LIMIT THEOREMS FOR QUEUES WITH TRAFFIC INTENSITY ONE[1]

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1. Introduction. We shall consider a single server queueing process in which customers are served in the order of their arrival. Let the instants customers arrive at the counter be denoted by $\tau_1, \tau_2, \cdots, \tau_n, \cdots$. (We shall follow, for the most part, the notation used in Takács [14].) Let $\theta_n = \tau_{n+1} - \tau_n$ ($n = 0, 1, \cdots; \tau_0 = 0$) denote the interarrival times and let χ_n be the service time of the nth customer. We shall assume that $\{\theta_n\}$ and $\{\chi_n\}$ are independent sequences of independent, identically distributed, positive random variables. In particular, we shall assume that θ_n has an exponential distribution with mean $1/\lambda$ (the input process is a homogeneous Poisson process with intensity λ) and that χ_n has an arbitrary distribution, H. This queueing process is often designated $M/G/1$.

There are three important characteristics of the queue which we shall be interested in studying. These characteristics are the queue size (number of customers in the system), the busy period (interval of time during which the server is busy), and the waiting time of the customers. Let $\xi(t)$ denote the number of customers in the system at time t; i.e., the number of customers waiting or being served at time t. Let $\eta(t)$, the so-called virtual waiting time at time t, be the time that a customer would wait before starting his service, if he joined the queue at time t. We shall let $\tau_1', \tau_2', \cdots, \tau_n', \cdots$ denote the instants at which customers complete their service and depart from the system. Then if we let $\xi_n \equiv \xi(\tau_n' + 0)$, ξ_n will be the number of customers left in the system at the moment the nth customer departs. Also, if we define $\eta_n \equiv \eta(\tau_n - 0)$, η_n will be the time the nth customer waits before starting his service.

In the study of queueing processes an important role is played by the traffic intensity ρ, which is defined as the ratio of the expected service time to the expected interarrival time. If we denote the mean of the service distribution by μ, then $\rho \equiv \lambda\mu$. If $\rho < 1$, it has been shown by Takács [14] and others that the stochastic processes $\{\eta(t): t \geqq 0\}$, $\{\eta_n : n \geqq 1\}$, $\{\xi(t): t \geqq 0\}$, and $\{\xi_n : n \geqq 1\}$ converge in distribution to non-degenerate limiting distributions. In this case it is usually said that the queueing process attains a steady-state. The great bulk of papers in the queueing literature deal exclusively with the steady-state situation. On the other hand, if $\rho \geqq 1$, the distributions of the processes mentioned above tend to zero as either n or t tend to infinity.

In this paper we shall treat exclusively the case $\rho = 1$. Without loss of generality we shall choose our unit of time so that $\lambda = \mu = 1$. We shall be interested in obtaining limit distributions for certain functionals of the stochastic processes mentioned in the last paragraph. The basic tool used in obtaining these limit

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theorems is the following result of Darling and Kac [3] which we proceed to describe.

Let $\{X(t): t \geqq 0\}$ be a Borel measurable Markov process with stationary transition probabilities

$$P(t; x, E) = \Pr\{X(t+s) \in E \mid X(s) = x\},$$

and state space $\mathcal{E}$, a metric space. If V is a non-negative measurable function on E, then define the functional $Z(t)$ as

$$Z(t) = \int_0^t V[X(\tau)]\, d\tau.$$

When V is the characteristic function of a set $E \subset \mathcal{E}$, then $Z(t)$ is called the occupation time of E; that is, $Z(t)$ is the amount of time in the interval $[0, t]$ that $X(\tau)$ resides in E. Darling and Kac studied the limiting distribution of the stochastic process $\{Z(t): t \geqq 0\}$ as t tends to infinity. If we let $p_s(x, E) = \int_0^\infty e^{-st} P(t; x, E)\, dt$, then their result is contained in the following

THEOREM (Darling and Kac). *If the*

$$\int_{\mathcal{E}} p_s(x, dy) \cdot V(y) \sim Ch(s) \quad as \quad s \to 0+ \tag{1}$$

uniformly on the set $\{x : x \in \mathcal{E},\ V(x) > 0\}$, *where* C *is a positive constant and* $h(s) = s^{-\alpha} L(1/s)(0 \leqq \alpha < 1$, L *slowly varying*[2]), *then the*

$$\lim_{t\to\infty} \Pr\{Z(t)/Ch(1/t) \leqq x\} = G_\alpha(x), \tag{2}$$

where $G_\alpha(x)$ *is the Mittag-Leffler distribution. For discrete time Markov chains with* n *step transition probabilities* $p_{ij}^{(n)}$, *condition* (1) *should be replaced by*

$$\sum_{n=0}^\infty \sum_j p_{ij}^{(n)} w^n V(j) \sim Ch(1-w) \quad as \quad w \to 1- \tag{3}$$

uniformly on the set $\{i: V(i) > 0\}$. *Then the conclusion* (2) *takes the form*

$$\lim_{n\to\infty} \Pr\left\{ \sum_{i=0}^n V(X_i)/Ch(1/n) \leqq x \right\} = G_\alpha(x).$$

The Mittag-Leffler distribution $G_\alpha(x)$ is given by

$$G_\alpha(x) = (1/\pi\alpha) \int_0^x \sum_{n=1}^\infty [(-1)^{n-1}/n!]\Gamma(\alpha n + 1) y^{n-1} \sin n\pi\alpha\, dy.$$

For $\alpha = 0$, $G_0(x) = 1 - e^{-x}$, $x \geqq 0$, and for $\alpha = \frac{1}{2}$, $G_{\frac{1}{2}}(x) = \pi^{-\frac{1}{2}} \int_0^x e^{-y^2/4}\, dy$, $x \geqq 0$.

We should remark that Darling and Kac also obtain a converse to this theorem which we shall not use. Furthermore, Kesten [9] has extended [3] in the discrete time Markov chain case to obtain limit theorems for functions $V(j)$ which vanish outside a finite set of states, but where $V(j)$ need not be non-negative. The condition required for Kesten's result is essentially (3). For our queueing process we shall state the results as applications of the Darling-Kac theorem. At the expense of introducing more machinery we could equally well apply Kesten's results.

To apply the Darling and Kac result to our queueing process we must essen-

[2] A function $L(t)$, $0 \leqq t < \infty$, is said to be slowly varying if it is continuous and $L(ct) \sim L(t)$ as $t \to \infty$ for all $c > 0$.

tially check conditions (1) or (3). These conditions cannot, of course, be expected to hold for all queueing processes that we consider. We shall have to assume that the service distribution H belongs to the domain of attraction of a stable law with exponent α $(1 < \alpha \leqq 2)$. Since the expected value (μ) of H is one, this means that the $\lim_{n\to\infty} H_n(B_n x + n) = F(x)$, where H_n is the n-fold convolution of H with itself, $B_n = n^{1/\alpha}L(n)$, and F is a stable law of exponent α $(1 < \alpha \leqq 2)$. We shall summarize this situation by saying that $H \varepsilon \mathfrak{D}(F, \alpha)$. (Since H is a distribution function on the positive half-line, the possible limit distributions F, for a fixed α, differ only in the specification of a particular constant.) This condition contains all distributions possessing a finite variance (as part of the case $\alpha = 2$) plus many others as well.

We briefly indicate the contents of this paper. Section 2 is devoted to basic lemmas required to establish conditions like (1) and (3). In Section 3 we shall apply the Darling-Kac results to obtain limit laws for the Markov processes $\{\xi_n : n \geqq 1\}$, $\{\eta_n : n \geqq 1\}$, and $\{\eta(t): t \geqq 0\}$. Section 4 deals with the distribution of the busy period, G. In particular we show that if $H \varepsilon \mathfrak{D}(F, \alpha)$, $1 < \alpha \leqq 2$, then $G \varepsilon \mathfrak{D}(F^*, 1/\alpha)$ for some stable law F^*. A similar result is obtained for the distribution of the number of customers served in a busy period. These results immediately enable us to apply some results of Dynkin [5] and Lamperti [10] to obtain further limit laws. Finally, in Section 5 we mention a number of possible extensions.

We conclude this introduction by mentioning related work of other authors. Karlin and McGregor [7] have obtained occupation time laws for birth and death processes. For the special case of an exponential service time distribution their results yield the occupation time law for the Markov process $\{\xi(t): t \geqq 0\}$.

If we let $X_n = \chi_n - \theta_n$ and $S_n = \sum_{i=1}^{n} X_i$, then it is well known that the distribution of $\max\{0, S_1, \cdots, S_n\}$ coincides with the distribution of η_n; see Spitzer [11], p. 330. Darling in [2] has shown that if $E[X_n] = 0$ (this is the case for $\rho = 1$) and X_n has a finite variance, say $\mu_2 - 1$, then $\eta_n/n^{\frac{1}{2}}$ converges in distribution as n tends to infinity to the truncated normal distribution with parameter $\mu_2 - 1$. In [1] Brody shows that $\eta(t)/t^{\frac{1}{2}}$ has the truncated normal limit law (although his constant is not correct) as t tends to infinity. It is easy to show that $\xi_n/n^{\frac{1}{2}}$ and $\xi(t)/t^{\frac{1}{2}}$ also have the same truncated normal limit law.

Furthermore, Darling shows that if $E[X_n] = 0$ and the distribution of X_n belongs to $\mathfrak{D}(F, \alpha)$, $1 < \alpha \leqq 2$, which is the case if $H \varepsilon \mathfrak{D}(F, \alpha)$, then $\eta_n/n^{1/\alpha}$ converges to a non-degenerate limit law as n tends to infinity. It seems clear in this case that $\eta(t)/t^{1/\alpha}$, $\xi_n/n^{1/\alpha}$, and $\xi(t)/t^{1/\alpha}$ will also have this same non-degenerate limit law as either n or t tend to infinity.

2. Preliminary lemmas. To obtain the limit theorems described in the introduction we shall have to verify the asymptotic relation (1) or (3). This will require a knowledge of the transient behavior of the stochastic processes $\{\xi_n : n \geqq 1\}$, $\{\eta_n : n \geqq 1\}$, $\{\eta(t): t \geqq 0\}$. In the transient analysis of this queueing process by Takács [14] a very important role is played by the following

LEMMA 1 (Takács, [14], p. 47). *If* $\Re(s) \geqq 0$ *and* $|w| \leqq 1$, *then* $z = \gamma(s, w)$, *the root of the equation*

$$z = w\psi[s + \lambda(1 - z)] \tag{4}$$

which has smallest absolute value, is

$$\gamma(s, w) = \sum_{j=1}^{\infty} (\lambda^{j-1} w^j / j!) \int_0^\infty \exp\{-(\lambda + s)x\} x^{j-1} \, dH_j(x), \tag{5}$$

where $\psi(s) = \int_0^\infty e^{-sx} \, dH(x)$.

This root is a continuous function of s *and* w *if* $\Re(s) \geqq 0$ *and* $|w| \leqq 1$ *and further* $z = \gamma(s, w)$ *is the only root of* (4) *in the unit circle* $|z| < 1$ *if* $\Re(s) \geqq 0$ *and* $|w| < 1$ *or* $\Re(s) > 0$ *and* $|w| \leqq 1$ *or* $\Re(s) \geqq 0$, $|w| \leqq 1$ *and* $\rho > 1$. *Specifically,* $\omega = \gamma(0, 1)$ *is the smallest positive real root of the equation,* $\omega = \psi[\lambda(1 - \omega)]$. *If* $\rho > 1$, *then* $\omega < 1$; *and if* $\rho \leqq 1$, *then* $\omega = 1$.

Following Takács we let $\gamma(s) = \gamma(s, 1)$ and $g(w) = \gamma(0, w)$. Then appealing to Equation (4), we clearly have

$$\begin{aligned} \gamma'(0) &= -\mu/(1 - \rho) && (\rho < 1) \\ &= -\infty && (\rho = 1), \end{aligned}$$

and

$$\begin{aligned} g'(1) &= 1/(1 - \rho) && (\rho < 1) \\ &= \infty && (\rho = 1). \end{aligned}$$

We shall only deal with the case $\rho = 1$ and shall be interested in the rate at which $\gamma'(s)$ diverges as $s \to 0+$ and at which $g'(w)$ diverges as $w \to 1-$. Equivalently, we could study the rate at which $1 - \gamma(s)$ approaches zero as $s \to 0+$ or at which $1 - g(w)$ approaches zero as $w \to 1-$. In carrying out this analysis we can choose to work with either Equation (4) or (5). We shall illustrate both techniques for different cases. The lemmas that follow derive these asymptotic expressions starting from various assumptions on the service distribution H.

We begin by stating some results that we shall need in this development. The first lemma is a standard Abelian result; see for example Doetch [4], p. 460.

LEMMA 2. *If* C *is a constant,* $L(n)$ *is slowly varying, and*

$$a_n \sim Cn^{-1/\alpha}/L(n) \quad as \quad n \to \infty, \qquad 1 < \alpha \leqq 2,$$

then

$$\sum_{n=0}^{\infty} a_n w^n \sim C\Gamma(1 - 1/\alpha)(1 - w)^{\alpha^{-1}-1}/L(1/(1 - w)) \quad as \quad w \to 1-.$$

The second lemma is a very elegant local limit theorem recently obtained by Stone [13] for non-lattice distributions. (The author is indebted to Professor Charles J. Stone for making this result available to him before publication and for discussions on applications of the result.)

LEMMA 3. *Let* H *be a non-lattice distribution function in the domain of attraction of a non-degenerate stable distribution function* F *with exponent* α, $0 < \alpha \leqq 2$. *Let* $\{A_n\}$ *and* $\{B_n\}$ *be constants such that the* $\lim_{n\to\infty} H_n(B_n x + A_n) = F(x)$, *for*

$-\infty < x < \infty$. *Then*

$$H_n[B_n(x+h)+A_n] - H_n[B_nx+A_n] - [F(x+h) - F(x)] = o_n(1)(h+B_n^{-1}),$$

where the $\lim_{n\to\infty} o_n(1) = 0$ *uniformly in* x *and* h.

Next we use (4) to study the behavior of $1 - g(w)$ in the neighborhood of $w = 1$.

LEMMA 4. *If the service distribution* H *has a finite second moment* μ_2, *then* $1 - g(w) \sim (2/\mu_2)^{\frac{1}{2}}(1-w)^{\frac{1}{2}}$ *and* $g'(w) \sim (2\mu_2)^{-\frac{1}{2}}(1-w)^{-\frac{1}{2}}$ *both as* $w \to 1-$.

PROOF. Since $\mu_2 < \infty$, we can write $\psi(s)$ as

$$\psi(s) = 1 - s + (\mu_2/2)s^2 + o(s^2) \quad \text{as} \quad s \to 0+.$$

Appealing to (4) yields

$$g(w) = w\{1 - (1-g(w)) + (\mu_2/2)(1-g(w))^2 + o[(1-g(w))^2]\} \text{ as } w \to 1-.$$

After rearrangement we have

$$(1-g(w))^2 = (2/\mu_2)g(w)(1-w) + o[(1-g(w))^2] \quad \text{as} \quad w \to 1-,$$

which gives the first result. The asymptotic relation for $g'(w)$ follows immediately from L'Hospital's rule.

It should be noted that this same argument was used by Brody [1], p. 78, to obtain the first result in the next lemma. The second result again follows immediately from the first. The reader is cautioned that Brody used the unconventional notation σ^2 for the second moment of H; we use μ_2.

LEMMA 5 (Brody). *If* μ_2 *is finite, then* $1 - \gamma(s) \sim (2/\mu_2)^{\frac{1}{2}}s^{\frac{1}{2}}$ *and* $-\gamma'(s) \sim (2\mu_2)^{-\frac{1}{2}}s^{-\frac{1}{2}}$ *both as* $s \to 0+$.

Lemmas 4 and 5 could also be proved using (5), a local central limit theorem for H_n, Laplace's method for integrals, and Lemma 2 above. We proceed now to the case where $H \in \mathfrak{D}(F, \alpha)$, $1 < \alpha < 2$.

LEMMA 6. *If* $H \in \mathfrak{D}(F, \alpha)$, $1 < \alpha < 2$, *then*

$$1 - g(w) \sim \alpha f(0)\Gamma(1 - 1/\alpha)L^{-1}(1/(1-w))(1-w)^{1/\alpha} \tag{6}$$

and

$$g'(w) \sim f(0)\Gamma(1-1/\alpha)L^{-1}(1/(1-w))(1-w)^{(1/\alpha)-1} \tag{7}$$

both as $w \to 1-$, *where* f *is the density of* F *and* L *is slowly varying.*

PROOF. We shall derive (7) from which (6) can be obtained by a standard Abelian argument. Since $g(w)$ is analytic for $|w| < 1$ we can write $g'(w) = \sum_{n=0}^{\infty} b_n w^n$ where $b_n = (1/n!)\int_0^\infty e^{-x}x^n\, dH_{n+1}(x)$. First we shall show that $b_n \sim f(0)B_n^{-1}$ as $n \to \infty$, where as usual $\{B_n\}$ is such that $H_n[B_nx + n] \to F(x)$ as $n \to \infty$. This result together with Lemma 2 yields (7).

Given $\epsilon > 0$, choose M so large that $\Phi(-M) < \epsilon$, where Φ is the standardized normal distribution function. We decompose b_{n-1} into three parts $I_1(n)$, $I_2(n)$, and $I_3(n)$ as follows.

$$b_{n-1} = (1/(n-1)!)\{\sum_{k=0}^{[n-Mn^{\frac{1}{2}}]} \int_k^{k+1} e^{-x}x^{n-1}\, dH_n(x) + \sum_{k=[n-Mn^{\frac{1}{2}}]+1}^{[n+Mn^{\frac{1}{2}}]} \int_k^{k+1} + \sum_{k=[n+Mn^{\frac{1}{2}}]+1}^{\infty} \int_k^{k+1}\}$$
$$= I_1(n) + I_2(n) + I_3(n).$$

The principal contribution to b_{n-1} will come from $I_2(n)$. Since for $[n - Mn^{\frac{1}{2}}] + 1 \leqq k \leqq [n + Mn^{\frac{1}{2}}]$ we have $e^{-k}k^{n-1} \sim e^{-(k+1)}(k+1)^{n-1}$ as $n \to \infty$, we can write for sufficiently large n that

$$I_2(n) \leqq [(1+\epsilon)/(n-1)!] \sum_{k=[n-Mn^{\frac{1}{2}}]+1}^{[n+Mn^{\frac{1}{2}}]} e^{-k}k^{n-1}[H_n(k+1) - H_n(k)].$$

At this point we must distinguish the two cases in which H is either a non-lattice distribution or a lattice distribution. For the non-lattice case we appeal to Lemma 3 and for the lattice case we use the Gnedenko local limit theorem for lattice random variables; see [6], p. 236. We shall only carry out the details for the non-lattice case.

Using Lemma 3 we obtain

$$I_2(n) \leqq [(1+\epsilon)/(n-1)!] \sum_{k=[n-Mn^{\frac{1}{2}}]+1}^{[n+Mn^{\frac{1}{2}}]} e^{-k}k^{n-1}[F((k+1-n)/B_n) - F((k-n)/B_n) + o_n(1)B_n^{-1}]$$

for n large. Since all stable laws are absolutely continuous this inequality can be written as

$$(8) \quad I_2(n) \leqq [(1+\epsilon)^2/(n-1)!]B_n^{-1} \int_{[n-Mn^{\frac{1}{2}}]+1}^{[n+Mn^{\frac{1}{2}}]} e^{-x}x^{n-1}f((x-n)/B_n)dx + o_n(1)B_n^{-1}.$$

Now if we let

$$J(n) = [1/(n-1)!] \int_{n-Mn^{\frac{1}{2}}}^{n+Mn^{\frac{1}{2}}} e^{-x}x^{n-1}f((x-n)/B_n)\, dx,$$

we shall show that $J(n) \leqq f(0) + \epsilon$ for n large. Making the change of variable $y = (x-n)/B_n$ in $J(n)$ we obtain

$$J(n) = [B_n/(n-1)!] \int_{-Mn^{\frac{1}{2}}B_n^{-1}}^{Mn^{\frac{1}{2}}B_n^{-1}} \exp\{-(B_n y + n)\}(B_n y + n)^{n-1} f(y)\, dy.$$

Since $n^{\frac{1}{2}}B_n^{-1} \to 0$ as $n \to \infty$ and f is continuous, we have

$$J(n) \leqq [B_n/(n-1)!][f(0)+\epsilon] \int_{-Mn^{\frac{1}{2}}B_n^{-1}}^{Mn^{\frac{1}{2}}B_n^{-1}} \exp\{-(B_n y + n)\}(B_n y + n)^{n-1}\, dy$$

for n large. Letting $x = B_n y + n$ again yields

$$(9) \qquad J(n) \leqq \{[f(0)+\epsilon]/(n-1)!\} \int_{n-Mn^{\frac{1}{2}}}^{n+Mn^{\frac{1}{2}}} e^{-x}x^{n-1}dx \leqq f(0) + \epsilon,$$

since $e^{-x}x^{n-1}/(n-1)!$ is the density of a random variable. Combining (8) and (9) we arrive at $\limsup_{n\to\infty} B_n I_2(n) \leqq (1+\epsilon)^2(f(0)+\epsilon)$. In a similar manner we obtain $\liminf_{n\to\infty} B_n I_2(n) \geqq (1-\epsilon)^3(f(0)-\epsilon)$. Thus since ϵ was arbitrary we have $I_2(n) \sim f(0)B_n^{-1}$ as $n \to \infty$. Turning now to $I_1(n)$ we write

$$I_1(n) \leqq [1/(n-1)!] \sum_{k=0}^{[n-Mn^{\frac{1}{2}}]} e^{-(k+1)}(k+1)^{n-1} \cdot [F((k+1-n)/B_n) - F((k-n)/B_n) + o_n(1)B_n^{-1}],$$

since $e^{-x}x^{n-1}$ is monotone increasing in x for $0 \leqq x \leqq [n - Mn^{\frac{1}{2}}]$ when n is large. The stable law F has a bounded density $f(x) \leqq K$ for all x which yields

$$B_n I_1(n) \leqq [K/(n-1)!] \sum_{k=0}^{[n-Mn^{\frac{1}{2}}]} e^{-(k+1)}(k+1)^{n-1} + o_n(1)$$

for n large. Furthermore

$$[1/(n-1)!] \sum_{k=0}^{[n-Mn^{\frac{1}{2}}]} e^{-(k+1)}(k+1)^{n-1} \leqq [1/(n-1)!] \int_0^{n-Mn^{\frac{1}{2}}+2} e^{-x}x^{n-1}dx \leqq \Phi(-M) + \epsilon$$

for n large. The last inequality is obtained by applying the central limit theorem to exponentially distributed random variables. Hence, we have shown that the $\lim_{n\to\infty} B_n I_1(n) = 0$. Similarly we can show that $\lim_{n\to\infty} B_n I_3(n) = 0$. Thus we have now shown that $b_{n-1} \sim f(0)B_n^{-1}$ as $n \to \infty$ which combined with Lemma 2 yields (7).

The next lemma is proved in a similar manner.

LEMMA 7. *If $H \varepsilon \mathfrak{D}(F, \alpha)$, $1 < \alpha < 2$,*

$$1 - \gamma(s) \sim \alpha f(0)\Gamma(1 - 1/\alpha)L^{-1}(1/s)s^{1/\alpha}$$

and

$$-\gamma'(s) \sim f(0)\Gamma(1 - 1/\alpha)L^{-1}(1/s)s^{(1/\alpha)-1}$$

both as $s \to 0+$.

Finally, we give the value of $f(0)$ in

LEMMA 8. *If f is the density of the stable law F appearing in Lemmas* 6 *and* 7, *then*

$$f(0) = \pi^{-1}\Gamma(1 + 1/\alpha)c^{-1/\alpha}\Re\{(1 + i\tan(\pi\alpha/2))^{-1/\alpha}\}, \tag{10}$$

where the logarithm of the characteristic function of F is given by

$$-c\,|t|^\alpha\left[1 + (it/|t|)\tan\left(\frac{\pi\alpha}{2}\right)\right], \qquad c > 0.$$

PROOF. The inversion formula for characteristic functions states that

$$f(0) = (1/2\pi)\int_{-\infty}^{\infty} \exp\{-c|t|^\alpha[1 + (it/|t|)\tan(\pi\alpha/2)]\}\,dt.$$

This integral has been evaluated by Stone [12], p. 336, to give (10).

There is one case which is not treated in Lemmas 4–7; namely, the case in which $H \varepsilon \mathfrak{D}(F, 2)$ but has an infinite second moment. The proof in this case would follow the lines of Lemma 6, however, the evaluation of the $\lim_{n\to\infty} J(n)$ is more involved. We leave this case to the interested reader.

3. Occupation time laws. In this section we shall apply the Darling and Kac results to the Markov processes $\{\xi_n : n \geqq 0\}$, $\{\eta_n : n \geqq 1\}$, and $\{\eta(t) : t \geqq 0\}$.

We begin by discussing the process $\{\xi_n : n \geqq 0\}$, where ξ_n is the number of customers left in the system at the moment the nth customer departs. We suppose there is a departure at time $t = 0-$ and let $\xi_0 \equiv \xi(0)$. It is well known (see Kendall [8]) that $\{\xi_n : n \geqq 0\}$ forms a stationary Markov chain with transition probabilities

$$p_{ij} = \Pr\{\xi_{n+1} = j \mid \xi_n = i\}, \qquad n = 0, 1, 2, \cdots$$

given by

$$\begin{aligned} p_{ij} &= p_{j-i+1} && \text{if } j \geqq i - 1 \text{ and } i = 1, 2, \cdots \\ &= p_j && \text{if } j \geqq 0 \text{ and } i = 0 \\ &= 0 && \text{if } j < i - 1 \text{ and } i = 2, 3, \cdots \end{aligned}$$

where

$$p_j = \int_0^\infty (e^{-x}x^j/j!)\,dH(x), \qquad j = 0, 1, 2, \cdots .$$

As usual we let $p_{ij}^{(n)}$ denote the n-step transition probabilities. The double generating function for $p_{ij}^{(n)}$ has been obtained by Takács and we state it as

LEMMA 9 (Takács, [14], p. 70). *The double generating function of* $p_{ij}^{(n)}$ *for the case* $\lambda = 1$ *is given by*

$$\sum_{n=0}^\infty \sum_{j=0}^\infty p_{ij}^{(n)} w^n z^j$$
$$= \{z^{i+1}[1 - g(w)] - (1 - z)w\psi(1 - z)[g(w)]^i\}/[1 - g(w)][z - w\psi(1 - z)],$$

where $g(w)$ *and* $\psi(s)$ *are defined in Section* 2.

This lemma together with Lemmas 4 and 6 enable us to obtain the Darling-Kac occupation time laws for the Markov chain $\{\xi_n : n \geqq 0\}$. Unfortunately, we have to restrict the functions V that we use to vanish outside a *finite* set of positive integers. We state the result in

THEOREM 1. *If* μ_2 *is finite and the non-negative function* V *vanishes outside a finite set, then*

$$\lim_{n\to\infty} \Pr\{[1/C(V)(\mu_2/2)^{\frac{1}{2}} n^{\frac{1}{2}}] \textstyle\sum_{i=0}^n V(\xi_i) \leqq x\} = G_{\frac{1}{2}}(x),$$

where $C(V)$ *is a constant depending only on* V, *and* $G_{\frac{1}{2}}$ *is the Mittag-Leffler distribution of order* $\frac{1}{2}$. *On the other hand, if* $H \in \mathfrak{D}(F, \alpha)$, $1 < \alpha < 2$, *then the*

$$\lim_{n\to\infty} \Pr\{[\alpha f(0)\Gamma(1 - \alpha^{-1})/C(V)L(n)n^{1/\alpha}] \textstyle\sum_{i=0}^n V(\xi_i) \leqq x\} = G_{1/\alpha}(x).$$

PROOF. Consider the generating function given in Lemma 9. If we rewrite it as

$$\sum_{n=0}^\infty \sum_{j=0}^\infty p_{ij}^{(n)} w^n z^j = z^{i+1}/[z - w\psi(1 - z)]$$
$$- \{w[g(w)]^i/[1 - g(w)]\} \cdot \{(1 - z)\psi(1 - z)/[z - w\psi(1 - z)]\}$$

it is clear that the first term on the right hand side is negligible as $w \to 1-$. Since the term $(1 - z)\psi(1 - z)/[z - \psi(1 - z)] = a_0 + a_1 z + a_2 z^2 + \cdots$, we can deduce that $\sum_{n=0}^\infty p_{ij}^{(n)} w^n \sim a_j w[g(w)]^i/[1 - g(w)]$ as $w \to 1-$. Hence we have

$$\sum_{n=0}^\infty \sum_j p_{ij}^{(n)} w^n V(j) \sim \sum_j a_j V(j) w[g(w)]^i/[1 - g(w)] \text{ as } w \to 1-.$$

If we let $C(V) = \sum_j a_j V(j)$, then the theorem follows immediately from Lemmas 4 and 6 and the Darling-Kac result. Note that we need to restrict the set $\{j: V(j) > 0\}$ to be finite in order to obtain the required uniform convergence in (2).

Observe that if in Theorem 1 the function V is chosen to be the indicator function of the state 0, that $C(V) = 1$ and the limit law tells us that of the first n customers of the order of $n^{1/\alpha}$ of them leave behind them an empty system (i.e., an idle server) as $n \to \infty$.

We turn our attention now to the Markov process $\{\eta_n : n \geqq 1\}$. If we let $\Omega_n(s)$ be the Laplace-Stieltjes transform of the distribution of η_n, then the generating function of $\Omega_n(s)$ has been obtained by Takács [14], p. 57. We state the result as

LEMMA 10 (Takács). *The generating function of $\Omega_n(s)$ for the case $\lambda = 1$ is given by*

$$\sum_{n=1}^{\infty} \Omega_n(s) w^n = w(1-s)\Omega_1(s)/(1-s-w\psi(s)) - sg(w)\Omega_1[1-g(w)]/[1-g(w)][1-s-w\psi(s)].$$

We shall restrict our attention now to functions V which are indicator functions of bounded sets. Then Lemmas 4, 6 and 10 yield

THEOREM 2. *If μ_2 is finite, then the*

$$\lim_{n\to\infty} \Pr\{[1/C(E)(\mu_2/2)^{\frac{1}{2}} n^{\frac{1}{2}}] \sum_{i=1}^{n} V(\eta_i) \leqq x\} = G_{\frac{1}{2}}(x),$$

where V is the indicator function of the bounded set $E \subset [0, \infty)$ and $C(E)$ is a constant. On the other hand, if $H \varepsilon \mathfrak{D}(F, \alpha)$, $1 < \alpha < 2$, then the

$$\lim_{n\to\infty} \Pr\left\{\frac{\alpha f(0)\Gamma(1-\alpha^{-1})}{C(E)L(n)n^{1/\alpha}} \sum_{i=1}^{n} V(\eta_i) \leqq x\right\} = G_{1/\alpha}(x).$$

PROOF. Let $W_n(x, \cdot)$ be the measure induced on $[0, \infty)$ by the distribution of η_n, given $\eta_1 = x$. Then the Darling-Kac condition (1) becomes

$$\sum_{n=1}^{\infty} W_n(x, E) w^n \sim Ch(1-w)$$

as $w \to 1-$ uniformly for all $x \varepsilon E$. From Lemma 10 we know that

$$(11) \quad \int_0^{\infty} e^{-sy} \sum_{n=1}^{\infty} W_n(x, dy) w^n = w(1-s)e^{-sx}/[1-s-w\psi(s)] - sg(w)e^{-[1-g(w)]x}/[1-g(w)][1-s-w\psi(s)].$$

To obtain $\sum_{n=1}^{\infty} W_n(x, E)w^n$ we use the complex inversion formula for Laplace-Stieltjes transforms; see Widder [19], p. 69. If we let $\varphi(E, s) = \int_E e^{st}\, dt$ and denote the right side of (11) by $f(s; w, x)$, then

$$(12) \quad \sum_{n=1}^{\infty} W_n(x, E) w^n = \lim_{T\to\infty} (1/2\pi i) \int_{c-iT}^{c+iT} [f(s; w, x)/s]\varphi(E, s)\, ds, \; c > 0.$$

Since we shall let $w \to 1-$ in Equation (12), it is clear that the first term on the right hand side of (11) will not contribute to the asymptotic behavior of (12).

Thus we can apply Lemmas 4 and 6 to obtain the desired results with

$$C(E) = \lim_{T\to\infty} (1/2\pi i) \int_{c-iT}^{c+iT} \varphi(E, s)ds/[1 - s - \psi(s)]ds.$$

Turning now to the virtual waiting time of the process $\{\eta(t): T \geqq 0\}$, we let $\Omega(t, s) = E\{e^{-s\eta(t)}\}$. Since the Laplace transform of $\Omega(t, s)$ has been computed by Takács [14], p. 53, we can in principle use the same technique employed in Theorem 2 to obtain the occupation time laws for $\{\eta(t): t \geqq 0\}$ for bounded sets. We omit the details.

4. The busy period. The server's time is composed of alternating idle and busy periods. The length of the successive idle and busy periods are independent random variables. Since we assume a Poisson input to the queue, the distribution of the idle periods is clearly exponential. If we assume, as we shall in this section, that the queue starts with an idle period, then the length of all busy periods has the same distribution. We denote the distribution of the length of a busy period by G. For the case $\rho = 1$, it is well known that G is a proper distribution in the sense that $G(+\infty) = 1$ and that the mean of G is infinite; see Takács [14], p. 58 and 64. The next theorem gives slightly more information about G.

THEOREM 3. *If μ_2 is finite then $G \varepsilon \mathfrak{D}(F^*, \frac{1}{2})$ for some stable law F^* and*

$$1 - G(x) \sim (2/\pi\mu_2)^{\frac{1}{2}} x^{-\frac{1}{2}} \quad as \quad x \to +\infty.$$

Similarly, if $H \varepsilon \mathfrak{D}(F, \alpha)$, $1 < \alpha < 2$, then $G \varepsilon \mathfrak{D}(F^*, 1/\alpha)$ for some stable law F^* and

$$1 - G(x) \sim \alpha f(0) L^{-1}(x) x^{-1/\alpha} \quad as \quad x \to +\infty.$$

PROOF. On page 58 of [14] Takács shows that the Laplace-Stieltjes transform of G is $\gamma(s)$. Dynkin [5], p. 179, has shown that the asymptotic behavior of $1 - \gamma(s)$ indicated in Lemmas 5 and 7 is a necessary and sufficient condition for G to belong to the domain of attraction of a stable law. The asymptotic behavior of $1 - G(x)$ also follows from Dynkin's result.

We turn now to the distribution of N, the number of customers served during a busy period. Consider again the Markov chain $\{\xi_n: n \geqq 0\}$, the number of customers in the system at the departure points, with n-step transition probabilities $p_{ij}^{(n)}$. In the usual way we let $f_{ij}^{(n)}$ denote the probability of first passage from state i to state j in n steps. Then clearly we have the $\Pr\{N = n\} = f_{00}^{(n)}$, $n = 1, 2, \cdots$. Let $F_{00}(n) = \Pr\{N \leqq n\}$. Then we obtain

THEOREM 4. *If μ_2 is finite, then $F_{00} \varepsilon \mathfrak{D}(F^*, \frac{1}{2})$ for some stable law F^* and*

$$1 - F_{00}(n) \sim (2/\pi\mu_2)^{\frac{1}{2}} n^{-\frac{1}{2}} \quad as \quad n \to \infty.$$

Similarly, if $H \varepsilon \mathfrak{D}(F, \alpha)$, $1 < \alpha < 2$, then $F_{00} \varepsilon \mathfrak{D}(F^*, 1/\alpha)$ for some stable law F^* and

$$1 - F_{00}(n) \sim \alpha f(0) L^{-1}(n) n^{-1/\alpha} \quad as \quad n \to \infty.$$

PROOF. From Lemma 9 we have the fact that $\sum_{n=0}^{\infty} p_{00}^{(n)} w^n = 1/[1 - g(w)]$.

By the standard relation between the generating functions of $\{p_{00}^{(n)}\}$ and $\{f_{00}^{(n)}\}$ we see that $\sum_{n=1}^{\infty} f_{00}^{(n)} w^n = g(w)$. Now

$$\sum_{n=0}^{\infty} (1 - F_{00}(n))w^n = \sum_{\nu=1}^{\infty} f_{00}^{(\nu)} \sum_{n=0}^{\nu-1} w^n = [1 - g(w)]/(1 - w).$$

Thus for $\mu_2 < \infty$,

$$\sum_{n=0}^{\infty} (1 - F_{00}(n))w^n \sim (2/\mu_2)^{\frac{1}{2}}(1 - w)^{-\frac{1}{2}} \quad \text{as} \quad w \to 1-,$$

and for $H \varepsilon \mathfrak{D}(F, \alpha)$ $(1 < \alpha < 2)$,

$$\sum_{n=0}^{\infty} (1 - F_{00}(n))w^n \sim \alpha f(0)\Gamma(1 - 1/\alpha)L^{-1}(1/(1 - w)) \cdot (1 - w)^{\alpha^{-1}-1} \quad \text{as} \quad w \to 1-.$$

Since $1 - F_{00}(n)$ is non-increasing, we obtain from Karamata's Tauberian theorem the asymptotic behavior of $1 - F_{00}(n)$ as $n \to \infty$. The fact that F_{00} belongs to the domain of attraction of a stable law again follows from Dynkin [5].

The results of Theorems 3 and 4 enable us to apply a number of results obtained independently by Dynkin [5] and Lamperti [10]. We proceed to describe their results.

Let $\{X_n : n \geqq 1\}$ be a sequence of independent identically distributed positive random variables with distribution function F. We define three new random variables in terms of the partial sums $S_n = \sum_{i=1}^{n} X_i$ and $S_0 = 0$ as

$$\begin{aligned} N(t) &\equiv \max\{n : S_n \leqq t\}, \\ \gamma(t) &\equiv t - S_{N(t)}, \\ \delta(t) &\equiv S_{N(t)+1} - t. \end{aligned} \tag{13}$$

These are the random variables commonly studied in renewal theory. In renewal theory language $N(t)$ is the number of renewals in the interval $(0, t]$, $\gamma(t)$ is the age (or slack), and $\delta(t)$ is the excess. For this set-up Dynkin and Lamperti show that

(i) $E[N(t)] \sim (\alpha\pi/A \sin \alpha\pi)t^{\alpha}/L(t)$ as $t \to \infty$,

(ii) $\lim_{t\to\infty} \Pr\{\gamma(t)/t \leqq x\} = (\sin \alpha\pi/\pi) \int_0^x u^{\alpha-1}(1 - u)^{-\alpha}\, du$, $0 < x < 1$,

(iii) $\lim_{t\to\infty} \Pr\{\delta(t)/t \leqq x\} = (\sin \alpha\pi/\pi) \int_0^x u^{-\alpha}(1 + u)^{-1}\, du$, $0 < x < \infty$,

if and only if $1 - F(x) \sim AL(x)x^{-\alpha}$ as $x \to \infty$ with $0 < \alpha < 1$.

Now the time in a queueing process can be conveniently divided into cycles. We shall consider a cycle to begin at the start of an idle period and to end at the completion of the next busy period. Again we assume that the server is initially idle. Since the length of an idle period is exponentially distributed and independent of the next busy period, the Laplace-Stieltjes transform of the cycle length is $(1 + s)^{-1}\gamma(s)$. Thus the distribution of the length of a cycle belongs to the domain of attraction of a stable law with exponent β $(0 < \beta < 1)$ if G

does. Hence we can immediately apply the Dynkin and Lamperti results to the sequence of cycle lengths. In this context the random variables defined in (13) have the following interpretations:

$N(t)$ $\equiv$ number of complete cycles in $(0, t]$;

$\gamma(t)$ $\equiv$ the time measured at t since the end of the last completed busy-period; and

$\delta(t)$ $\equiv$ the time measured at t until the end of the current cycle.

In a similar manner we can consider the sequence of random variables $\{N_n : n \geqq 1\}$, where N_n is the number of customers served in the nth cycle. For this case the random variables of (13) are

$N(n)$ $\equiv$ number of cycles completed at the moment the nth customer departs,

$\gamma(n)$ $\equiv$ number of customers served in the current cycle at the moment the nth customer departs, and

$\delta(n)$ $\equiv$ number of customers still to be served in the current cycle at the moment the nth customer departs.

Again we can apply the Dynkin and Lamperti results with the help of Theorem 4.

5. Extensions. There are a number of ways in which the above results can be generalized. The first and perhaps simplest extension would be to allow customers to leave the queue without being served with positive probability q rather than waiting. Takács considers this problem in [16]. The analysis is very similar to the usual $M/G/1$ queue.

Secondly, we could analyze the queue $GI/M/1$ by essentially the same techniques as we have used here for $M/G/1$. Furthermore, at the expense of greater technical details we could also handle the queues $M/G/1$ and $GI/M/1$ in which the customers are served in batches of size m in the order of their arrival; see Takács [14], p. 81–112 and 125–139. Also, the queue $E_m/G/1$ (interarrival times have a gamma distribution) can be analyzed as $M/G/1$ with customers served in batches of size m; see Takács [15].

Finally, we mention the situation for the queue $M/G/1$ in which the distribution of the service time for a customer initiating a busy period is different than that of the other customers. In [17] Welch has analyzed that problem along the lines of Takács' work. In a second paper Welch [18] uses the results of [17] to investigate preemptive resume queues. Welch's results can thus be used to study the case $\rho = 1$ which we have considered here.

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