameter of any logistic distribution is studied for small sample sizes up to 20. Tables are given for estimating these parameters for (i) complete samples and also for (ii) singly and doubly censored samples.

(Abstracts of papers not connected with any meeting of the Institute.)

1. Contributions to the Theory of Order Statistics. PETER J. BICKEL, University of California, Berkeley.

Let $Z_{1n} < \cdots < Z_{nn}$ be the order statistics of a sample from a population with distribution F, density f which is continuous and strictly positive on $\{x \mid 0 < F(x) < 1\}$. Necessary and sufficient conditions are given for the existence of the moments of quantiles $Z_{[n\alpha]n}$, $0 < \alpha, < 1$, and the convergence of the suitably normalized moments to those of a normal distribution. This extends results of Blom, Hotelling and Chu, Sen. A limit theorem of the Donsker type is proved for the quantile function restricted to proper closed subintervals of (0, 1) and is applied to obtain the asymptotic normality and moment convergence for linear combinations of quantiles and for linear combinations involving the extremes to an extent comparable to that of the sample mean. Some other facts of general interest are also derived.

2. Robust Inference in Some Linear Models with One Observation Per Cell (Preliminary Report). KJELL DOKSUM, University of California, Berkeley.

For the independent observations $X_{i\alpha}$, $(i = 1, \cdots, r; \alpha = 1, \cdots, n)$, assume the model $X_{i\alpha} = \nu + \xi_i + \mu_\alpha + Y_{i\alpha}$, where the Y's have the common continuous distribution F, and the ξ's, μ's and ν are constants satisfying $\sum \xi_i = \sum \mu_\alpha = 0$. Let W_{ij} denote the median of the n quantities $\{X_{i\alpha} - X_{j\alpha}; \alpha = 1, \cdots, n\}$. For inference about the contrast $\theta = \sum c_i\xi_i = \sum d_{ij}(\xi_i - \xi_j)(\sum c_i = 0)$, it is shown that procedures obtained by replacing $X_{i\cdot} - X_{j\cdot}$

in the classical procedures of analysis of variance by $Z_{ij} = W_{i\cdot} - W_{j\cdot}$, has the asymptotic relative efficiency (ARE) $e(F) = 12\sigma^2(\int f^2)^2 r/(r+1)$ to these classical procedures. Here, $\sigma^2(f)$ is the variance (density) of F. For testing $\xi_1 = \cdots = \xi_r$, the distribution-free (DF) statistic corresponding to the procedures based on the Z's (which are not DF) is $T = \sum_i (\sum_j a_{ij} S_{ij})^2$, where the a_{ij} are defined by $\sum_j a_{ij}(X_{i\cdot} - X_{j\cdot}) = X_{i\cdot} - X_{\cdot\cdot}$, $S_{ij} = U_{i\cdot} - U_{j\cdot}$, and U_{ij} = number of α's such that $(X_{i\alpha} - X_{j\alpha}) > 0$. Thus the ARE of T to the usual $\mathfrak{F}$-statistic is $e(F)$, and the *ARE* of T to the Friedman [*J. Amer. Statist. Assoc.* **32** 675–701] statistic is unity.

3. The Compound Pascal Distributions. Satya D. Dubey, Proctor and Gamble Co.

In this paper the *Pascal-Beta* and the *Pascal-Gamma* distributions have been derived by compounding the simple Pascal distribution with the Beta distribution and the Gamma distribution. These two compound distributions include Pascal-Uniform, Geometric-Beta, Geometric-Uniform, Pascal-Exponential, Geometric-Gamma and Geometric-Exponential as special cases. The methods of computing the moments of these compound distributions have been given. The moment estimators of the parameters of the Pascal-Beta distribution have been derived explicitly and the necessary condition has been pointed out for the asymptotic normality of these estimators. The essential expressions have been derived to obtain the maximum likelihood estimators of the parameters of the Pascal-Beta distribution numerically. These expressions involve polygamma functions. The moment estimators of the parameters of the Binomial-Beta distribution have been also derived explicitly. It has been pointed out that under certain conditions the Pascal-Beta distribution can be approximated by the Pascal distribution, the Geometric-Beta distribution can be approximated by the Geometric distribution, and the Binomial-Beta distribution can be approximated by the Binomial distribution. Finally, some interesting identities have been established.

4. Hyper-Efficient Estimator of the Location Parameter of the Weibull Laws. Satya D. Dubey, Procter and Gamble Co.

In this paper the author has obtained the *hyper-efficient* estimator of the location parameter of the Weibull laws when it is known *a priori* that their shape parameter falls within the semi-closed interval (0, 1]. The hyper-efficient estimator turns out to be the minimum of sample observations. Its probability density function has been derived which is shown to obey a different Weibull law. The existence of the hyper-efficient estimator of the location parameter has been further exploited toward obtaining the maximum likelihood estimators for the scale and the shape parameters of all possible Weibull laws. The asymptotic properties of these estimators are discussed and the expressions for their asymptotic variances and covariance are derived which involve polygamma functions in some cases. When the scale and the shape parameters are known, an unbiased hyper-efficient estimator of the location parameter is proposed which does not tend to a normal distribution. The results of this paper have good possibility of applications in diverse areas of human endeavor since the shape parameter of the Weibull distribution seems to fall within the semi-closed interval (0, 1] in many cases.

5. On a Result of Hoel and Levine. J. Kiefer and J. Wolfowitz, Cornell University.

Let $\{F_0, \cdots, F_{m-1}\}$ and $\{F_0, \cdots, F_m\}$ be Chebyshev systems of continuous real functions on $(-\infty, \infty)$. Let X^* be the set of $m+1$ points of maximum absolute deviation of

the best Chebyshev approximation on $[-1, 1]$ to F_m of the form $\sum_0^{m-1} c_iF_i$. A specified number of uncorrelated homoscedastic observations may be taken at levels x in $[-1, 1]$ only, with expectation $\sum_0^m \theta_iF_i(x)$, where the θ_i are unknown. As usual in the approximate design theory, a design is a probability measure ξ on $[-1, 1]$, which specifies the proportions of observations to be taken at various levels, and $V(y, \xi)$ is the variance of the BLE of the regression at $y \varepsilon (-\infty, \infty)$ when ξ is used. It is shown that the unique optimum (extrapolation) design for estimating the regression $\sum_0^m \theta_iF_i(y)$ at $y \varepsilon (-\infty, \infty) - [-1, 1]$ is supported by X^*; and that, under slight additional conditions, for y sufficiently large this design minimizes $\max_{-k \leqq t \leqq y} V(t, \xi)$, with k constant. These results generalize recent ones of Hoel and Levine for the case $F_i(y) = y^i$. The $(m + 1)$-dimensional set of vectors $(a_0, a_1, \cdots, a_m)$ for which $\sum_0^m a_i\theta_i$ is optimally estimated by a design supported by X^* is also characterized.

6. Dynamic Programming for Countable Action Spaces (Preliminary Report). ASHOK MAITRA, Mathematisch Centrum, Amsterdam.

We consider a system with a finite state space $S = \{1, 2, \cdots, S\}$. The states of S are labeled by s or s'. Once a day, we observe the current state s of the system and then choose an action a from a countable set A of actions. As a result of this we receive an immediate income $i(s, a)$ and the system moves to a new state s' with probability $q(s' \mid s, a)$. Assume that there is a finite $M > 0$ such that $|i(s, a)| \leqq M$ for all $s \varepsilon S$ and $a \varepsilon A$. Further there is specified a discount factor β, $0 \leqq \beta < 1$, so that the value of unit income n days in the future is β^n. For a fixed β, denote by $V(\pi)$ the vector of total expected reward from the policy π. It is proved that given $\epsilon > 0$, there is a stationary policy $f_\epsilon^{(\infty)}$ such that $V(f_\epsilon^{(\infty)}) \geqq V(\pi) - \epsilon e$ for all policies π, where e is the $S \times 1$ column vector with all coordinates unity. (This result remains true for countable S). In the case of $\beta = 1$, we restrict ourselves to stationary policies and take as our criterion the average reward per transition. Denote by $x(f)$ the vector of average reward per transition from the stationary policy $f^{(\infty)}$. We prove that given $\epsilon > 0$ there exists a policy $g_\epsilon^{(\infty)}$ such that $x(g_\epsilon) \geqq x(f) - \epsilon e$ for all stationary policies $f^{(\infty)}$.

7. On Stable Transformations. ASHOK MAITRA, Mathematisch Centrum, Amsterdam.

Let $(\Omega, \mathfrak{A}, P)$ be a probability triple and let T be a measure preserving transformation of Ω into itself. Following Rényi (*Sankhyā Ser. A* **25** (1963) 293–302), we will say that T is *stable* if for every $A \varepsilon \mathfrak{A}$, the sequence $\{T^{-n}A, n = 1, 2, \cdots\}$ of sets is stable, that is, if for every $A, B \varepsilon \mathfrak{A}$, $\lim_{n\to\infty} P(T^{-n}A \cap B)$ exists. Criteria are given for a measure-preserving transformation to be stable. The concept of a stable transformation generalises the notion of a mixing transformation; indeed, if T is stable, then T is mixing if and only if $\mathcal{I}$, the σ-field of invariant sets, is trivial under P. The theorems of Rényi for stable sequences of sets are extended to stable transformations.

8. A Remark on Doob's "Elementary Gaussian Processes." V. S. MANDREKAR, University of Minnesota. (Introduced by G. B. Kallianpur.)

From the general representation theory developed by the author with G. Kallianpur (Tech. Report 51, Univ. Minnesota), an explicit representation for continuous parameter multivariate wide-sense Markov (not necessarily stationary) processes is obtained. In stationary case, it yields a more precise form of J. Doob's well-known theorem on the characterization of purely non-deterministic stationary Gaussian Markov processes (*Ann. Math. Statist.* **15** Theorem 4.3). In fact, the number of ones occuring in the matrix, as stated in his result, is equal to the multiplicity of the process.

9. Nonparametric Regression with Missing Observations. I. Estimation of Kendall's Tau. HANS K. URY, California State Department of Public Health, Berkeley.

Four procedures for estimating Kendall's tau are given for the case in which the observations underlying one of the rankings (the x_i) are equally spaced and some observations for the other ranking (the y_i) are missing. Since a function of tau can serve as a nonparametric analogue for the classical estimate of the regression slope, these methods can be used in obtaining nonparametric regression estimates with missing values. The procedures consist of (I) utilizing all available pairwise comparisons among the y_i ; (II) estimating any missing j-step comparison by means of the available ones, where j-step comparison denotes a comparison of a pair of y_i for which the corresponding ranks of the x_i differ by j; (III) and (IV) estimating the missing 1-step comparisons and, from these, the missing j-step comparisons by means of models which express the latter in terms of the former. (III) and (IV) can also be used in the case of unequally spaced x_i . The estimates are compared in some numerical examples, and some small-sample comparisons of their variances are carried out; in each case, procedure (II) has the smallest variance.

10. A Note on Taking a Covariable into Account. HANS K. URY, California State Department of Public Health, Berkeley.

In the application of Bross' COVAST test (*J. Amer. Statist. Assoc.*, September, 1964) for detecting differences between two treatments with dichotomous response variable in the presence of a covariable, with a monotone relationship between covariable and response, pairwise comparisons are made between subjects receiving different treatments. In the process, the set of all such comparisons is partitioned into two subsets potentially favoring one or the other treatment. Under random allocation of the treatments (with respect to the covariable), the expected proportion of each of these subsets, $E(r)$ and $E(1 - r)$, will be 0.5; in actual experimental situations, however, r will not usually be equal to 0.5. Approximations are given for the expected value of the test statistic conditional on r, both under the "Grand Null Hypothesis" (neither treatment nor covariable effect) and under the hypothesis of no treatment differences. Some procedures are suggested for dealing with the case $r \neq 0.5$.