ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Western Regional meeting, Berkeley, California, July 19–21, 1965. Additional abstracts appeared in earlier issues.)

15. The estimation of the parameters of a mixture of distributions. MIR M. ALI and A. B. M. LUTFUL KABIR, University of Western Ontario.

In this paper a general method of constructing estimators for the parameters of a mixture of a finite number of distributions has been developed. The method is applicable to a finite mixture of each of the following distributions: binomial, Poisson, negative binomial, logarithmic series, geometric, exponential, normal and Weibull. The special case of a mixture of two binomial distributions has been studied at length. To this end, two different sets of estimators for the three parameters have been constructed. The estimators are shown to be consistent and asymptotically normally distributed. The expressions for the asymptotic covariance matrices are also derived explicitly. Finally, tables are prepared furnishing the asymptotic efficiencies of the estimators.

16. Inadmissibility of the classical estimator of the multiple regression function. A. J. BARANCHIK, Columbia University. (By title)

Let $Z_1, \cdots, Z_n$ be a sample of size n from $Z = (Y, X_1, \cdots, X_p)'$, a $(p + 1)$-dimensional multivariate normal random variable. Writing the regression function as $E(Y \mid X) = a + b'X$ it is shown that, for $p \geqq 3$, the classical estimator $\hat{a} + \hat{b}'X$ has everywhere greater risk (for the loss function given by C. Stein in *Multiple Regression*. Contributions to Prob. and Stat., Stanford Univ. Press, 1960) than $\tilde{a} + \tilde{b}'X$, where $(\hat{a}, \hat{b})$ is the maximum likelihood estimator of (a, b), $\tilde{a} = \bar{Y} - [1 - c(1 - R^2)/R^2]\hat{b}'\bar{X}$, $\tilde{b} = [1 - c(1 - R^2)/R^2]\hat{b}$, R is the sample correlation coefficient, and $0 < c < 2(p - 2)/(n - p + 2)$.

17. An optimization problem in quality control. EBERHARD BAUR, Aerojet-General Corporation.

In industry management often faces the problem to optimize quality control procedures with respect to test expenses, discrepencies, and fixed obligations to the customer. This paper discusses the case where the customer requires the mean values of production runs to exceed not a given value with given significance. The producer controls with samples drawn from the runs, and he may use fixed sample size procedures or sequential testing. Introducing the distribution function of the true means as a parameter, mathematical formulations are discussed which relate sample size of the quality control procedure and discrepancy. The variance of the distribution under test is assumed to be known. For the application by sequential testing, Wald's test for the mean of a normal population with known variance is used. Some considerations are given to a comparison between the fixed sample size and the sequential procedure; this will involve several parameters as the error of the first kind and the average sample size of the sequential test, and the fraction to accept the null hypothesis. Knowing all the relations between sample size and discrepancy the producer can balance the involved costs for a maximum profit.

18. Application of the statistical decision theory to signal waveform extraction. M. BEHARA and N. K. LOH, University of Waterloo.

The theory of statistical decisions is used in the extraction of the waveform of a signal in the presence of Gaussian white noises. The manner (i.e. additive or multiplicative) in which the noises mix with the signal is assumed to be known. An optimal computer which makes decision according to the decision rule δ, is used to produce the optimum output of the extraction system, thereby the average risk of extraction is minimized. Two cases are considered: (a) For the case in which the receiver of the extraction system has memory, the minimum risk has been found to be $R^*(\delta) = \int_{\Omega[\mathbf{v}^*(\mathbf{z}),\mathbf{u},\mathbf{y},\mathbf{x}]} c[i, \mathbf{x}, \mathbf{v}^*(\mathbf{z})]p(\mathbf{x})[\prod_{i=1}^{n} p(y_i \mid x_i)] \cdot [\prod_{i=1}^{n} p(u_i \mid \mathbf{y}_i)][\prod_{i=1}^{n} p(v_i^*(z) \mid u_i)]d\mathbf{v}^*(z)\, d\mathbf{u}\, d\mathbf{y}\, d\mathbf{x}$, (b) when the receiver is memoryless, the minimum risk is $R'^*(\delta) = \int_{\Omega[\mathbf{v}^*(\mathbf{z}),\mathbf{u},\mathbf{y},\mathbf{x}]} c[i, \mathbf{x}, \mathbf{v}^*(z)]p(\mathbf{x})[\prod_{i=1}^{n} p(y_i \mid x_i][\prod_{i=1}^{n} p(u_i \mid y_i] \cdot [\prod_{i=1}^{n} p(v_i^*(z) \mid u_i)]d\mathbf{v}^*(z)\; d\mathbf{u}\; d\mathbf{y}\; d\mathbf{x}$, where $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{z}$ and $\mathbf{v}$ are defined as vector spaces of transmitted signal, noise obscured input signal into receiver, output signal of receiver, noise obscured input signal into computer and output (estimated) signal of computer, respectively. As expected both $R^*(\delta)$ and $R'^*(\delta)$ decrease as the number of performance steps n increases. For the case of the receiver having no memory, the minimum risk $R'^*(\delta)$ has been found to be 2.83 and 0.353 for $n = 0$ and the standard deviation σ of the signals equals to 1 and 0.5 respectively. These risks fall down to approximately 6.3% for $n = 30$ of those for $n = 0$.

19. The limit of the n**th power of a density.** ROBERT J. BUEHLER, University of Minnesota.

If $f(\cdot)$ is a bounded density of a variate z, then the powers $f^2, f^3, \cdots$, can be normalized to define new variates $z_2, z_3, \cdots$. Typically, z_n will converge to the mode (say m) of $f(\cdot)$, and it is shown that if f is unimodal, $f'(m) = 0, f''(m) \neq 0$, then $n^{\frac{1}{2}}(z_n - m)$ will tend in distribution to a normal variate with zero mean and variance equal to $-f(m)/f''(m)$. Known results for gamma, beta, t and F variates are examples. A more general result is given wherein the density of z_n has the form $c_n\{f(z)\}^n k(z)$ where $k(z)$ is continuous at $z = m$ and bounded for all z, and where the above conditions on f are weakened. A limiting density is obtained having the form $c \cdot \exp\{-|y|^\gamma\}$ where γ is the order of the first non-vanishing term in the Taylor expansion of $f(z) - f(m)$. As an example, the (known) result is obtained that the $(\alpha n + 1)$th order statistic from a sample of size $(\alpha + \beta)n + 1$ is asymptotically normal as n tends to infinity. The proofs differ from those known for special cases in that explicit evaluation of the normalization constants c_n is avoided by appealing to the dominated convergence theorem. Scheffe's theorem is then applicable.

20. Identification of state-calculable functions of finite Markov chains (preliminary report). J. W. CARLYLE, University of California, Los Angeles.

If $\{Y_n, n \geq 1\}$ is a function of a finite-state Markov chain, then (Blackwell and Koopmans, *Ann. Math. Statist.* **28** 1011–1015; Gilbert, *Ann. Math. Statist.* **30** 688–697) the probability law of $\{Y_n\}$ is uniquely determined by the joint distribution of $Y_1, Y_2, \cdots, Y_k$, where k can be taken to be a function only of the number of states of the chain; problems of identification (i.e., explicit construction, based only on the distribution of $Y_1, Y_2, \cdots, Y_k$, of the class of possible underlying chains and functions) have been solved in certain special cases (previous ref's., and Dharmadhikari, *Ann. Math. Statist.* **34** 1022–1041) in which $\{Y_n\}$ is subject to additional hypotheses having no simple structural interpretation. We have previously (*J. Math. Anal. Appl.* **7** 167–175; *Information and Control* **7** 385–397) extended these results to stochastic automata; in the present paper, we study a rather natural structural restriction on $\{Y_n\}$ (suggested by analogies in automata theory) under which unique minimal-state identifications are immediately obtained and related structural questions are resolved.

21. Regenerative methods in systems reliability. BENJAMIN EPSTEIN. Private Consultant, Palo Alto. (By title)

A class of stochastic processes known as regenerative processes occur in diverse fields of application. It is our purpose, in this paper, to show how a variety of systems reliability problems can be treated simply and elegantly, when considered as appropriate regenerative processes.

22. On certain properties of the exponential-type families. G. P. PATIL and RICHARD SHORROCK, Pennsylvania State University and McGill University.

Certain structural properties of the exponential-type families are studied under three headings: (1) mean value function and the exponential-type families, (2) a characterization of the gamma family, and (3) the equality of the first two Bhattacharya bounds and the exponential-type families. Theorems that are proved include: THEOREM. *Let S be a sequence with a limit point in Ω_ν. If $\mu(\omega)$ is given on S, the family is determined among all exponential-type families.* THEOREM. *Let $\{X_\omega : \omega \in \Omega_\nu\}$ be an exponential-type family and let $\{Y_\omega : \omega \in \Omega_\nu\}$ be the family arising from the change of variable $Y_\omega = \exp[X_\omega]$. Let S be an infinite sequence of points in arithmetic progression with common difference unity. Assume $E[Y_\omega]$ exists for some $\omega \in S \cap \Omega_\nu$; and for $\omega \in S \cap \Omega_\nu$, assume that if $E[Y_\omega]$ exists, then $\mathrm{Var}(Y_\omega)$ exists and $\mathrm{Var}(Y_\omega) = E[Y_\omega]$. Then $\Omega_\nu = (-c, \infty)$ for some $c \geqq 0$ and the family $\{Y_\omega : \omega > -c\}$ is gamma with mean $\omega + c$.* THEOREM. *If in an exponential-type family a parametric function is given for which the first two Bhattacharya bounds are equal on some open interval in the parameter space, the linear orbit to which the family belongs is uniquely determined.*

23. Bayes risk efficiency. HERMAN RUBIN and J. SETHURAMAN, Michigan State and Stanford Universities and Indian Statistical Institute and Stanford University.

Let the statistic T_n satisfy $\log P_\theta\{n^{\frac{1}{2}}|T_n - \theta| > c(\log n)^{\frac{1}{2}}\} \sim -\frac{1}{2}c\tau^2(\theta)\log n$ uniformly in θ, with $\tau^2(0) = \sigma^2$, $\tau^2(\theta) \leqq A$. Consider the problem of testing $\theta = 0$ against $\theta \neq 0$. Let the *a priori* distribution of θ have a positive mass at 0 and a density function in a neighborhood of θ excluding 0. Let the loss function behave like $\alpha|\theta|^\lambda$, $\lambda > -1$ in a neighborhood of θ. Then the minimum Bayes risk test is of the form, reject if $|T_n| > a\,\sigma^2(\log n/n)^{\frac{1}{2}}$, and the risk is of the order $b(\sigma^2 \log n/n)^{(\lambda+1)/2}$ where a, b do not depend on n and σ. Let T_n^* be another statistic satisfying the above theorem with σ^* instead of σ. Then the Bayes risk efficiency of T_n wrt T_n^* is σ^{*2}/σ^2. This efficiency is independent of the actual parameters in the *a priori* distribution and the loss function provided that they are of the general form described. This efficiency compares the overall power. Note that the significance level corresponds to a deviation of order $(\log n/n)^{\frac{1}{2}}$. Probabilities of such deviations are estimated in Rubin and Sethuraman, "Probabilities of moderate deviations." The above concept of Bayes risk efficiency lends itself easily to multivariate and composite hypothesis and leads to similar conclusions. Furthermore, the initial conditions can be considerably weakened.

24. Probabilities of moderate deviations. HERMAN RUBIN and J. SETHURAMAN, Michigan State and Stanford Universities and Indian Statistical Institute and Stanford University.

Let $X_1, X_2, \cdots$ be independently and identically distributed with means 0 and variance σ^2 and $E(|X|^q) < \infty$ for some $q > 2 + c^2/\sigma^2$. Then $P\{|X_1 + \cdots + X_n|/n > c(\log n/n)^{\frac{1}{2}}\} \sim 2\sigma n^{-c^2/2\sigma^2}(2\pi c^2 \log n)^{-\frac{1}{2}}$. It is also shown that the condition $E(|X|^q) < \infty$ for all $q < 2 + c^2/\sigma^2$ is necessary for the above result. We propose to call a deviation of the order of $(\log n/n)^{\frac{1}{2}}$, as in the above as a moderate deviation. Similar moderate deviation results have been obtained for U-statistics and the Kolmogorov-Smirnov statistic. The latter result may be stated as $(1/\log n)P\{\sup_x|F_n(x) - F(x)| > c(\log n/n)^{\frac{1}{2}}\} \to -2c^2$. Finally, a transformation theorem is valid for the calculus of the probability of moderate deviations: Let $f(\alpha, \beta)$ have continuous second-order derivatives at $(0, 0)$. Let $P\{|aX_n + bY_n| > c(\log n/n)^{\frac{1}{2}}\} \sim \theta_n = 2\sigma(2\pi c^2 \log n)^{-\frac{1}{2}} n^{-c^2/2\sigma^2}$ for $a = (\partial f/\partial\alpha)(0, 0)$, $b = (\partial f/\partial\beta)(0, 0)$. Then $P\{|f(X_n, Y_n) - f(0, 0)| > c(\log n/n)^{\frac{1}{2}}\} \sim \theta_n$. In fact the above transformation result is valid for deviations of all orders that are $o(n^{-1/3})$. Probabilities of moderate deviations find immediate applications in Bayesian risk efficiencies for parametric and non-parametric problems. See Rubin and Sethuraman, "Bayes risk efficiency."

25. A characterization of the multivariate normal distribution. V. SESHADRI, McGill University.

The following theorem is established: *Let X and Y be two $(n \times 1)$ independent random vectors with continuous probability density functions $f(x)$ and $g(y)$ with the functions non-vanishing at the origin. If the conditional distribution of X given $(X + Y)$ is multivariate normal with mean $C(X + Y)$ and covariance matrix V, where $V^{-1}C$ is symmetric and the characteristic roots of C lie in the open interval $(0, 1)$, then both X and Y are multivariate normal.*

Conversely if (X, Y) has a multivariate normal distribution with covariance matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ *then the matrix of partial regression coefficients $A(A + B)^{-1}$ in the conditional distribution of X given $(X + Y)$, has its characteristic roots in the open unit interval.*

26. Minimax prediction of random probabilities. MORRIS SKIBINSKY, Brookhaven National Laboratory.

The conditional distributions of a random variable given another random variable (independent of the underlying probability measure) are known. Only the first random variable is observable. The second, while not observable, is *a priori* known to have a distribution which is a member of some specified class. It is desired to predict the second random variable from an observation on the corresponding value of the first. A structure is defined to provide a general framework for the problem and a foundation for more explicit development of the prediction of random probabilities. Under what conditions will a predictor of the second random variable, which is optimal in a minimax sense when the specified class of distributions is unrestricted, continue to be optimal in the same way relative to a restricted class of distributions? Game theoretic results yield sufficient conditions for an answer to this question. These are applied to prediction of random probabilities (distributed so that they have at least $1 - \alpha$ chance to fall in an interval $[a, b]$), from an observable which, given the probability, is conditionally equal to 1 with this probability and otherwise 0. Minimax predictors are classified under conditions of negative answer to the above question. Application of results to empirical Bayes situations is considered.

27. On an asymptotic approximation to the non-central t-distribution. William M. Stone and Bonita Peura, Oregon State University and University of California, San Diego.

The non-central t-distribution continues to be of interest. The Locks, Alexander, Byars table (Rocketdyne, Canoga Park, Calif.) affords broader coverage of the three parameters, plus other advantages. Amos [*Biometrika* **51** (1964) 451–458] has achieved a representation in terms of classical hypergeometric functions which is also well-adapted for computation. Resnikoff [*Ann. Math. Statist.* **33** (1962) 580–586] has published tables to facilitate the use of the well-known Johnson-Welch tables to obtain percentage points. The present paper is concerned with the development of an asymptotic relationship between $\Pr(t > t_0 \mid f, \delta)$ and the error function, after the procedure introduced by Hotelling-Fraenkel [*Ann. Math. Statist.* **9** (1938) 87–96] and placed on a firm foundation by Wasow (*Proc. Sym. Appl. Math.*, **7** (1956), McGraw-Hill]. The method seems to be well adapted for the unusually large values of f, the number of degrees of freedom, which appear in various branches of information theory.

(*Abstracts of papers to be presented at the Annual meeting, Philadelphia, Pennsylvania, September 8–11, 1965. Additional abstracts appeared in the June and August issues.*)

9. Probability distributions for subgraphs of random graphs. C. T. Abraham and S. P. Ghosh, Thomas J. Watson Research Center, IBM.

Directed and undirected random graphs with n vertices and N edges have been defined. The probabilities that a random graph will have various types of component subgraphs with and without the deletion of edges have been computed. These probabilities have been used to develop tests of hypotheses concerning the extent of agreement between schemes of associations for the same word list.

10. Bayes and minimax procedures for estimating mean of a population with two-stage sampling. Om P. Aggarwal, Iowa State University.

Simple random sampling and stratified sampling procedures were considered from Bayes and minimax point of view by the author [Om P. Aggarwal. Bayes and minimax procedures in sampling from finite and infinite populations—I. *Ann. Math. Statist.* **30** (1959) 206–218], taking the loss in estimating the mean as a linear function of the squared error of the estimator and the cost of observations. Two-stage sampling procedure is discussed in this paper using the same approach. It is shown that in the case of equal-sized clusters (first-stage units) and equal subsampling from each cluster, the simple mean of the cluster means (i.e., the overall sample mean) is a minimax estimator. The results are obtained in the case of both infinite and finite populations. Extension is made to the case of unequal-sized clusters when the clusters are chosen with equal probability. It is shown that the overall sample mean is not ordinarily a minimax estimator. However, if the cost of sampling a cluster is inversely proportional to the variance within an infinite-sized cluster, it is proved that the simple mean of the cluster means (not the overall mean) is a minimax estimator.

11. Correlation between ranks and variate-values in a singly truncated bivariate normal distribution. M. A. Aitkin and M. W. Hume, Virginia Polytechnic Institute.

The correlation between ranks and variate-values is found for the two marginal distributions of the truncated bivariate normal distribution. This correlation is quite insensitive to both the underlying correlation ρ and the severity of truncation, and in practice does not fall below 0.96, suggesting that variate-values can be replaced by ranks in computations with little loss of efficiency.

12. The exact evaluation of the operating characteristic function and the average sample number of truncated sequential tests. LEO A. AROIAN, TRW, Inc.

Methods are given for the exact evaluation of the operating characteristics function, OC, and the average sample number, ASN, for any truncated sequential test, once the test region is known. The sequential test is interpreted as a random walk, which is a Markov chain. The probability of remaining in the test region is determined by the test statistic and its statistical distribution. The exact evaluation of the OC and ASN follows, based on these principles. The method is basic for the exact determinations of the OC and ASN, replacing difficult mathematical techniques by a simple procedure combined with high speed computing equipment. It is general and may be used iteratively, over changing test regions, to establish an optimum region, depending on the definition of optimality. As examples, the truncated sequential for p_1 vs. p_0 in the binomial case, and the mean with known variance in the normal case are discussed.

13. Admissible minimax estimation of the mean of a multivariate normal random variable. A. J. BARANCHIK, Columbia University.

Let X be a p-variate ($p \geqq 3$) normal random variable with unknown mean vector θ and known covariance matrix I, the $p \times p$ identity matrix. With sum-of-squares loss X, although minimax, is not an admissible estimator of θ. In this paper a family of formal Bayes estimators (estimators which minimize the Bayes risk with respect to infinite prior measures) is obtained, each member of which is minimax. X, for example, is formal Bayes with respect to the prior measure which distributes θ uniformly over p-dimensional space. For the prior measure with differential element $(\sum_1^p \theta_i^2)^{(2-p)/2}\prod_1^p d\theta_i$, the resulting formal Bayes estimator is shown to be both admissible and minimax.

14. Some distribution-free statistics and their application to the selection problem. NOEL S. BARTLETT and ZAKKULA GOVINDARAJULU, Standard Oil Company and Case Institute of Technology.

Let $X_{i,j}$ ($j = 1, \cdots, n_i$; $i = 1, \cdots, k$) be independent random samples drawn from populations π_i having continuous cumulative distribution functions (cdfs) $F(x - \theta_i)$, ($i = 1, \cdots, k$). Also let $R(X_{i,j})$ denote the rank of $X_{i,j}$ in the pooled $\sum n_i = N$ observations. Let H be any cdf, and let $Z(j)$, ($j = 1, \cdots, N$) denote the jth order statistic of a random sample of size N from H. Define the non-randomized rank-sum statistics by $S'_{N,i}(H) = n_i^{-1}\sum E(Z(R(X_{i,j})) \mid H)$ and the randomized rank-sum statistics by $T'_{N,i}(H) = n_i^{-1}\sum Z(R(X_{i,j}))$, ($i = 1, \cdots, k$). It is shown that when $\theta_1 = \theta_2 = \cdots = \theta_k$, $T'_{N,1}(H), \cdots, T'_{N,k}(H)$ are jointly distributed as the means of independent samples of sizes $n_1, \cdots, n_k$ drawn from populations having a common cdf H, which generalizes a result of Bell and Doksum [*Ann. Math. Statist.* **36** (1965) 203–214]. It is also shown that, under very general conditions, the $S'_{N,i}(H)$ and the $T'_{N,i}(H)$ have the same asymptotic joint distribution, namely a k-variate normal distribution, which broadens the results of several authors. Let $\theta_{[1]} \leqq \theta_{[2]} \leqq \cdots \leqq \theta_{[k]}$ denote the ordered values of the location parameter, and define that population as best which has $\theta_{[k]}$. Gupta [Ph.D. Thesis (1956) Mimeo Series 150, Univ. of N. Carolina] and others have discussed a procedure, based on sample means, for selecting a subset containing the best population. It is shown, in the present paper, that two similar procedures for the subset selection problem, based on the $S'_{N,i}(H)$ and the

$T'_{N,i}(H)$, are asymptotically equally efficient, and the asymptotic efficiency of either relative to the means procedure is equal to that of the same statistics in the hypothesis testing problem, which is known to be good. The procedure based on the $T'_{N,i}(H)$ has an advantage for finite sample sizes.

15. A secondarily Bayes approach to the two-means problem. MORTON B. BROWN, Princeton University.

The problem considered is that of discriminating between the means of two Gaussian populations with unknown variances on the basis of two independent samples. Previous solutions have been obtained by fiducial, Bayesian, and asymptotic confidence techniques; none has received widespread acceptance. The approach used here is a secondarily Bayes technique in which a prior distribution is assumed for the ratio of the variances and the distribution of the test statistic is taken conditionally on the ratio of the observed sample variances; it is not necessary to assume a prior distribution for the unknown means in this approach, hence the term, secondarily Bayes. Several different priors, reflecting different amounts of prior information, are used and their effect shown.

16. Maximum likelihood estimation of two stochastically ordered distributions. H. D. BRUNK, W. E. FRANCK, D. L. HANSON, and R. V. HOGG, University of Missouri, University of Missouri, University of Missouri, and University of Iowa.

The maximum likelihood estimates $\hat{F}$ and $\hat{G}$ of two independent distribution functions F and G are found, subject to the restrictions that $\hat{F}(x) \geqq \hat{G}(x)$ for all x and that $\hat{F}$ and $\hat{G}$ are of the discrete type. Random samples $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ are taken from the respective distributions and these $m + n$ values are ordered according to magnitude, with x's before y's in cases of equality. These $n + m$ values are divided into a number of blocks as follows. The first block ends after the y value that is determined so that the ratio of the number n_1 of y values in that block to the number m_1 of x values there is a maximum. If this block does not include all $n + m$ values, then (excluding the first block) the second block and the numbers n_2 and m_2 are determined in a similar manner. This can be continued to construct a (possible) third block, and so on. Let $h_1(x_i)$ and $h_2(y_j)$ be the respective frequencies of x_i and y_j. If x_i and y_j are in the kth block, say, then the maximum likelihood estimates assign the following probabilities to x_i and y_j : $\hat{f}(x_i) = h_1(x_i)(m_k + n_k)/m_k(m + n)$ and $\hat{g}(y_j) = h_2(y_j)(m_k + n_k)/n_k(m + n)$. These assignments provide consistent estimators of F and G.

17. On a class of admissible partitions. THEOPHILOS CACOULLOS, University of Minnesota and New York University.

Consider a k-variate ($k \geqq 2$) spherical normal distribution (unit variance in any direction) centered at the origin. Let ω denote an arbitrary but fixed system of $k + 1$ convex polyhedral k-dimensional cones $\omega_1, \cdots, \omega_{k+1}$ with the same vertex which form a partition of the k-space. It is shown that given a probability vector $\alpha = (\alpha_1, \cdots, \alpha_{k+1})$, there exists a unique translation $\tau(\omega) = (\tau(\omega_1), \cdots, \tau(\omega_{k+1}))$ of the system ω such that the probability content of $\tau(\omega_i)$ under the normal distribution is equal to α_i, $i = 1, \cdots, k + 1$. Also, given a probability vector α and a translation $\tau(\omega)$ of ω there exists a unique normal distribution under which the probability of $\tau(\omega_i)$ is α_i. It is noted that the family of all translations of ω looked upon as partitions of the sample space corresponding to an observation X from a spherical normal distribution with mean μ defines a class of admissible partitions in the

$k+1$-decision problem of locating μ into one of the $k+1$ cones $\omega_1, \cdots, \omega_{k+1}$ [see the author's paper, "Comparing Mahalanobis Distances," *Sankhyā ser. A* **27**, (to appear)].

18. On the moments of some one-sided stopping rules. Y. S. Chow, Purdue University.

Let x_n be a sequence of random variables with $E|x_n| < \infty$ for $n \geqq 1$, and $\mathfrak{F}_n$ be the Borel field generated by $x_1, \cdots, x_n$ and $\mathfrak{F}_0 = \{\varphi, \Omega\}$. Put $S_n = \sum_1^n x_j$, $S_0 = 0$, $m_n = E(x_n \mid \mathfrak{F}_{n-1}$ and $T_n = \sum_1^n m_j$. Assume that $\lim_{n\to\infty} T_n/n = \mu$ uniformly for some $0 < \mu < \infty$. For $0 < c < \infty$ and $0 \leqq p < 1$, define $s =$ first $n \geqq 1$ such that $S_n \geqq cn^p$. Theorem 1. If $E([(x_n - m_n)^+]^\alpha \mid \mathfrak{F}_{n-1}) \leqq K < \infty$ *for some* $\alpha > 1$, *then* $Es < \infty$ *and* $\lim_{c\to\infty} Es/(cEs^p) = 1/\mu$. Theorem 2. *If* $Ex_n^2 < \infty$ *and* $E((x_n - m_n)^2 \mid \mathfrak{F}_{n-1}) \leqq K < \infty$, *then* $Es^2 < \infty$ *and* $\lim_{c\to\infty} Es^2/(c^2Es^{2p}) = 1/\mu^2$. *When* $p = 0$, *Theorems* 1 *and* 2 *reduce to the elementary renewal theorem for the first and second moments respectively.*

19. Estimation of non-unique quantiles. Dorian Feldman and Howard G. Tucker, Michigan State University and University of California, Riverside.

If F is a distribution function and $0 < p < 1$, then a pth quantile x_p is defined to be a number such that $F(x_p - 0) \leqq p \leqq F(x_p + 0)$. The general problem under consideration is to estimate particular pth quantiles when x_p is not unique (i.e., $F(x) = p$ over an interval of positive length), and in particular the smallest pth quantile. If $\{X_n\}$ is a sequence of independent observations on F, the following results are obtained. The sample pth quantile crosses over the pth quantile interval infinitely often with probability one. If $p = \frac{1}{2}$ and if F is continuous and symmetric about a median, then the sample median of the $n(n+1)/2$ random variables $\{(X_i + X_j)/2, 1 \leqq i \leqq j \leqq n\}$ converges not only in probability but *with probability one* to the center median. If the symmetry condition is relaxed, it is shown that the sample median of this set of averages of pairs need not converge, and even if it did converge it is possible for it to converge to a number which is not a median of F. For the problem of estimating the smallest pth quantile in the general case of arbitrary p and arbitrary continuous F, conditions are obtained for weak and strong consistency of suitably chosen order statistics.

20. The application of functions of finite-state Markov chains to communication channels with memory (preliminary report). Bruce D. Fritchman, Philco Corporation.

A function of a finite-state Markov Chain is considered as a binary communication channel. Several useful distributions are derived including the error-free and error cluster distributions. It is shown that for a stationary N-state chain, with k-error free states and $(N-k)$ error states, the error free run distribution is the sum of at most k-exponentials, and the error cluster distribution is the sum of at most $(N-k)$-exponentials. The problem of estimating transition probabilities is considered and experimental results presented. Moreover, block coding and interlacing techniques are evaluated on the functional Markov channel. Here it is shown that for stationary, recurrent, aperiodic chains the effect of interlacing, in the limit, is to cause the channel to appear as binary symmetric channel (BSC) to the decoder.

21. Estimation of the Cauchy location and gamma scale. J. K. GHOSH and RAJINDER SINGH, University of Illinois.

We consider two separate problems. In the first we look for an unbiased estimate of θ in $f^x(x) = \pi^{-1}[1 + (x - \theta)^2]^{-1}$. We show (Theorem 1) that there exists no unbiased estimate with finite variance. The proof uses the Hilbert space representation of estimators, first used by Barankin and Stein. We are not able to settle the existence of an unbiased estimate with infinite variance but we show (Theorem 2) that completeness is incompatible with the existence of an unbiased estimator. Partial results about completeness are given (Theorem 3). In the second problem we consider estimation of λ (not λ^{-1} as is usually done) in $f^x(x) = [\lambda^n/\Gamma(n)] \exp^{(-\lambda x_x n-1)}$. We show (Theorem 4) for $n \leqq 1$ there does not exist an unbiased estimate and for $n \leqq 2$ there does not exist one with finite variance; the proof is similar to that of Theorem 2. For $n > 2$, $(n - 1)/X$ is the best unbiased estimate. However, it is inadmissible compared with $(n - 2)/x$ with squared error loss. We show that the latter estimate is admissible (Theorem 5).

22. Pseudo-estimates versus pseudo-inverses for singular sample covariance matrices. THOMAS J. HARLEY, JR., Philco Corporation.

In applying statistical classification procedures to populations with multivariate normal probability density functions it is necessary to compute the inverse of the covariance matrix for each class. When the population parameters are not known, it is usual to estimate them for each class from the distribution of a sample of finite size. However, in many circumstances, because of the nature of the specific problem, the size of the sample available is not greater than the number of random variables. Under these conditions the sample covariance matrix will be singular. A geometrical interpretation of the singularity leads to the conclusion that the Moore Penrose pseudo-inverse should not be applied in these circumstances. A novel procedure is presented for computing a nonsingular pseudoestimate, V, of the covariance matrix by diluting the sample matrix, S: $V(f) = fS + (1 - f)$ (Trace S/n) I. The dilution procedure reduces the eccentricity of the equiprobability contours of the estimated distribution. The pseudoestimate has worked effectively for the author in dealing with singular conditions.

23. Non-preemptive priorities in machine interference. VINCENT HODGSON, Florida State University.

We consider a machine interference model which is the analog of the priority queueing model of Kesten and Runnenburg [*Proc. Akad. Wet. Amst. A*, **60** (1957), 312–336]. We suppose that one repairman is assigned to maintain k batteries of machines with m_i machines in the ith battery. The repairman can only repair one machine at a time and the sequence of repair is governed by a non-preemptive priority discipline such that the repair of a machine is never interrupted by the breakdown of a machine of higher priority but on the completion of each repair, the repairman will next repair a machine from the highest priority battery not having all machines working. All working times and all repair times are mutually independent. For the ith battery, working times are exponentially distributed with mean $1/\lambda_i$ and repair times have a general distribution $S_i(t)$. Multivariate binomial moments of the number of machines working in each battery in the steady state immediately after a repair is completed, are found to satisfy a system of difference equations. This system of equations is solved. We note that there is an inversion formula for deriving probabilities from binomial moments.

24. A statistical design with a bounded statistic. EDMUND H. INSELMANN, Pitman-Dunn Research Laboratories.

A statistical design is suggested for solving the life testing problem of computers. The test consists of choosing a statistic for terminating the test. This statistic is the total accumulation run time of the computer. The assumption is made that the distribution between times between breakdowns is exponential and the test statistic for this design is the number of breakdowns of the computer. It is shown that this test has the property that it is the uniformly most powerful test for null hypotheses and their respective alternatives given in Lehmann's book, *Testing Statistical Hypotheses* (page 125).

25. On one-sample nonparametric test for the location parameter (preliminary report). M. V. JOSHI and Z. GOVINDARAJULU RAMACHANDRAMURTY, Case Institute of Technology.

For a random variable X the lower limit θ is defined by the relation: $\theta = \text{lub}\{\theta_1 : P[X \leqq \theta_1] = 0\}$. On the basis of the smallest order statistic $X_{1,N}$ of the random sample $X_1, \cdots, X_N$, one-sample nonparametric tests of the location hypothesis $H_0 : \theta = 0$ against $H_1 : \theta > 0$ are derived. The class $\mathfrak{F}_\lambda^c$ of the distribution functions is defined by: $\mathfrak{F}_\lambda^c = \{F: F(x) = 0 \text{ when } x < 0 \text{ and } F(x)/x^\lambda \rightarrow c \text{ as } x \rightarrow 0\}$. It is shown that for distribution functions of this class the asymptotic size $-\alpha$ critical regions for testing the above mentioned location hypothesis are given by: $N^{1/\lambda}x_{1,N} > (-(1/c)\ln \alpha)^{1/\lambda}$. Asymptotic approximations to the powers of these critical regions are derived. Some distributions of the class $\mathfrak{F}_\lambda^c$ are discussed. The chi-distribution (1 degree of freedom), defined by the density $(2/\pi)^{\frac{1}{2}} \exp(-x^2/2)$ when $x > 0$ and zero otherwise, belongs to the class $\mathfrak{F}_1^{(2/\pi)^{\frac{1}{2}}}$. For chi-distribution (1 df) the asymptotic powers obtained from the general theory are compared with the exact powers. Approximation is found to be reasonably good. For this distribution, exact size $-\alpha$ critical regions are derived when $N = 2(1)10(5)20$ and when $\alpha = .01, .025, .05$ and $.1$.

26. A test for goodness-of-fit based on discriminatory information. B. K. KALE, Iowa State University.

Let $u_1, u_2, \cdots, u_n$ be the order statistic of a random sample of size n from a continuous cdf F. Let F_n be the empirical df. Define $I(F, F_n) = (n+1)^{-1}\sum_{i=1}^{n+1} \log\{[F(u_i) - F(u_{i-1})](n+1)\}$ with $u_0 = -\infty$, $u_{n+1} = +\infty$. Then $I(F, F_n)$ is the discriminatory information (in the sense of Kullback) provided by F_n against F. Obtaining moment generating function of $I(F, F_n)$ and exploiting the distribution theory of the U statistic, it is shown that the asymptotic distribution of $I(F, F_n)$ is normal with mean $r - \frac{1}{2}(n+1)$ and variance $[2/(n+1)](\pi^2/6 - 1) - (n+1)^{-2}$ where r is Euler's constant. For testing $H_0 : F = F_0$, a test is proposed on the basis of the statistic $I(F_0, F_n)$. The performance of this test is compared with χ^2-test, Kolmogorov's test and its modifications, by considering 15 random samples of size 50 each, five being from normal, five exponential and five from Laplace.

27. The log-0-Poisson distribution. S. K. KATTI, Florida State University.

A number of discrete distributions have been proposed to describe populations that are labelled as being "contagious." In his M.S. Thesis, *Some Families of Contagious Distributions*, Iowa State University (1958), the author generated a number of distributions, approximately 40 in number, following the compounding and generalizing processes underlying the proposed distributions and compared them using skewness and kurtosis. It was observed that one distribution, which is referred to here as Log-0-Poisson, has more flexibility than all of these distributions. Martin, D. C. and Katti, S. K. collected 35 sets of empirical distributions from the journals such as *Biometrics*, *Biometrika* and *Journal of Entomology* and published maximum likelihood fits of Poisson, Poisson with Zeros, Neyman Type A, Negative Binomial and Poisson Binomial in *Biometrics*, March (1965). The Log-0-Poisson has now been fitted to these sets and it is observed that the fits of Log-0-Poisson are comparable with the best of these fits. The Log-0-Poisson distribution is obtained by adding zeros to the logarithmic distribution and then generalizing with Poisson. Its probability generating function is given by $g(z) = 1 - p_1 \log(q_2 - p_2 \exp[\lambda(z - 1)])$ where $\lambda \geqq 0$, $p_2 \geqq 0$, $q_2 = 1 + p_2$ and p_1 is such that the probability of the zero count lies in [0, 1].

28. Replicated (or interpenetrating) samples of unequal sizes. J. C. Koop, North Carolina State University.

We consider k sets of independent replicated or interpenetrating samples each containing m_i $(i = 1, 2, \cdots, k)$ first stage units which are selected with equal probabilities and without replacement after each draw from a finite universe of N. For the sake of generality, the sample design beyond this stage is not defined, except that in each unit the structure of the sample thereafter is the same. In all, there are $k\bar{m} = \sum_1^k m_i$ first stage units, where $\bar{m}$ is an integer. Let V_u be the variance of the estimator based on the k unbiased estimates of the universe total for a given characteristic, each weighted by the reciprocal of its true variance. Alternatively consider k sets of independent replicated samples each of size $\bar{m}$ drawn in a similar way, and let V_u be the variance of the estimator for the same characteristic based on the mean of the k estimates, each based on equal sized first stage samples. Then it is shown that $V_e/V_u = (1/k\bar{m})(1 - \bar{f}\alpha)\sum_1^k[m_i/(1 - f_i\alpha)] > 1$, where $\bar{f} = \bar{m}/N$, $f_i = m_i/N$ $(i = 1, 2, \cdots, k)$ and $\alpha = S^2/(S^2 + E(u))$ in which S^2 is the true between first stage unit variance for the total value of the characteristic in a unit, and $E(u)$ is the variance beyond the first stage. Thus surprisingly, replicated samples of unequal sizes are more efficient than those with equal sizes. The practical implications of this result in sample survey work are far reaching.

29. Split subsamples vs. independent replicated samples of equivalent size. J. C. Koop, North Carolina State University. (By title).

The paper examines the precision of the mean of k subsample means, each based on samples of size m, obtained by randomly splitting a sample of size $mk = n < N$, drawn without replacement and with equal probabilities at each draw from a finite universe of N elements. If S^2 is the universe variance of the characteristic of interest, it is demonstrated that the variance of the mean based on split samples is augmented by $(S^2/m)(1 - m/n)[1 - (k - 1)/(C_m{}^n - 1)]$. This extra variance is due to splitting the initial sample. Further it is shown that the variance of the mean of k means, based on independent replicated samples, each of size m drawn in a similar way, is always less than that of the mean based on split samples, excepting cases where $n = N - 1$ and $k = (N - 1)/2$ is an integer, when the two variances will be equal. For infinite populations the relative efficiency of split samples is always lower. Clearly the practice of splitting samples, often resorted to for various purposes, suffers from these theoretical disadvantages.

30. Some parametric empirical Bayes techniques. R. G. Krutchkoff and J. R. Rutherford, Virginia Polytechnic Institute.

In the parametric empirical Bayes situation there is a random variable Λ with unknown distribution $G(\lambda)$, and a random variable X with a known family of conditional distribution functions $F(X \mid \lambda)$. We assume that X is a vector of $k \geqq 1$ iid random variables with a known sufficient statistic $T(X)$. For the class of exponential and range dependent continuous density functions $f(t \mid \lambda)$, we find general formulations of the Bayes decision function which are amenable to empirical Bayes techniques. We then demonstrate that these techniques provide asymptotically optimal decision functions in both hypothesis testing and estimation problems.

31. On a generalized goal in fixed-sample ranking and selection problems (preliminary report). DESU M. MAHAMUNULU, University of Minnesota.

Consider $k \geqq 2$ populations π_i on the real line with cdf $F(x \mid \theta_i)$ $(i = 1, \cdots, k)$, where $\theta_i \varepsilon \Theta$. Let $\theta_{[1]} \leqq \cdots \leqq \theta_{[k]}$ be the unknown ordered θ-values and suppose $\theta_{[k-t]} < \theta_{[k-t+1]}$. An experimenter's goal is to select a subset of size $s(< k)$ which contains at least c of the t best populations (those with largest θ-values). The experimenter specifies two positive constants d^* and P^* where $\{\binom{k}{s}^{-1}\sum_{i \geqq c}\binom{t}{i}\binom{k-t}{s-i}\} < P^* < 1$ and he desires to have a procedure R such that $P(CS \mid R) \geqq P^*$ whenever $d(\theta_{[k-t+1]}, \theta_{[k-t]}) \geqq d^*$; here CS denotes correct selection and d-function is an appropriate distance measure. A procedure R_s based on real-valued statistics T_i, computed from a random sample of size n $(i = 1, \cdots, k)$, is proposed; it states that the subset of populations, which gave the s largest T-values, is to be selected. The problem is the determination of the common sample size n, so as meet the above probability requirement. It has been solved by assuming that the resulting distributions $G_n(\cdot \mid \theta)$ of T is an absolutely continuous and stochastically increasing family, when indexed by θ. Some properties of the procedure R_s are considered. These results generalize the earlier results of the author [*Ann. Math. Statist.* **36** (1965) 728, abstract #7]. Some related problems are considered.

32. The integral of an invariant unimodal function over an invariant convex set —an inequality and applications. GOVIND S. MUDHOLKAR, University of Rochester.

In this paper we have obtained an extension of an inequality due to T. W. Anderson [*Proc. Amer. Math. Soc.* **6** 170–176]. Anderson proved that, for $0 \leqq k \leqq 1$, $\int_E f(x + ky)dx \geqq \int_E f(x + y)dx$, if E is a convex set in n-space, symmetric about origin, and $f(x) \geqq 0$ is a function satisfying (i) symmetry condition: $f(x) = f(-x)$, (ii) unimodality condition, i.e. convexity of $K_u = \{x \mid f(x) \geqq u\}$ for each u, and (iii) $\int_E f(x)dx < \infty$. Now let $G = \{g_i\}$ be a group of N linear transformations of n-space onto itself preserving measure of Lebesgue measurable sets. Let E be a convex set invariant under G, i.e. satisfying: $x \varepsilon E \Longrightarrow g_i \varepsilon E$ for each $g_i \varepsilon G$. Let $f(x) \geqq 0$ be a function which, in addition to Anderson's conditions (ii) and (iii) above, satisfies the invariance condition: $f(g_i x) = f(x)$ for each $g_i \varepsilon G$. For a set $\alpha = \{\alpha_i\}$ of N nonnegative reals, $\sum \alpha_i = 1$, and an n-vector y let $\alpha(y) = \sum \alpha_i g_i y$. Then the extension of Anderson's inequality is $\int_{E+\alpha(y)} f(x)dx \geqq \int_{E+y} f(x)dx$. Probabilistic applications and the problems of extending the result to infinite groups have been discussed.

33. An extension of Ferguson's characterization of the geometric distribution. V. K. MURTHY and V. R. RAO UPPULURI, Douglas Aircraft Company, Inc. and Oak Ridge National Laboratory.

Ferguson has shown that if X_1 and X_2 are independent, nondegenerate and discrete random variables, then $\min(X_1, X_2)$ and $X_1 - X_2$ are independent if, and only if, X_1 and X_2 both have geometric distributions. We will extend this result as follows, when we have more than two variables: DEFINITION. A random variable X is said to be degenerate at

x_0 if $P(X = x_0) = 1$. Let $X_1, X_2, \cdots, X_n$ be n nondegenerate, independent, and identically distributed discrete random variables, with $P(X_i = x) = f(x) \neq 0$, $x = 0, 1, 2, \cdots$. Let $Y = \min(X_1, X_2, \cdots X_n)$ and $Z = \sum_{i=1}^{n}(X_i - Y)$. THEOREM. $f(x) = (1 - \theta)\theta^x$, $x = 0, 1, 2, \cdots, 0 < \theta < 1$ *if and only if*, Y *and* Z *are independent.*

34. On confidence limits for the reliability of systems. J. M. MYHRE and S. C. SAUNDERS, Claremont Men's College and Boeing Scientific Research Laboratories.

An application of the asymptotic chi-square distribution of the log-likelihood ratio is made to obtain approximate confidence intervals for the reliability of any structure of the class of structures which can be represented by a monotone Boolean function of Bernoulli variates. This is whenever we have available only a number of Bernoulli trials of the adequate performance or inadequate performance of the components. Some special methods are obtained for the subclass of structures which fail if and only if a certain prescribed number of their components fail. This generalizes the results of A. Madansky (Rand P-2401) for series and parallel systems. Computational procedures are given and some Monte-Carlo studies are reported to assess how accurate the confidence intervals are as a function of the sample size. Some comparisons are made with exact results in the cases where they are known.

35. Central confidence intervals for the minor means in large samples. NILAN NORRIS, Hunter College.

Let $x_i = x_1, x_2, \cdots, x_n$ be a set of random, positive, and independent variables with the same distribution function, where the expectations $E(x_i)$ and $E(x_i^2)$ exist and where $\sigma^2 = E\{[x_i - E(x_i)]^2\} > 0$. In random samples of n the estimate of the population geometric mean, θ_1, is the sample geometric mean $G = (x_1x_2\cdots x_n)^{1/n}$, and the estimate of the population harmonic mean, θ_2, is the sample harmonic mean $H = [(1/n)\sum(1/x_i)]^{-1}$. By one of the forms of the central limit theorem (Laplace-Lyapunov theorem), under certain simple conditions, the limiting distribution of $n^{\frac{1}{2}}(G - \theta_1)$ is normal with an arithmetic mean of 0 and a variance of $\theta_1^2\sigma_{\log x}^2$; and the limiting distribution of $n^{\frac{1}{2}}(H - \theta_2)$ is normal with an arithmetic mean of 0 and a variance of $\theta_2^4\theta_{1/x}^2$. Let the normal deviate $z = (G - \theta_1)/\sigma_G$, where $\sigma_G = (\theta_1\sigma_{\log x})/n^{\frac{1}{2}}$ is the standard error of G; and let the normal deviate $z = (H - \theta_2)/\sigma_H$, where $\sigma_H = (\theta_2^2\sigma_{1/x})/n^{\frac{1}{2}}$ is the standard error of H. A confidence of $(1 - \alpha)$ is associated with a central interval established with a given value of z, where α denotes the probability of a Type I error. For θ_1 the central confidence interval is given by: $G - z[(\theta_1\sigma_{\log x})/n^{\frac{1}{2}}] < \theta_1 < G + z[(\theta_1\sigma_{\log x})/n^{\frac{1}{2}}]$, and for θ_2 the central confidence interval is given by: $H - z[(\theta_2^2\sigma_{1/x})/n^{\frac{1}{2}}] < \theta_2 < H + z[(\theta_2^2\sigma_{1/x})/n^{\frac{1}{2}}]$.

36. Maxima of stationary Gaussian processes. JAMES PICKANDS IIII, Virginia Polytechnic Institute.

Let $\{X_N, N = 0, \pm 1, \pm 2, \cdots\}$ be a stationary Gaussian stochastic process, with means zero, variances one, and covariance sequence $\{r_N\}$. Let $Z_N = \max_{1 \leq k \leq N} X_k$. Limit properties are obtained for Z_N, as N approaches infinity. A double exponential limit law is known to hold if the random variables X_i are mutually independent, that is $r_N \equiv 0$, $N \neq 0$. Berman has shown that the same law holds in the case of dependence, provided r_N approaches zero "sufficiently fast." Specifically sufficient conditions are that either $\lim_{N\to\infty} r_N \log N = 0$, or $\sum_{N=1}^{\infty} r_N^2 < \infty$. In the present work, it is shown however, that $\lim_{N\to\infty} r_N = 0$ is not sufficient. A corresponding law is obtained for a separable measurable version of a continuous parameter process. Sufficient conditions are obtained for the "strong laws of large numbers," $Z_N - [2 \log N]^{\frac{1}{2}} \to 0$, a.s., and $Z_N/[2 \log N]^{\frac{1}{2}} \to 1$, a.s. in both discrete, and continuous time.

37. A note on estimation of ratios by Quenouille's method. J. N. K. RAO, Texas A & M University. (Invited)

Quenouille (*Biometrika* **43** 353–60) has proposed a simple method of reducing bias from $O(n^{-1})$ to $O(n^{-2})$. Suppose $\hat{R}$ denotes an estimator based on n observations whose bias is $cn^{-1} + O(n^{-2})$ where c is a constant. Let the sample be divided at random into g groups each of size m, where $n = mg$ and let $\hat{R}_j$ denote the estimator calculated from the sample after omitting the jth group. Then the estimator $\hat{R}_Q = g\hat{R} - [(g-1)/g] \sum \hat{R}_j$ has a bias of order n^{-2} at most. Durbin (*Biometrika* **46** 477–80) has shown that, if the estimators are ratio estimators of the form $r = y/x$ and the regression of y on x is linear and x is normally distributed, $\hat{R}_Q$ with $g = 2$ has asymptotically a smaller variance than $\hat{R}$. However, the advantages of one choice of g over another have not been investigated. In this note the bias and the variance of $\hat{R}_Q$, for general g, to $O(n^{-3})$ is derived assuming that the regression of y on x is linear and x is normally distributed. It is shown that *both* the bias and the variance of $\hat{R}_Q$ are in fact monotonically decreasing functions of g so that $g = n$ would be the optimum choice.

38. The effect of truncation on the F-test for a class of PBIB designs. P. V. RAO, University of Florida.

In an earlier paper [*Ann. Inst. Statist. Math.* **15** 25–36] the author studied the effect of departures from assumptions on the F-test for two associate PBIB designs with $\lambda_1 = 0$ and $\lambda_2 = 1$. It was found that as in the case of randomized blocks, latin squares and PBIB designs, the F-test works reasonably well as an approximation to the randomization test. In this paper, the behaviour of F-test is studied for the same class of PBIB designs when the observations are assumed to have arisen from a truncated normal population. The conclusion of this investigation is as follows: Truncation of the population alters the situation in the sense that F-test will be over-estimating the true treatment differences. Furthermore, symmetric truncation may often help improve the standard F-test as an approximation to the randomization test.

39. Regression analysis when the least-squares estimate is not asymptotically efficient (preliminary report). HENRY R. RICHARDSON, Daniel H. Wagner, Associates.

Consider a real, discrete parameter, vector valued process $\{y_t\} = \{x_t + m_t\}$, where $Ey_t = m_t$ and the residual process $\{x_t\}$ is weakly stationary with a continuous positive-definite spectral density. Suppose $m_{t,n} = \sum_{s=1}^{S} c_s \phi_{t,n}^{(s)}$, where the regression sequences are given in analytic form by $\phi_{t,n}^{(s)} = t^{d(s)} \int_{-\pi}^{\pi} e^{it\lambda} d\kappa_n^{(s)}(\lambda)$, ($d(s)$ an integer) and the vector c is to be estimated. It is well known that there are situations where the least-squares estimate of c is not asymptotically efficient as compared with the minimal covariance Markov estimate. The usefulness of the Markov estimate in these cases is limited, however, because it requires detailed knowledge of the covariance structure of the residual process and is difficult to handle numerically. We obtain an estimate, called the almost-best-linear-unbiased (ABLU) estimate of c, which is asymptotically efficient when the least-squares estimate is not but, in contrast to the Markov estimate, requires only limited knowledge of the spectral density and is easy to compute. Further, under more stringent assumptions on $\{x_t\}$ (including a strong mixing condition), it is shown that the ABLU estimate is asymptotically normal even in the case where the relevant values of the spectral density are not assumed to be known but are replaced by their estimates.

40. Some further results in Bayes risk efficiency. HERMAN RUBIN and J. SETHURAMAN, Michigan State University and Stanford University, and Indian Statistical Institute and Stanford University. (Invited)

Let T_n be a sequence of statistics such that $\log P(T_n - \theta > c, (\log n/n)^{\frac{1}{2}} \mid \theta, \varphi) \sim c^2 \log n/2\sigma^2(\theta, \varphi)$, and similarly for the other tail. Consider the testing of $\theta < 0$ against $\theta > 0$ where (θ, φ) is assumed to have an *a priori* distribution with a non-zero continuous density. Let the loss function for deciding $\theta \leqq 0$ be $A\theta^\alpha$ if $\theta > 0$, and if the other decision is made, $B(-\theta)^\beta$ if $\theta \leqq 0$, and that conditions similar to those of Rubin and Sethuraman, "Bayes risk efficiency," are satisfied. If $\alpha = \beta$, the asymptotic Bayes procedure based upon T_n is a fixed level test of $\theta = 0$ and the Bayes risk is proportional to $E_\varphi n^{-[(\alpha+1)/2]}\sigma^{\alpha+1}(0, \varphi)$; if $\alpha > \beta$, the critical point is on the order of $\sigma(0, \varphi)[(\alpha - \beta)\log n/n]^{\frac{1}{2}}$ and the risk is proportional to $(\sigma^2(0, \varphi)\log n/n)^{\alpha+1}$. The proportionality constants depend only on the problem, and hence the efficiency is inversely proportional to $\sigma^2(0, \varphi)$.

41. Empirical Bayes estimation of prior and posterior distribution. J. R. RUTHERFORD and R. G. KRUTCHKOFF, Virginia Polytechnic Institute.

We assume that there is a random variable Λ distributed according to a specific but unknown prior distribution G from a class G_p. G_p is the class of distributions which have finite first p moments; and have density functions with respect to Lebesque measure which are completely specified by a continuous function of its first p moments. Λ $(=\lambda)$ is unobservable but another random variable T $(=t)$, distributed with known conditional distribution function $F(t \mid \lambda)$, is observable. $\{F(t \mid \lambda)\text{: for all } \lambda\}$ is any class of distributions for which we know measurable functions $h_k(t)$, $k = 1, 2, \cdots, p$, such that $E[h_K(T) \mid \lambda] = \lambda^k$. We construct estimators $G_n(\lambda)$ of $G(\lambda)$ such that $\lim E[(G_n(\lambda) - G(\lambda))^2] = 0$, a.e. λ. We use $G_n(\lambda)$ to estimate the posterior distribution $G(\lambda \mid t)$ and hence to construct consistent estimators of posterior confidence intervals.

42. Tests of linear hypotheses using a generalized inverse matrix. S. R. SEARLE, Cornell University.

A generalized inverse of the matrix $X'X$ can be defined as any matrix G for which $X'XGX'X = X'X$. One such matrix can be developed from reducing $X'X$ to diagonal form; in so doing, G is symmetric and satisfies $GX'XG = G$. Solutions to normal equations $X'Xb = X'y$ derived for the linear model $E(y) = Xb$ can then be expressed as $\hat{b} = GX'y$. If $H = GX'X$ the hypothesis $Q'b = m$ can be tested provided $Q'H = Q'$. On the basis of normality assumptions the F-value for testing the hypothesis is $F = (Q'\hat{b} - m)'(Q'GQ)^{-1} \cdot (Q'\hat{b} - m)/s\sigma^2$, where s is the rank and order of Q' and $\hat{\sigma}^2 = (y'y - \hat{b}'X'y)/(n - r)$, n being the number of observations and r the rank of X.

43. On a class of c-sample weighted rank-sum tests for location and scale. P. K. SEN and Z. GOVINDARAJULU, University of California, Berkeley.

A class of c-sample non-parametric tests ($c \geqq 2$) for the homogeneity of either location or scale parameters is proposed and its various properties studied. These tests are based on a family of congruent interquantile numbers, and may be regarded as a c-sample extension of a similar class of two sample tests proposed and studied by Sen [*Ann. Inst.*

Statist. Math. **15** (1963), 117–135]. Sufficient conditions for the joint asymptotic normality of the class of statistics proposed are obtained. With the aid of these, the asymptotic power-efficiency of the class of tests is studied, and comparison is made with other test procedures. Testing for equality of scale parameters when the locations are unknown and unequal is also considered.

44. Optimal balanced fractional plans for main effects in the presence of odd ordered interactions. J. N. SRIVASTAVA and DONALD A. ANDERSON, University of Nebraska. (By title)

It has been pointed out by Srivastava [North Carolina Institute of Statistics Mimeo Series No. 301] and Bose and Srivastava [Analysis of irregular fractions. *Sankhyā* **26**] that (1, 0) symmetry of a 2^n factorial fraction implies that all effects involving an odd number of factors can be estimated clear of those with an even number of factors. If A is a set of b assemblies, and $\bar{A}$ is obtained from A by interchanging the symbols 0 and 1, then the set of $2b$ assemblies $A + \bar{A}$ is (1, 0) symmetric. Plans for main effects are thus constructed by taking the set of assemblies $A + \bar{A}$, where A is the incidence matrix of a BIB, with equal or unequal block sizes, having parameters (v, b, r, λ). This choice of A makes the plan balanced in the sense that M, the covariance matrix of the estimates, is symmetric wrt the factors. It is shown next that if A corresponds to a BIB with parameters (v, b, r, λ) with $b = 4t + \theta$, and $0 \leqq \theta \leqq 3$; then tr M^{-1} is a minimum among the class of balanced designs if $\theta = 0, 1, 2$ and $r - \lambda = t$; or $\theta = 3$ and $r - \lambda = t$ or $t + 1$ according as $t \leqq$ or $> \frac{3}{2}[v/4 - 1]$. Plans optimal in the above sense have been obtained for a large set of values of the pair (v, b) including in particular most of the pairs with $7 \leqq v \leqq 10$ and $7 \leqq b \leqq 20$. Also certain series of designs with an infinite number of pairs (v, b), including cases with $v = b$, are obtained.

45. Tests for the dispersion, and the modal vector, of a unimodal distribution on a sphere. MICHAEL A. STEPHENS, McGill University.

A point P on the circumference of a unit sphere, center 0, has polar coordinates θ, ϕ Suppose the density function of θ, ϕ is $c(\mathcal{K}) \exp(\mathcal{K} \cos \theta) \sin \theta \, d\theta \, d\phi$; the parameter $\mathcal{K}$ is then a measure of the dispersion of the vectors OP. The paper gives tests and significance points to test hypotheses of two types: (1) concerning $\mathcal{K}$, whether the modal vector $\theta = 0$ is known or unknown, and (2) concerning the polar vector, when $\mathcal{K}$ is known.

46. A non-linear optimum stochastic control problem (preliminary report). CHARLOTTE STRIEBEL, University of Minnesota. (By title)

The problem considered is that of optimum control of a vector-valued dynamic system with white noise driving functions. Observations are taken to be linear in the system vector and in the observation error process which is also a white noise. The covariance matrices of the white noise processes and the coefficients (time varying) of the dynamic system and of the observation equation are all assumed to be known. The loss function is a linear combination of the square of a one-dimensional final miss distance and the integral (or sum) over time of the absolute value of the control vector. The optimum control is to be determined as a function of all available observations. By attacking the discrete control problem (control is restricted to a finite number of previously specified times) it is shown that the optimum control is a function only of the "best" estimate of the final miss and that control is to be applied in such a way that this estimate remains within a certain boundary. Thus, the optimum control problem is reduced to that of determining this boundary. For the discrete case, recursion equations are given by which the boundary can be computed. By formally passing to the continuous case, under certain regularity conditions it is shown that this boundary is the solution to a free boundary problem for the heat equation.

47. Bounds and rates of convergence for the extended compound estimation problem in the sequence case. DONALD D. SWAIN, U. S. Naval Academy.

Let $\mathfrak{F} = \{p_\theta : -\infty < \alpha \leqq \theta \leqq \beta < \infty\}$ be a family of probability density functions; $\theta = (\theta_1, \theta_2, \cdots, \theta_i, \cdots)$ be an arbitrary unknown vector with $\alpha \leqq \theta_i \leqq \beta$; X_1 be an observation with density p_{θ_1} and $\varphi_1(X_1)$ be an estimate of θ_1. In general let X_i be an independent observation with density p_{θ_i} and $\varphi_i(X_1, X_2, \cdots, X_i)$ be an estimate of θ_i. After n estimates are made, the average risk for an estimating procedure $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_i, \cdots)$ is $R_n(\varphi, \theta) = \sum_{i=1}^n E[(\varphi_i - \theta_i)^2/n]$. A sequence of standards $R_k(\theta_n)$, $k = 1, 2, \cdots$, based on the Bayes envelope function evaluated at the k dimensional "empirical distribution function" generated by θ_n, is considered. It is shown that $\lim\inf_{n\to\infty}[R_k(\theta_n) - R_{k+1}(\theta_n)] \geqq 0$, with strict inequality holding whenever θ has "patterns" of length $k + 1$ or greater. For many families $\mathfrak{F}$, including the Poisson, negative binomial, binomial, normal, and gamma, estimating procedures φ^k are given for which upper bounds for the difference $R_n(\varphi^k, \theta) - R_k(\theta_n)$ are found for all n. These bounds are independent of θ and approach zero as n approaches ∞. For the Poisson and negative binomial families it is shown that these bounds are of the order $\log^k n/n^{1/4}$ uniformly in θ. For the binomial family the order is $\log n/n^{1/4}$.

48. Sufficient statistic and the complete class of decision functions. KEI TAKEUCHI and HIROKICHI KUDŌ, University of Tokyo and Osaka City University.

Let $(\mathfrak{X}, \mathfrak{B})$ be a sample space and $\mathfrak{C}$ a σ-subfield of $\mathfrak{B}$. A decision function (for abbreviation, df) δ is called to be $\mathfrak{C}$-measurable, if $\delta(A:x)$ is $\mathfrak{C}$-measurable in x when A is a fixed measurable subset of the action space. In some cases, e.g. estimation problems of a parameter under the absence of nuisance parameter [Bahadur, R. R., *Amer. Math. Soc.* **26** (1955)] and problems of testing hypothesis [Kudō, H., *Sci. Rep. Ochanomizu Univ.* **4** (1954)], it is proved that if the class of the $\mathfrak{C}$-measurable df's is a complete class then $\mathfrak{C}$ is sufficient in $\mathfrak{B}$. An attempt of extending these results was made, but unfortunately it contains a wrong description in condition [Sugaku **8** (1957), in Japanese; Math. Rev. **20** p. 461]. A corrected condition for the above implication will be given. The main part of the condition is: for any n sample distributions there is a partition $\mathfrak{X} = \bigcup_\nu \mathfrak{X}_\nu$ such that the loss function is irreducible.

49. Higher moments of randomly stopped sums. HENRY TEICHER, Purdue University.

Let t be a stopping variable of the stochastic process $\{x_n, n \geqq 1\}$ so that $S_t = \sum_{i=1}^t x_i$ is a randomly stopped sum. If the x_n are independent and for some positive integer m, $Ex_n^j = c_j < \infty$ (independent of n), $1 \leqq j \leqq 2m$ and further $Et^m < \infty$, an identity for ES_t^{2m} (necessarily finite) is given, thus generalizing results of Chow, Robbins and Teicher [*Ann. Math. Statist.* **36** 789–799] which, in turn, generalized the theorem on cumulative sums of Wald [*Ann. Math. Statist.* **15** 283–296]. An analogous expression for ES_t^{2m} is rendered in the case where the x_n are the summands of a martingale.

50. Asymptotic normality of binomial sequential stopping rules. M. T. WASAN, Queen's University, Kingston. (By title)

Let $X_1, X_2, \cdots, X_i \cdots$ be a sequence of independent random variables each with the same family of possible probability functions $p(X_i = 1) = p$, $p(X_i = 0) = 1 - p$, $0 < p < 1$ and $i = 1, 2, \cdots$. The observations are denoted by $x_1, x_2, \cdots$ and cumulative sum of first m observations by u_m. Sampling continues until a certain inequality is satisfied, of the form $u_m \geqq k(m)$ or $u_m \leqq k(m)$, where $k(m)$ is a function of m determined in advance. The value of m for which the inequality is first satisfied is denoted by n. p is estimated by $\hat{p} = k(n)/n$. It is proved that $\hat{p}$ is normally distributed for the stopping rule $u = \frac{1}{2}n[1 \pm (1 - 4an)^{\frac{1}{2}}]$ when $a \to 0$. It is also shown that for the boundary $u = \frac{1}{2}n[1 \pm (1 - 4cn^2)^{\frac{1}{2}}]$, $\hat{p}$ is normally distributed when $c \to 0$.

51. The moments of uniform order statistics. JOHN S. WHITE, General Motors Research Laboratories.

Let $U(i, n)$, $U(j, n)$ be the ith and jth order statistics of a sample of size n from a uniform $(0, 1)$ distribution. Simple recursion relations for the moments of $U(i, n)$ and the cross moments of $U(i, n)$ and $U(j, n)$ are derived. For example, if $m(k) = E(U(i, n) - p)^k$ where $p = i/(n+1) = E(U(i, n))$ then $m(k+1) = k((1-2p)m(k) + p(1-p)m(k-1))/(n+1+k)$. These results may be used to obtain approximations to the moments of order statistics $X(i, n)$ for any population with analytic distribution function $F(x)$ by expanding $X(i, n)$ as a Taylor series in powers of $U(i, n)$ [e.g., David and Johnson, *Biometrika* **41** (1954) 228–240].

52. Asymptotic theory of tests based on bivariate admissible points. GEORGE G. WOODWORTH, University of Minnesota.

Let $\{X_1, \cdots, X_n\}$ be a sample from a continuous bivariate population. The point X_j is kth layer admissible iff exactly $k - 1$ points of the sample have both coordinates smaller than the corresponding coordinates of X_j. Let $A(k)$ denote the number of kth layer admissible points. The distribution of $\{A(1), \cdots, A(n)\}$ under independence of the components of X was derived by Sobel and Barndorff-Nielsen [accepted for publication in *Teor. Veroyatnost. i Primenin.*, (1965)]. In this paper, statistics of the form $T(\varphi_n) = n^{-\frac{1}{2}}\sum_{k=1}^{n}\varphi_n(k/n)A(k)$ are investigated, where φ_n has a derivative φ_n' on $(0, 1)$. It is shown for a certain class of distributions of X that if φ_n' approaches in qm some φ' whose integral is bounded on $(0, 1)$, then $T(\varphi_n)$ approaches normality uniformly. For any family $\{F_\theta\}$ of bivariate distributions, where $\theta = 0$ corresponds to independent marginals, an expression for the asymptotically locally most powerful φ is given. The efficacy of the optimal φ, from which one can compute relative efficiency and approximate large sample power, is also derived. For a certain class of families $\{F_\theta\}$ including the correlated bivariate normal φ is given explicitly and for the bivariate normal the numerical value of the efficacy is computed.

53. On a sequential nonparametric estimation procedure (preliminary report). ELIZABETH H. YEN, The Catholic University of America.

Consider the problem of using a Wilcoxon two-sample statistic $U_{N/2,N/2}$, in a sample of total N (even number) observations, to estimate the parameter $\Pr(X > Y) = p$. When $G = F^\theta$ that is, $p = (1 + \theta)^{-1}$, the variance of this statistic is a function of N and of the unknown parameter θ. The author obtained some asymptotic optional properties for the two-stage estimation in a more general set-up. [*Ann. Math. Statist.* **33** (1964) 1099–1114].

This motivates the study of a sequential procedure for the present special case. The procedure utilizes the information about θ at each stage and allocates the N observations between the two populations. The resulting statistic $U_{m,n}$ $(m + n = N)$ have smaller variance than that of $U_{N/2,N/2}$ under the conditions that θ is a large integer (or by symmetry, θ^{-1} is a larger integer) and that $N > 4$. Furthermore, the magnitude of the squared bias of this estimate is negligible compared to its variance. Exact distribution is computed on $N =$ 6, 8, 10, 12, etc., and $\theta = 5, 10, 20, 50, 100$ and 200, etc. In general, when $\theta > N$ and N is not too small, $\text{Var}(U_{N/2,N/2})/\text{Var}(U_{m,n})$ is greater than 1. When θ is near $2N^2$, the ratio can be shown to be near 2. Possible application is also indicated.

54. Maximum likelihood estimation for the mixed analysis variance model. H. O. HARTLEY and J. N. K. RAO, Texas A & M University.

In this paper we develop a procedure for maximum likelihood estimation for the general mixed model in analysis of variance involving any number of fixed and/or random factors and/or interactions of any order. No experimental balance is required. The system of equations reached for the maximum likelihood estimators of all constants and variances (of variables) occurring in the model is, however, involved and must, in general, be solved by iterative methods. We also derive the asymptotic variances and co-variances of the estimators and approximate test procedures for (a) variance component and (b) subsets of effect constants arising in the model. A general proof for the convergence of the iterative procedure is given. A feature of particular interest is that the maximum likelihood estimators of all variance components can be made to be positive. By contrast the unbiased type of estimators which are customarily used for balanced analysis of variance data may attain negative values. Comparisons are made with the customary least square estimation procedures which are appropriate only if the model is a fixed one. It can be shown that the estimators of a variance components obtained by this method will in general not be efficient.

(Abstracts of papers not connected with any meeting of the Institute.)

1. Stochastic approximation and nonlinear regression—IV. A ALBERT, N. BARNERT, and L. A. GARDNER, JR., ARCON Inc. and Lincoln Laboratory, M.I.T.

Let $y_1, y_2, \cdots$ be an observable scalar-valued time series with uniformly bounded variances where $\mathcal{E}y_n = F_n(\theta)$ is known up to the (column) p-vector θ. We investigate estimation schemes of the form $t_{n+1} = t_n + a_n(t_1, \cdots, t_n)(y_n - F_n(t_n))$ where t_1 is arbitrary and $a_n(\cdot, \cdots, \cdot)$ is a vector-valued function of n vector arguments. Letting $g_n(x) = \text{grad } F_n(x), \gamma_n = \inf_x \|g_n(x)\|$ and $\Gamma_n^2 = \gamma_1^2 + \cdots + \gamma_n^2$, we consider (in particular) gain vectors for which $\inf_{x_1,\cdots,x_n} \|a_n(x_1, \cdots, x_n)\| \geqq a\gamma_n/\Gamma_n^2$ and $\sup_{x_1,\cdots,x_n} \|a_n(x_1, \cdots, x_n)\| \leqq b\gamma_n/\Gamma_n^2$ for some $0 < a \leqq b < \infty$. We assume (1) $\limsup_n \sup_x \|g_n(x)\|/\gamma_n \leqq K < \infty$, (2) $\Gamma_n^2 \to \infty$ and (3) $\gamma_n^2/\Gamma_n^2 \to 0$. Let $h_n(x) = g_n(x)/\|g_n(x)\|$ and assume (4) there exist integers $1 = n_1 < n_2 < \cdots$ with $p_k = n_{k+1} - n_k$ satisfying $p \leqq p_k \leqq q < \infty$ $(k = 1, 2, \cdots)$ such that $\liminf_k \lambda_k/p_k \geqq \tau > 0$ where $\lambda_k = \inf_{X_k} \lambda_{\min}\{H_k(X_k)H_k(X_k)'\}$, $H_k(X_k)$ is the $p \times p_k$ matrix $[h_{n_k}(x_1), h_{n_k+1}(x_2), \cdots, h_{n_{k+1}}(x_{p_k})]$, and X_k denotes the set of p_k vector arguments. Let J_k be the index set $\{n_k, n_k + 1, \cdots, n_{k+1} - 1\}$ and suppose $\liminf_k \min_{j\varepsilon J_k} \gamma_j^2/\max_{j\varepsilon J_k} \gamma_j^2 \geqq c > 0$. Finally, set $b_n(x_1, \cdots, x_n) = a_n(x_1, \cdots, x_n)/\|a_n(x_1, \cdots. x_n)\|$ and $\varphi_n = \inf_{x_1,\cdots,x_n,x_{n+1}} b_n'(x_1, \cdots, x_n)h_n(x_{n+1})$, and assume $\liminf_k \min_{j\varepsilon J_k}\varphi_j > \{(1 - \tau)/[1 - (1 - \rho^2)\tau]\}^{\frac{1}{2}}$ where $\rho = ac/Kb$. Then, if $\{y_n\}$ is a process of independent random variables, $t_n \to \theta$ with probability one and in mean square. If (2) is strengthened to (2′) $\sum(1/\Gamma_n^2) < \infty$, then the conclusion holds true without the assumption of independence.

2. Over estimation of binomial probabilities by Poisson probabilities. T. W. ANDERSON, Columbia University.

Let $b_n(x) = \binom{n}{x}(\lambda/n)^x(1 - \lambda/n)^{n-x}$, $p(x) = e^{-\lambda}\lambda^x/x!$, $B_n(y) = \sum_{x=0}^{y} b_n(x)$, and $P(y) = \sum_{x=0}^{y} p(x)$. THEOREM 1. *$P(y) > B_n(y)$, $0 \leqq y < c$, and $1 - P(y-1) > 1 - B_n(y-1)$, $c < y$ for some c.* THEOREM 2. *$b_n(x)$ is monotonically increasing in n for $0 \leqq x \leqq \lambda + \frac{1}{2} - (\lambda + \frac{1}{4})^{\frac{1}{2}}$ and for $\lambda + \frac{1}{2} + (\lambda + \frac{1}{4})^{\frac{1}{2}} \leqq x$.* THEOREM 3. *$P(y) - B_n(y)$ is positive and monotonically decreasing in n for $0 \leqq y \leqq \lambda + \frac{1}{2} - (\lambda + \frac{1}{4})^{\frac{1}{2}}$, and $1 - P(y-1) - [1 - B_n(y-1)]$ is positive and monotonically decreasing in n for $\lambda + \frac{1}{2} + (\lambda + \frac{1}{4})^{\frac{1}{2}} \leqq y$.*

3. On the F-test in the intrablock analysis of two associate partially balanced incomplete block designs (preliminary report). ROBERT CLÉROUX, Université de Montréal.

The distribution of the statistic U = (treatment sum of squares)/(treatment + error-sums of squares) is considered over all possible random assignments of the treatments on the experimental plots for two associate PBIB designs under the null hypothesis of absence of treatment effects. The first two moments of U are computed and the distribution of U is approximated by a beta distribution with degrees of freedom chosen in such a way that the means and the variances of the two distributions are equal respectively. A similar work has been carried out for two associate PBIB designs with $\lambda_1 = 0$ and $\lambda_2 = 1$ by P. V. Rao [*Ann. Tokio Inst. Math. Statist.* **15** (1963) 25–36] and for two associate PBIB designs with $\lambda_1 = 0$ and $\lambda_2 > 1$ by N. C. Giri [*J. Amer. Statist. Assoc.* **60** (1965) 285–293]. In this paper, results are obtained for the two associate PBIB designs in general and it is found that a reasonable approximation of the randomization test of the null hypothesis of absence of treatment effects, when the normal theory assumptions are in doubt, is obtained by modifying the degrees of freedom of the actual normal theory F-test.

4. Integrals of products of multivariate-t densities (preliminary report). J. M. DICKEY, Yale University.

The complete integral, over p-dimensional Euclidean space, of $f(\mathbf{x})g(\mathbf{x})$ is considered, where $f(\mathbf{x})$ is a product of K t-like factors, $[1 + (\mathbf{x} - \mathbf{y}_k)'\mathbf{M}(\mathbf{x} - \mathbf{y}_k)]^{-a_k}$, $a_k > 0$, $\mathbf{M}_k \geqq \mathbf{0}$, and where $g(\mathbf{x})$ is a polynomial in the coordinates of $\mathbf{x}$. If $K = 2$ and $g(\mathbf{x}) \equiv 1$, the integral is proportional to a multivariate generalization of the Behrens-Fisher density. These integrals give the normalizing constants and moments of some Bayesian posterior distributions [Tiao and Zellner, *Biometrika* **51** (1964)]. Results are limited to cases in which the $\mathbf{M}_k$ are simultaneously diagonalizable (not necessarily orthogonally). With each factor of $f(\mathbf{x})$ expressed as a gamma mixture of normal densities, the p-dimensional integrals reduce to $(K-1)$-dimensional integrals. A special-function (Appell's F_1) representation is given for the usual Behrens-Fisher density. Arguments are given favoring a general class of joint prior distributions for the parameters of a multivariate normal distribution such that the marginal posterior density for the mean vector (or regression vector) is proportional to $f(\mathbf{x})$.

5. On the interpolation of the renewal function. S. EHREFELD, New York University.

Some useful conditions are explored for determining a renewal process from knowledge of the renewal function $H(t)$ at values $H(h)$, $H(2h)$, $\cdots$ for some $h > 0$. Furthermore, these conditions are used for expressing $H(t)$ as an infinite interpolation series involving

$H(h)$, $H(2h)$, $\cdots$ and is useful in estimation problems involving attribute observations of renewal processes. One condition making $H(t)$ sufficiently smooth is when the Laplace transform of the probability density function of the inter-event random variable is analytic at infinity. The class of renewal processes where this is the case is fairly wide and includes many of those used in practice.

6. Estimation of parameters for a mixture of normal distributions. Victor Hasselblad, University of Washington.

n observations are taken from a mixture of K (known) normal subpopulations. It is assumed that these n observations are given as N frequencies from equally spaced intervals. Initial guesses of the K means, K variances, and $K - 1$ proportions are made using the truncated maximum likelihood estimates for a single normal population as derived by Hald. Then an approximation to the likelihood function of the entire sample is used, and attempts to maximize this yield two iteration formulas. The method of steepest descent is shown to always converge very fast when it converged at all. Special cases of equal variances and variances proportional to the square of the mean are also considered.

7. A problem in minimax variance polynomial extrapolation. Arnold Levine, University of Buenos Aires.

When the variables y_{t_1}, y_{t_2}, $\cdots$, y_{t_n} in the regression of a kth degree polynomial are uncorrelated, Hoel and Levine [*Ann. Math. Stat.* **35** (1964)] have found the observation points and respective proportion of observations at each point in the observation interval $[-1, 1]$ in order to obtain the minimax variance over the interval $[-1, t]$ for all $t > t_1 > 1$; t_1 is the point outside the interval $[-1, 1]$ at which the Chebyshev polynomial of degree k is equal to the maximum value of the variance of the least squares estimate inside $[-1, 1]$. Using the same observation points and corresponding proportions, it is found here that the maximum of the variance of the least squares estimate in $[-1, 1]$ is found at -1 for all $t > 1$. As a consequence, an equation is developed which permits the computation of t_1 for all k. Moreover, it is shown that $t_1 \to 1$ as $k \to \infty$ so that, for large k, these observation points and proportions yield an approximation to the minimax variance over the interval $[-1, t]$, all $t > 1$. The same techniques yield the exact determination of the minimax variance for linear regression.

8. A k-sample analogue of Watson's U^2-statistic. Urs R. Maag, Université de Montréal.

In this paper, extensions are given of the goodness-of-fit tests on the circle, U_N^2 and $U_{N,M}^2$, introduced by Watson [*Biometrika* **48** (1961) 109–114 and **49** (1962) 57–63]. Let X_{ij}, $i = 1, \cdots, n_j$, $j = 1, \cdots, k$ be k independent samples of independent random variables where the set X_{ij}, $i = 1, \cdots, n_j$, j fixed, stems from the continuous cumulative distribution function $F_j(x)$. Let $S_j(x)$ stand for the corresponding sample cdf, $S_N(x)$ for the sample cdf of the k pooled sets. To test the goodness-of-fit hypothesis $F_1(x) = \cdots = F_k(x) = G(x)$ (assumed to be known and continuous) the statistic $U'_{k,N} = \int_{-\infty}^{+\infty}\{\sum_{j=1}^{k} n_j[S_j(x) - G(x) - \int_{-\infty}^{+\infty}[S_j(y) - G(y)]dG(y)]^2\}dG(x)$ is proposed. To test the homogeneity hypothesis $F_1(x) = \cdots = F_k(x)$ the statistic $U_{k,N} = \int_{-\infty}^{+\infty}\{\sum_{j=1}^{k} n_j[S_j(x) - S_N(x) - \int_{-\infty}^{+\infty}[S_j(y) - S_N(y)]\cdot dS_N(y)]^2\}dS_N(x)$ is proposed. It is shown that under the null hypotheses the limiting distributions (all $n_j \to \infty$) of the homogeneity test for k samples and of the goodness-of-fit test for $k - 1$ samples are identical. This limiting distribution is derived and the result is tabulated for $k = 1, \cdots, 5$.

9. Minimum bias designs for the use of polynomial approximations to univariate exponential models (preliminary report). ALLISON R. MANSON, Virginia Polytechnic Institute and Oak Ridge Institute of Nuclear Studies.

For the model $y_u = \alpha + \beta x_u{}^\gamma + \epsilon_u$ or alternately $y_u = \alpha + \beta e^{-\gamma z_u} + \epsilon_u$ $(u = 1, 2, \cdots, n)$ where (α, β) lie on the real line $(-\infty, \infty)$, γ is a positive integer, and the ϵ_u are distributed independently as $N(0, \sigma^2)$; the sets of x_u which minimize the bias due to the inherent inability of polynomial approximations to fit such models are given. These designs are determined for specific values of n, resulting in a maximum protection level for the parameter γ. For example for $n = 13$ the set of x_u is given which will minimize the bias which occurs from using the approximation polynomial $\hat{y}_u = b_0 + b_1 x_u$ for all values of γ in the model from one to ten, inclusively. These same designs will minimize the bias due to the use of higher degree polynomial approximations; although for given n, the maximum value of γ for which the protection holds decreases gradually as the degree of the approximation polynomial increases. The criteria for obtaining the minimum bias designs follow along lines similar to those given by Box and Draper [*J. Amer. Statist. Assoc.* **54** (1959) 622]. Further work is in progress to obtain minimum "variance plus bias" designs and to extend the same to models containing two independent variables.

10. Some Bayesian decision problems in a Markov chain. JAMES J. MARTIN, United States Navy.

Some Bayesian decision models which involve a finite Markov chain with rewards in which the transition probabilities are uncertain are studied in this thesis. The principal theoretical features of these models are derived and various questions of numerical computation are considered. It is assumed throughout that the family of prior distributions of the matrix of transition probabilities is closed under the operation of Bayes' theorem. Some properties of such closed families of distributions are derived and used to find expressions for the prior means, variances, and covariances of the n-step transition probabilities, the steady-state probabilities, the total discounted reward vector, and the process gain. Two sequential sampling models of a Markov chain with alternative transition probabilities and rewards are formulated as sets of functional equations; one model includes an explicit sampling cost, while the other does not. In both cases a unique bounded solution is shown to exist and methods of successive approximations are considered. The prior-posterior and preposterior analysis of a finite Markov chain with uncertain transition probabilities is also developed for a fixed sample size. This analysis leads to the introduction of the Whittle, the matrix beta, and the beta-Whittle probability distributions.

11. Rank order probabilities; two-sample normal shift alternatives. ROY C. MILTON, Florida State University.

Assume that random variables $X_1, \cdots, X_m$ $(Y_1, \cdots, Y_n)$ are normally and independently distributed with mean $O(D)$ and variance 1. Let $U = (U_1, \cdots, U_{m+n})$, $U_1 < \cdots < U_{m+n}$, denote the order statistics of the random variables $(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$, and let $Z = (Z_1, \cdots, Z_{m+n})$ denote a random vector of zeros and ones where the ith component Z_i is $O(1)$ if U_i is an $X(Y)$. Denote by $f(x \mid D)$ the normal density with mean D and variance 1. If $z = (z_1, \cdots, z_{m+n})$ is a fixed vector of m zeros and n ones, then the probability of the rank order z, $\Pr(Z = z)$, is given by $P_{m,n}(z \mid D) = m!n! \int \cdots \int_R \prod_{i=1}^{m+n} f(t_i \mid Dz_i) dt_i$, where the integration is over the region $R = -\infty < t_1 \leqq \cdots \leqq t_{m+n} < \infty$. A method for computing $P_{m,n}(z \mid D)$ is described which involves a form of indefinite integration augmented by extrapolation to the limit. This method is used to calculate $P_{7,7}(z \mid D)$ and $P_{12,1}(z \mid D)$;

values for smaller sample sizes are obtained by Savage's back-recursive scheme [*Ann. Math. Statist.* **31** (1960) 519–520]. Values of $P_{m,n}(z \mid D)$ to nine decimal places are presented in tabular form for all z for $1 \leqq n \leqq m \leqq 7$ and $n = 1$, $m = 8(1)12$; $D = 0(.2)1,1.5,2,3$. This table enlarges and generalizes in several respects the unpublished tables of Teichroew [working paper of the National Bureau of Standards, Los Angeles, (1954)] and Klotz [doctoral dissertation, Univ. of California, Berkeley, 1961].

12. Power of two-sample nonparametric tests against the normal shift alternative. Roy C. Milton, Florida State University.

The tables described in the above abstract are used to find the exact power of the Wilcoxon, c_1, median and Kolmogorov-Smirnov two-sample tests for location against the normal shift alternative for sample sizes $1 \leqq n \leqq m \leqq 7$ and for one-sided and two-sided tests at nominal levels of significance $\alpha = .25, .10, .05, .025, .01, .005$. Selected power and efficiency comparisons are made among these tests and with the two-sample Student's t-test. The most powerful rank test is also considered.

13. Sequential two-sample rank tests of the normal shift hypothesis. Roy C. Milton, Florida State University.

Sequential two-sample rank tests based on the likelihood ratios of the probabilities of the vector z and of the rank sum $\sum_{i=1}^{m+n} iz_i$ are described, for notation see second abstract above. This extends the work of Wilcoxon, Rhodes and Bradley [*Biometrics* **19** (1963) 58–84] to the important case of the normal shift hypothesis. Tables are presented which facilitate the use of these tests, and values of the operating characteristic functions and average sample number functions are given.

14. A generalized selection procedure for normal populations. Roy C. Milton, Florida State University.

M. Sobel has suggested a multiple decision or ranking procedure for selecting a subset of r populations from among k normal populations with common variance σ^2 such that at least t_0 of the t populations with largest means are among the r populations selected. This generalizes the work of Bechhofer [*Ann. Math. Statist.* **25** (1954) 16–39]. The populations corresponding to the r largest sample means are selected. The probability of a correct selection in the so-called least favorable configuration (i.e., when δ^* is the difference between the ith and jth largest population means, $i \leqq t, j \geqq t + 1$) is given by $P(t_0; r, k - t, t, d)$ where $d = \delta^* N^{\frac{1}{2}}/\sigma$ and N is the common sample size. From the tables described in the third abstract above, values of $P(n_1; r, m, n, D)$ are given for $1 \leqq n \leqq m \leqq 7$ and $n = 1$, $m = 8(1)12$; $r = 1(1)[m/2]$; $n_1 = 1(1)n$; and $D = 0(.2)1,1.5,2,3$.

15. A nonparametric population selection procedure. D. W. Patterson, Bell Telephone Laboratories, Inc. and Rutgers University.

Given k populations with CPFs $F_1(x - a_1) = F_2(x - a_2) = \cdots = F_k(x - a_k)$, define $F_i(x)$ as "best" if $\max(a_1, a_2, \cdots, a_k) = a_i$. A nonparametric procedure is given for selecting a subset S which contains the best population with probability not less than a specified P^*. The procedure is based on a novel idea introduced by Bell and Doksum [*Ann. Math. Statist.* (1965)]. Let X_{ij}, $(i = 1, 2, \cdots, k; j = 1, 2, \cdots, n_i)$ be independent random variables from $F_i(x)$ and let $R[X_{ij}]$ denote the rank of X_{ij} in the combined observations. Let $Z(1), Z(2), \cdots, Z(N)$ be $N = \sum n_i$ order statistics from a CPF $G(z)$. The procedure consists of replacing X_{ij} by $Z(R[X_{ij}])$ and computing $\bar{Z}_{i\cdot} = [\sum_j Z(R[X_{ij}])]/n_i$. Population j is included in S if $\bar{Z}_{j\cdot} \geqq \max[\bar{Z}_{1\cdot}, \bar{Z}_{2\cdot}, \cdots, \bar{Z}_{k\cdot}] - b$, where $b > 0$ is chosen to achieve the desired probability of a correct selection $P(CS)$. It is shown that the infimum of $P(CS)$ occurs when $a_1 = a_2 = \cdots = a_k$. Furthermore the expected size of S is shown to be maximum when all a_i are equal. An attractive feature of the procedure is that standard tables may be employed in its application.

16. Multi-sample scale problem: unknown location parameters. MADAN L. PURI, Courant Institute of Mathematical Sciences, New York University.

In this paper, a class of tests proposed by the author (abstract, *Ann. Math. Statist.* June, 1965) for the problem of testing the equality of scale parameters of c continuous probability distribution functions are modified for the case when the location parameters are unknown. The modified test statistics are shown to retain their original asymptotic distributions (cf. the abstract cited above) under certain regularity conditions. The modification is done by expressing the sample observations as deviations from the respective consistent estimators of the unknown location parameters and basing the tests on these centered observations.

17. Large-sample sign tests for trend in dispersion. HANS K. URY, California State Department of Public Health and University of California, Berkeley.

For quickly testing trend in dispersion, Cox and Stuart [*Biometrika* **42** (1955), 80–95] have investigated several sign tests applied to the ranges of subsets of k observations. For a particular trend model, and against normal alternatives, they computed the asymptotic relative efficiency (ARE) of the best weighted (S_1) and unweighted (S_3) sign tests compared to the ML test. The ARE's depend on k and were given as 0.74 and 0.71 for S_1 and S_3, respectively, for subgroups of size 5 to 9. (This was intentionally conservative; actually, ARE's of 0.77 and 0.74 can be attained for $k = 8$.) In this note, S_1 and S_3 applied to variances of subsets are considered. The ARE is shown to increase with k, the limits being 0.86 for S_1 and 0.83 for S_3. These values can be almost attained for $k \geqq 49$, while subgroups of 15 will still yield ARE's of 0.83 for S_1 and 0.80 for S_3. Thus for large samples which permit the use of sizable subgroups, the resulting reduction in the number of sign tests and the increased ARE should compensate for the loss of simplicity incurred by using variance-based, rather than range-based, sign tests. "Efficiency" comparisons of different subgroup sizes are carried out for samples of 150 and 1500.