ADMISSIBILITY AND BAYES ESTIMATION IN SAMPLING FINITE POPULATIONS. III

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1. Introduction. In the Part I of this paper in Theorem 4.1 the Horvitz-Thomson estimate (H.T. estimate for short) of the population total was shown to be admissible in the class of all unbiased estimates. The restriction of unbiasedness was removed in Part II, but there the estimate shown to be admissible in the entire class, is different from the H.T. estimate. In Section 9 of Part I however the H.T. estimate was shown to be inadmissible in the entire class if the sampling design was not of fixed sample size, as defined there. Now in this part of the paper it is shown that for any sampling design of fixed sample size, the H.T. estimate is admissible in the class of all estimates satisfying a certain "regularity" (refer to Theorem 3.1) condition. This result, thus is a generalization of the Theorem 8.1 in Part I, where the H.T. estimate was proved to be admissible in the class of all linear estimates. As in Theorem 8.1 of Part I the present result is proved for a more general class of estimates of which the H.T. estimate is a particular case. Actually, for this general class but *excluding* the H.T. estimate, the admissibility is established following an argument due to the referee, among all 'measurable' estimates, thus relaxing the above referred to conditions of 'regularity.' In this connection we refer to Theorems 4.2 and 4.3.

One may note that the results, in this part of the paper are weaker than the result proved in the Part II, in the sense that they need the regularity or measurability conditions for their validation; and they are true for the fixed sample-size designs only, while the result of Part II is true regardless of any such restrictions.

ADDED AT PROOF STAGE: It is now clear to the author that due to a property of Laplace Transforms the results of this paper are valid *without* any reference to the *regularity* condition. However this and the measurability condition in this paper would be discussed in a subsequent publication.

2. Notation. The notation followed here is the same as formulated in Section 2 of Part I with the slight modification as adopted in Section 2 of Part II. The definitions and preliminaries in Section 2 of Part I are also applicable to the following discussion.

3. Admissibility of the estimate. As in Theorem 8.1 of Part I we shall prove the admissibility for a more general estimate of which the Horvitz-Thomson estimate is a particular case. We now denote the estimate by $\hat{e}(s, x)$. In Theorem 8.1 of Part I $\hat{e}(s, x)$ was defined by $\hat{e}(s, x) = \sum_{r \varepsilon s} b_r x_r$, where the coefficients b_r satisfy (i) $b_r \geqq 1$, $r = 1, 2, \cdots, N$, and (ii) $\sum_1^N 1/b_r = m$. Retaining condition

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(ii) we now, for the purpose of the following Theorem 3.1, replace condition (i) by the more stringent one, namely $b_r > 1$, $r = 1, 2, \cdots, N$. We now state the following

THEOREM 3.1. *If*

(a) *the sampling design is one of fixed sample size* m, i.e. *it satisfies* $p(s) = 0$ *whenever the sample size (Definition 2.4 of Part I)* $n(s) \neq m$, *a fixed integer and*

(b) $\hat{e}(s, x)$ *is an estimate given by*

$$\hat{e}(s, x) = \sum_{r \varepsilon s} b_r x_r, \tag{1}$$

where the coefficients b_r *satisfy*

(i) $b_r > 1$, $r = 1, 2, \cdots, N$ and

(ii) $\sum_{r=1}^{N} (b_r)^{-1} = m$,

then the estimate $\hat{e}(s, x)$ *is weakly admissible for the population total* $T(x)$ *in the sense that there exists no regular*[2] *estimate* $e'(s, x)$ *such that*

$$\sum_{s \varepsilon S} p(s)(e'(s, x) - T(x))^2 \leqq \sum_{s \varepsilon S} p(s)(\hat{e}(s, x) - T(x))^2 \tag{2}$$

for almost all $x \varepsilon R_N$ *(Lebesgue measure) and further the strict inequality in (2) holds for a non-null (Lebesgue) set in* R_N. *Further if any estimate* $e'(s, x)$ *satisfies* (2), a.e. *in* R_N *(Lebesgue measure) then for all* s *for which* $p(s) \neq 0$, $e'(s, x) = \hat{e}(s, x)$, a.e. *(Lebesgue measure) in* R_N.

NOTE. Throughout the rest of this paper the measure considered will be the Lebesgue measure for the space R_N. So also for any k dimensional sub space R_k of R_N the measure considered will be the Lebesgue measure for the k dimensional sub space. These points will not be repeated each time. The measure will be made explicit only when there is any possibility of confusion.

PROOF. Let $e'(s, x)$ be an estimate which satisfies (2). We make a transformation of the variates by putting

$$x_r = y_r / b_r, \qquad r = 1, 2, \cdots, N. \tag{3}$$

Further let

$$f(s, y) = e'(s, x) - \sum_{r \varepsilon s} x_r. \tag{4}$$

Substituting (3), (4) and (1) in (2) we have

$$\sum_{s \varepsilon S} p(s)(f(s, y) - \sum_{r \notin s} (y_r / b_r))^2 \leqq \sum_{s \varepsilon S} p(s)(\sum_{r \varepsilon s} (1 - (b_r)^{-1}) y_r - \sum_{r \notin s} (y_r / b_r))^2. \tag{5}$$

We now take expectations of both sides of (5) wrt a prior distribution on R_N, such that all the variates y_r, are distributed independently with common mean θ, the distribution being absolutely continuous in R_N. Then putting

$$A(s) = \sum_{r \notin s} (b_r)^{-1}, \tag{6}$$

[2] An estimate e is said to be *regular* if for every s with $p(s) > 0$, the expected value (s held fixed) of $e(s, x)$ wrt the frequency function L in (14) is differentiable wrt θ, under the integral sign. Evidently all the linear estimates (in (4) of Part I) are in this sense regular.

we have

$$E(f(s, y) - \sum_{r \notin s} (y_r/b_r))^2 = E(f(s, y) - \theta \cdot A(s) - \sum_{r \notin s} ((y_r - \theta)/b_r))^2$$
$$= E(f(s, y) - \theta \cdot A(s))^2 + E(\sum_{r \notin s} ((y_r - \theta)/b_r))^2$$

the product term vanishing due to the independence of the variates. Similarly in the right hand side of (5),

$$E(\sum_{r \varepsilon s} (1 - b_r^{-1})y_r - \sum_{r \notin s} (y_r/b_r))^2$$
$$= E(\sum_{r \varepsilon s} (1 - b_r^{-1})y_r - A(s) \cdot \theta)^2 + E(\sum_{r \notin s} ((y_r - \theta)/b_r))^2.$$

Thus by taking expectations of both sides of (5) we get by cancelling out the common term,

$$(7) \quad \sum_{s \varepsilon \bar{S}} p(s) E(f(s, y) - A(s) \cdot \theta)^2$$
$$\leqq \sum_{s \varepsilon \bar{S}} p(s) E(\sum_{r \varepsilon s} (1 - b_r^{-1})y_r - A(s) \cdot \theta)^2,$$

where $\bar{S}$ denotes the subset of S consisting of all those samples s for which $p(s) \neq 0$. We introduce this subset to avoid the bothersome repetition of the condition $p(s) \neq 0$ which otherwise becomes necessary. Hereafter all samples s considered will be such that $s \varepsilon \bar{S}$.

Now using condition (ii) in clause (b) of the Theorem, since each $s \varepsilon \bar{S}$ contains m units, we have in (6)

$$(8) \qquad A(s) = m - \sum_{r \varepsilon s} b_r^{-1} = \sum_{r \varepsilon s} (1 - b_r^{-1}).$$

We now put in (7),

$$(9) \qquad \bar{y}(s) = \sum_{r \varepsilon s} (1 - b_r^{-1}) y_r / A(s)$$

and

$$(10) \qquad g(s, y) = (A(s))^{-1} \cdot f(s, y)$$

and get

$$(11) \quad \sum_{s \varepsilon \bar{S}} p(s) A^2(s) E(g(s, y) - \theta)^2 \leqq \sum_{s \varepsilon \bar{S}} p(s) A^2(s) E(\bar{y}(s) - \theta)^2,$$

where $A^2(s)$ as usual denotes $(A(s))^2$.

We next make the further assumption regarding the prior distribution on R_N, namely that each variate y_r, $r = 1, 2, \cdots, N$, is distributed normally with variance σ_r^2 proportional to

$$(12) \quad (1 - b_r^{-1})^{-1}, \quad \text{so that} \quad \sigma_r^2 = k/(1 - b_r^{-1}) \quad \text{where } k \text{ is a constant} > 0.$$

Note that this assumption is permissible because by the assumed condition (i) in clause (b) of the Theorem, $1 - (b_r)^{-1} > 0$ for all r. This (incidentally) explains why for the present Theorem it was necessary to modify the corresponding condition in Theorem 8.1, Part I and exclude cases in which any $b_r = 1$. It is now easily seen that for $\bar{y}(s)$ in (9)

$$E(\bar{y}(s) - \theta)^2 = k/A(s). \tag{13}$$

For $E(g(s, y) - \theta)^2$ in the left hand side, we apply the Cramér-Rao lower bound. Now for the variates y_r for which $r \varepsilon s$ the frequency function is

$$L = ((2\pi)^{m/2})^{-1} \cdot \prod_{r \varepsilon s} \sigma_r^{-1} \exp(-\tfrac{1}{2} \sum_{r \varepsilon s} ((y_r - \theta)^2/\sigma_r^2)).$$

Hence

$$\begin{aligned} E(\partial \log L/\partial\theta)^2 &= E(\sum_{r \varepsilon s} ((y_r - \theta)/\sigma_r^2))^2 \\ &= \sum_{r \varepsilon s} \sigma_r^{-2} \\ &= \sum_{r \varepsilon s} (1 - b_r^{-1})/k \\ &= A(s)/k. \end{aligned} \tag{14}$$

Now, let $Eg(s, y) = \theta + b(s, \theta)$ where $b(s, \theta)$ is the bias of the estimate. Then from (14) by the Cramér-Rao inequality (validated by the preceding footnote 2),

$$\operatorname{Var} g(s, y) = E(g(s, y) - Eg(s, y))^2 \geqq (k/A(s))(1 + b'(s, \theta))^2,$$

hence

$$E(g(s, y) - \theta)^2 \geqq (k/A(s))(1 + b'(s, \theta))^2 + b^2(s, \theta). \tag{15}$$

Now substituting (13) and (15) in (11) we have

$$\sum_{s \varepsilon \bar{S}} p(s)A^2(s)b^2(s, \theta) + k \sum_{s \varepsilon \bar{S}} p(s)A(s)(1 + b'(s, \theta))^2 \leqq k \sum_{s \varepsilon \bar{S}} p(s)A(s). \tag{16}$$

We now define the weighted mean bias $\bar{b}(\theta)$ by

$$\bar{b}(\theta) = \sum_{s \varepsilon \bar{S}} p(s)A(s)b(s, \theta)/\sum_{s \varepsilon \bar{S}} p(s)A(s). \tag{17}$$

(Note that by (6) $A(s) > 0$ for all $s \varepsilon \bar{S}$.) Then

$$1 + \bar{b}'(\theta) = \sum_{s \varepsilon \bar{S}} p(s)A(s)(1 + b'(s, \theta))/\sum_{s \varepsilon \bar{S}} p(s)A(s),$$

and hence using the usual property of the mean,

$$\begin{aligned} \sum_{s \varepsilon \bar{S}} p(s)A(s)(1 + b'(s, \theta))^2 &= (1 + \bar{b}'(\theta))^2 \sum_{s \varepsilon \bar{S}} p(s)A(s) \\ &\quad + \sum_{s \varepsilon \bar{S}} p(s)A(s)(b'(s, \theta) - \bar{b}'(\theta))^2 \\ &\geqq (1 + \bar{b}'(\theta))^2 \sum_{s \varepsilon \bar{S}} p(s)A(s). \end{aligned} \tag{18}$$

Now using (18) in (16) and noting that by (12), $k > 0$, we have

$$(\sum_{s \varepsilon \bar{S}} p(s)A(s))^{-1}k^{-1} \sum_{s \varepsilon \bar{S}} p(s)A^2(s)b^2(s, \theta) + (1 + \bar{b}'\theta))^2 \leqq 1. \tag{19}$$

Now proceeding as in Problem 1 in the paper by Hodges and Lehmann (1951), we show that $\bar{b}(\theta)$ vanishes for all θ. Since neither term on the left side of (19) can be negative $|b(s, \theta)|$ for each $s \varepsilon \bar{S}$ is bounded above and hence so is $|\bar{b}(\theta)|$. Hence as $|\theta| \to \infty$, $\bar{b}'(\theta) \to 0$. Hence from (19) as $|\theta| \to \infty$, $|b(s, \theta)|$ for each $s \varepsilon \bar{S}$ and consequently $|\bar{b}(\theta)| \to 0$. But from (19) $\bar{b}'(\theta)$ is never positive. Since $\bar{b}(\theta)$ is monotonic, $\bar{b}(\theta) \to 0$ when $\theta \to \pm\infty$ implies that for all θ, $\bar{b}(\theta) = 0$ and consequently $\bar{b}'(\theta) = 0$ for all θ. It then follows from (19) that for all $s \varepsilon \bar{S}$, $b(s, \theta) = 0$. It next follows from (15) and (14) that $g(s, y)$ is an unbiased, efficient estimate of θ, and similarly from (13) $\bar{y}(s)$ is an unbiased, efficient estimate of θ. Hence by the usual Cramér-Rao theory, (see for example p. 483, Cramér (1951)) $g(s, y) = \bar{y}(s)$, a.e. in R_N. Using (3), (4) and (10) it then follows that $e'(s, x) = \hat{e}(s, x)$ a.e. in R_N. Hence the strict inequality in (2) can hold at most on a null set. This completes the proof of the Theorem.

4. Generalization of Theorem 3.1. Using the result proved in Theorem 3.1 we can now replace condition (ii) in clause (b) of that Theorem by the more general condition $\sum_{r=1}^{N} b_r^{-1} \geqq m$. To prove this, we consider the hyperplanes in R_N obtained by assigning fixed values to some k of the variates. Let Q_{N-k}^{α} be the hyperplane in which say, the last k variates x_{N-k+t}, $t = 1, 2, \cdots, k$ have fixed values $= \alpha_{N-k+t}$ respectively. Let $\bar{S}_k$ be the subset of $\bar{S}$, $\bar{S}_k \subset \bar{S}$, consisting of all the samples s which include each of the last k units u_{N-k+t}, $t = 1, 2, \cdots, k$, i.e. $s \varepsilon \bar{S}_k$ if and only if all $u_{N-k+t} \varepsilon s$ for $t = 1, 2, \cdots, k$, and $s \varepsilon \bar{S}$. Now suppose that for $x \varepsilon Q_{N-k}^{\alpha}$ and $s \varepsilon \bar{S}_k$, a regular estimate $e'(s, x)$ exists such that

$$(20) \qquad \sum_{s\varepsilon\bar{S}_k} p(s)(e'(s, x) - T(x))^2 \leqq \sum_{s\varepsilon\bar{S}_k} p(s)(\hat{e}(s, x) - T(x))^2$$

a.e. in Q_{N-k}^{α} and further such that $h(s, x) = e'(s, x) - \hat{e}(s, x) \neq 0$ for a non-null set in Q_{N-k}^{α}. Now $e'(s, x)$ is defined in (20) only for $x \varepsilon Q_{N-k}^{\alpha}$ and $s \varepsilon \bar{S}_k$. We next extend the definition to other points $x \varepsilon R_N$ and to samples $s \not\varepsilon \bar{S}_k$ as follows:

$$(21) \qquad \text{for } s \not\varepsilon \bar{S}_k \text{ and all } x \varepsilon R_N, \quad e'(s, x) = \hat{e}(s, x).$$

Next let $Q_{N-k}^{\alpha'}$ be the hyperplane $\subset R_N$, given by $x_{N-k+t} = \alpha'_{N-k+t}$, $t = 1, 2, \cdots, k$. We now establish a 1–1 correspondence between the points $x' \varepsilon Q_{N-k}^{\alpha'}$ and $x \varepsilon Q_{N-k}^{\alpha}$ by putting

$$(22) \qquad x_r' = x_r + a/b_r, \qquad r = 1, 2, \cdots, N - k$$

the constant a being so fixed that for $s \varepsilon \bar{S}_k$,

$$(23) \qquad \hat{e}(s, x') - T(x') = \hat{e}(s, x) - T(x).$$

This can always be done because noting that every $s \varepsilon \bar{S}_k$ includes each of the last k units and hence $m - k$ of the first $N - k$ units, for $x' \varepsilon Q_{N-k}^{\alpha'}$ and $s \varepsilon \bar{S}_k$,

$$\hat{e}(s, x') = \hat{e}(s, x) + a(m - k) + \sum_{r=N-k+1}^{N} (\alpha_r' - \alpha_r)b_r$$

and

$$T(x') = T(x) + a \sum_{r=1}^{N-k} b_r^{-1} + \sum_{r=N-k+1}^{N} (\alpha_r' - \alpha_r),$$

so that (23) is satisfied if

$$(24) \qquad a\left(\sum_{r=1}^{N-k} b_r^{-1} - (m - k)\right) = \sum_{r=N-k+1}^{N} (\alpha_r' - \alpha_r)(b_r - 1).$$

In the left hand side of (24) $\sum_{r=1}^{N-k} b_r^{-1} - m + k = k - \sum_{r=N-k+1}^{N} b_r^{-1} > 0$, using condition (i) in clause (b) of Theorem 3.1. Hence the constant a satisfying (24) and therefore (23) can be always found for any set of constants α_r', $r = N - k + 1, 2, \cdots, N$. Next for $x' \varepsilon Q_{N-k}^{\alpha'}$ and $s \varepsilon \bar{S}_k$ we define $e'(s, x')$ by

$$(25) \qquad e'(s, x') - T(x') = e'(s, x) - T(x),$$

which requires $e'(s, x') = e'(s, x) + \sum_{r=1}^{N-k} (a/b_r) + \sum_{r=N-k+1}^{N} (\alpha_r' - \alpha_r)$. From (23), (25) and (20) it follows that for $x' \varepsilon Q_{N-k}^{\alpha'}$,

$$(26) \quad \sum_{s\varepsilon\bar{S}_k} p(s)(e'(s, x') - T(x'))^2 \leqq \sum_{s\varepsilon\bar{S}_k} p(s)(\hat{e}(s, x') - T(x'))^2$$

a.e. in $Q_{N-k}^{\alpha'}$. It also follows that for $x' \varepsilon Q_{N-k}^{\alpha'}$, $h(s, x') = e'(s, x') - \hat{e}(s, x')$ is $\neq 0$ if and only if for the corresponding point $x \varepsilon Q_{N-k}^{\alpha}$, $h(s, x) \neq 0$. Hence corresponding to the non-null set $P^{\alpha} \subset Q_{N-k}^{\alpha}$ such that $h(s, x) \neq 0$ if $x \varepsilon P^{\alpha}$ we have a non-null set $P^{\alpha'} \subset Q_{N-k}^{\alpha'}$ such that $h(s, x') \neq 0$ if $x' \varepsilon P^{\alpha'}$.

(27) Thus every hyperplane $Q_{N-k}^{\alpha'}$ has a non-null set $P^{\alpha'}$ and hence the set $P = \bigcup_{\alpha'} P^{\alpha'}$ in R_N such that $h(s, x) \neq 0$ for $x \varepsilon P$ is a non-null set.

But from (20), (21) and (26) the estimate $e'(s, x)$ satisfies

$$\sum_{s\varepsilon\bar{S}} p(s)(e'(s, x) - T(x))^2 \leqq \sum_{s\varepsilon\bar{S}} p(s)(\hat{e}(s, x) - T(x))^2$$

a.e. in R_N. Hence by Theorem 3.1 the set $P \subset R_N$ on which $h(s, x) \neq 0$ must be a null set, thus contradicting (27). Hence in the original hyperplane Q_{N-k}^{α} the set P^{α} must be a null set. We thus have

THEOREM 4.1. *If a regular estimate* $e'(s, x)$ *satisfies* (20) a.e. *in* Q_{N-k}^{α}, *then* $e'(s, x) = \hat{e}(s, x)$ a.e. *in* Q_{N-k}^{α}.

Next consider the particular case of the hyperplane Q_{N-k}^{0} given by putting $x_{N-k+t} = 0$ for $t = 1, 2, \cdots, k$. For $x \varepsilon Q_{N-k}^{0}$ the population total $T(x)$ becomes $\sum_{i=1}^{N-k} x_i$ equal to the population total $T^*(x)$ for the subpopulation U^* consisting of the first $N - k$ variates, i.e. $U^* = (u_1, u_2, \cdots, u_{N-k})$. Let s^* denote the sample obtained by omitting from each $s \varepsilon \bar{S}_k$ the last k units $u_{N-k+1}, \cdots, u_N$. Then for $x \varepsilon Q_{N-k}^{0}$, $\hat{e}(s, x) = \hat{e}(s^*, x)$ for all $s \varepsilon \bar{S}_k$. For the subpopulation U^* let the sampling design d^* be such that $p(s^*) = p(s)/p(\bar{S}_k)$ where $p(\bar{S}_k) = \sum_{s\varepsilon\bar{S}_k} p(s)$. Suppose now there exists an estimate $e'(s^*, x)$ such that

$$(28) \quad \sum_{s^*\varepsilon\bar{S}^*} p(s^*)(e'(s^*, x) - T^*(x))^2 \leqq \sum_{s^*\varepsilon\bar{S}^*} p(s^*)(\hat{e}(s^*, x) - T^*(x))^2$$

a.e. in the space R_{N-k} of the first $N - k$ variates, $\bar{S}^*$ denoting the subset of all samples s^* for which the sample $s \varepsilon \bar{S}_k$, s being the sample obtained by adding the last k variates to s^*. Since $p(s^*)$ is proportional to $p(s)$ and since for $x \varepsilon Q_{N-k}^{0}$, $\hat{e}(s, x) = \hat{e}(s^*, x)$ and $T(x) = T^*(x)$, putting $e'(s, x) = e'(s^*, x)$ we have from (28) $\sum_{s\varepsilon\bar{S}_k} p(s)(e'(s, x) - T(x))^2 \leqq \sum_{s\varepsilon\bar{S}_k} p(s)(\hat{e}(s, x) - T(x))^2$, a.e. in Q_{N-k}^{0}. It then follows from Theorem 4.1 that $e'(s, x) = \hat{e}(s, x)$ a.e. in Q_{N-k}^{0} which implies that $e'(s^*, x) = \hat{e}(s^*, x)$ a.e. in the space R_{N-k} of the first $N - k$ variates. Thus Theorem 3.1 holds for the estimate $\hat{e}(s^*, x)$. Clearly the sampling design d^* obtained by putting $p(s^*) = p(s)/p(\bar{S}_k)$ is a design of fixed sample size $m - k$ from the subpopulation U^*. Further, it is clear that any given sampling design d^* of fixed size $m - k$ for the subpopulation U^* can be obtained in this way by taking for the population U, a sampling design d of fixed size m such that for every sample s which consists of the sample s^* from U^* and in addition the last k units, $p(s) = c \cdot p(s^*)$ where c is a constant, $0 < c < 1$. It thus follows that Theorem 3.1 holds for every sampling design of fixed size $m - k$ from the subpopulation U^* and for the estimate $\hat{e}(s^*, x)$. But for the subpopulation U^*, $\sum_{r=1}^{N-k} b_r^{-1} = m - \sum_{r=N-k+1}^{N} b_r^{-1}$ which is $>$ the fixed sample size $m - k$ as $b_r > 1$ for all r. From this it follows that Theorem 3.1 holds also for the population U, and for any estimate $\hat{e}(s, x)$ for which the coefficients b_r satisfy $\sum_{r=1}^{N} b_r^{-1} > m$. Let

$$(29) \qquad \sum_{r=1}^{N} b_r^{-1} = m + c \qquad \text{where} \quad n - 1 \leqq c < n,$$

n being an integer. We then determine additional coefficients $b_{N+1}, \cdots, b_{N+t}$ such that

$$(30) \qquad b_{N+r} > 1, \qquad r = 1, 2, \cdots, t \quad \text{and} \quad \sum_{N+1}^{N+t} b_r^{-1} = t - c.$$

This can always be done by taking $t \geqq n$. Then from (29) and (30), $\sum_{r=1}^{N+t} b_r^{-1} = m + t$. We now add conceptual units $u_{N+1}, u_{N+2}, \cdots, u_{N+t}$ to the population U and obtain a population U^{**}. For this latter population, by considering a sampling design of fixed size $m + t$, it follows as before that Theorem 3.1 holds for the subpopulation U and for the estimate $\hat{e}(s, x)$ whose coefficients b_r satisfy (29). We thus have,

THEOREM 4.2. *Theorem* 3.1 *continues to hold if the condition* (ii) *in clause* (b) *of that theorem is replaced by the more general condition* $\sum_1^N b_r^{-1} \geqq m$.

Now the Theorem 4.2, through its dependence on Theorem 3.1, restricts itself to the class of regular estimates. But *in case* $\sum_1^N b_r^{-1} > m$, according to a method suggested by the referee, the regularity restriction can be relaxed as follows: since $\sum_1^N b_r^{-1} > m$ we have from (6), $A(s) > \sum_{r\varepsilon s} (1 - b_r^{-1})$. Next put $\sigma^{-2} = k^{-1}[A(s) - \sum_{r\varepsilon s} (1 - b_r^{-1})]$. Then from (7) we get

$$\sum_{s\varepsilon \bar{S}} p(s) \int_{-\infty}^{\infty} E[f(s, y) - A(s)\theta]^2 (2\pi\sigma^2)^{-\frac{1}{2}} \exp(-\theta^2/2\sigma^2)\, d\theta$$
$$(31) \qquad = \sum_{s\varepsilon \bar{S}} p(s) \int_{-\infty}^{\infty} E[\sum_{r\varepsilon s} y_r(1 - b_r^{-1}) - A(s)\theta]^2 \cdot (2\pi\sigma^2)^{-\frac{1}{2}} \exp(-\theta^2/2\sigma^2)\, d\theta.$$

Next assuming that the distribution of $(y_r, r \varepsilon s)$ is given by (14) and noting that the distribution of θ *given* $(y_r, r \varepsilon s)$ is normal with variance

$$k^{-1}[\sum_{r\varepsilon s} (1 - b_r^{-1}) + \sigma^{-2}]^{-1} = (kA(s))^{-1}$$

and expectation $\sum_{r\varepsilon s} y_r(1 - b_r^{-1})/A(s)$, we see that the Equation (31) above implies

$$(32) \qquad \sum_{s\varepsilon \bar{S}} p(s) \int \cdots \int [f(s, y) - \sum_{r\varepsilon s} y_r(1 - b_r^{-1})]^2\, dF \leqq 0$$

where F is a non-degenerate normal distribution. Evidently in deriving (32) it is enough to assume that in (4) $f(s, y)$ and therefore $e'(s, x)$, for every fixed s, $s \varepsilon \bar{S}$, are measurable functions of $(x_r, r \varepsilon s)$. Hence from (4) and (32) we have

THEOREM 4.3. *Theorem* 3.1 *continues to hold by replacing in it the condition* (ii) *of its clause* (b) *by* $\sum_1^N b_r^{-1} > m$ *and the words, "regular estimate" by the words "estimate which for every fixed s,* $s \varepsilon \bar{S}$, *is a measurable function of* $(x_r, r \varepsilon s)$."

NOTE. Now in the subsequent discussion whenever the estimate satisfies the condition $\sum_1^N b_r^{-1} > m$, the word "regular" should be replaced by the word "measurable" in the sense of the Theorem 4.3 above. This is particularly so in Theorem 5.2 to follow.

5. Strict admissibility of the estimate. We now complete the argument by stating the

THEOREM 5.1. *Weak admissibility in the sense defined in Theorem* 3.1 *of the estimate* $\hat{e}(s, x)$ *in Theorem* 4.2 *implies its strict admissibility in the sense that there does not exist any regular estimate* $e'(s, x)$ *such that* (2) *is satisfied for all* $x \varepsilon R_N$ *and in addition the strict inequality in* (2) *holds for at least one* $x \varepsilon R_N$. *Further if any* $e'(s, x)$ *satisfies* (2) *then* $e'(s, x) = \hat{e}(s, x)$ *for all* $x \varepsilon R_N$ *and all* $s \varepsilon \bar{S}$.

PROOF. Let $e'(s, x)$ be a regular estimate for which (2) is satisfied for all points $x \varepsilon R_N$ and let $E \subset R_N$ denote the set of all the points $x \varepsilon R_N$ for which $h(s, x) = e'(s, x) - \hat{e}(s, x) \neq 0$. By Theorem 4.2, E is a null set and we have to show that it is also empty. Suppose it is not empty; then there must exist at least one point $x = a = (a_1, a_2, \cdots, a_N)$ and one sample $s_0 \varepsilon \bar{S}$ such that $h(s_0, a) = h_0 \neq 0$. Without loss of generality we may suppose the sample s_0 to consist of the first m units $u_1, u_2, \cdots, u_m$. Now consider the $(N - m)$-dimensional hyperplane P_{N-m}^a defined by

$$x \varepsilon P_{N-m}^a \text{ if and only if } x_i = a_i, \qquad i = 1, \cdots, m. \tag{33}$$

Now for every $x \varepsilon P_{N-m}^a$, by (33) $e'(s_0, x) = e'(s_0, a) = \hat{e}(s_0, a) + h_0$ and $\hat{e}(s_0, x) = \hat{e}(s_0, a)$, and hence

$$\begin{aligned} &\{e'(s_0, x) - T(x)\}^2 - \{\hat{e}(s_0, x) - T(x)\}^2 \\ &\quad = \{\hat{e}(s_0, a) + h_0 - \textstyle\sum_{i=1}^m a_i - \sum_{i=m+1}^N x_i\}^2 \\ &\qquad - \{\hat{e}(s_0, a) - \textstyle\sum_{i=1}^m a_i - \sum_{i=m+1}^N x_i\}^2. \end{aligned} \tag{34}$$

Next we define a subset $Q_{N-m}^a \subset P_{N-m}^a$ as follows:

$$\begin{aligned} &\text{if } h_0 > 0, \quad x \varepsilon Q_{N-m}^a \text{ if and only if} \\ &\textstyle\sum_{i=m+1}^N x_i < \hat{e}(s_0, a) + h_0/2 - \sum_{i=1}^m a_i \\ &\text{and if } h_0 < 0, \quad x \varepsilon Q_{N-m}^a \text{ if and only if} \\ &\textstyle\sum_{i=m+1}^N x_i > \hat{e}(s_0, a) + h_0/2 - \sum_{i=1}^m a_i. \end{aligned} \tag{35}$$

As $h_0 \neq 0$, the set Q_{N-m}^a is always defined. From (35) it easily follows that for every point $x \varepsilon Q_{N-m}^a$, the right hand side of (34) >0 and hence at each such point $h(s, x) \neq 0$ for at least one other $s \varepsilon \bar{S}$, with $s \neq s_0$, as otherwise at the point x, the left hand side of (2) would be >0.

In the following for any subspace R_k of R_N of $k < N$ dimensions, we shall denote the Lebesgue measure in k dimensions defined on R_k by μ_k. Clearly the set Q_{N-m}^a has infinite measure (μ_{N-m}).

We next partition Q_{N-m}^a into (not necessarily) disjoint subsets indexed by the samples $s \varepsilon \bar{S}$. Let for a specified sample $s \varepsilon \bar{S}$, $L_{N-m}^{a,s}$ be the subset consisting of all those points $x \varepsilon Q_{N-m}^a$, for which $h(s, x) \neq 0$, i.e. $x \varepsilon L_{N-m}^{a,s}$, if and only if, $x \varepsilon Q_{N-m}^a$ and $h(s, x) \neq 0$. Then from the definition of Q_{N-m}^a it follows that

$$Q_{N-m}^a = \bigcup_{s\varepsilon\bar{S}} L_{N-m}^{a,s}. \tag{36}$$

Since Q_{N-m}^a has infinite measure (μ_{N-m}), at least one of the subsets in the right hand side of (36) must be non-null (μ_{N-m}). Further since for every $x \varepsilon Q_{N-m}^a$, $h(s, x) \neq 0$ for some $s \varepsilon \bar{S}$ such that $s \neq s_0$, there must be at least one non-null (μ_{N-m}) subset in the right hand side of (36) such that $s \neq s_0$, i.e. the sample s does not include all the m units $u_1, \cdots, u_m$. If there are more than one such non-null (μ_{N-m}) subset we select any one of them arbitrarily. Let the subset selected be L_{N-m}^{a,s_1} where the sample s_1 includes some k ($k < m$) out of the first m units, the remaining ($m - k$) units in s_1 being from the last ($N - m$) units $u_{m+1}, u_{m+2}, \cdots, u_N$. Then take any point $a^1 \varepsilon L_{N-m}^{a,s_1}$. Since $a^1 \varepsilon P_{N-m}^a$, we have

$$a^1 = \{a_1, a_2, \cdots, a_m, a_{m+1}^1, a_{m+2}^1, \cdots, a_N^1\}.$$

Then for the point a^1 we define as in (33) a ($N - m$)-dimensional hyperplane, $p_{N-m}^{a^1}$ by

$$\begin{aligned} x_i &= a_i, \qquad \text{for } i \varepsilon s_1, \quad i \leqq m, \\ x_i &= a_i^1, \qquad \text{for } i \varepsilon s_1, \quad i > m, \end{aligned} \tag{37}$$

and a set $Q_{N-m}^{a^1} \subset P_{N-m}^{a^1}$ by

$$\textstyle\sum_{i \notin s_1} x_i > \quad \text{or} \quad < \quad \text{a certain number} \quad c. \tag{38}$$

The sign $>$ or $<$ in (38) being taken according as $h(s_1, a^1) < 0$ or > 0. (Here $i \varepsilon s_1$ means $u_i \varepsilon s_1$.)

Now in (38) assign to all coordinates x_i, for $i \notin s_1$ and $i > m$, fixed values equal to the corresponding coordinates of the point a^1, i.e. for i satisfying $i \notin s_1$, and $i > m$, $x_i = a_i^1$. Then from $Q_{N-m}^{a^1}$ we get a subset $Q_{m-k}^{a^1}$, defined by

$$\textstyle\sum_{i\notin s_1, i \leqq m} x_i > \quad \text{or} \quad < c - \sum_{i \notin s_1, i > m} a_i^1. \tag{39}$$

(The role of (39) is made further explicit in the Remark at the end of this proof.)

Clearly (39) defines a subset $Q_{m-k}^{a^1}$ of infinite measure (μ_{m-k}) in the hyperplane $p_{m-k}^{a^1}$, i.e. $Q_{m-k}^{a^1} \subset p_{m-k}^{a^1}$, the hyperplane $p_{m-k}^{a^1}$ being defined by

$$\begin{aligned} x_i &= a_i, \qquad \text{for } i \varepsilon s_1, \quad i \leqq m, \\ x_i &= a_i^1, \qquad \text{for } i > m. \end{aligned} \tag{40}$$

The hyperplane $p_{m-k}^{a^1}$ is wholly orthogonal to p_{N-m}^{a} and hence the set Q_{N-m}^{a} with the set $Q_{m-k}^{a^1}$ defined by (39) for each $a^1 \varepsilon Q_{N-m}^{a}$ determine a set $D_{N-k}^{a} \subset p_{N-k}^{a}$, p_{N-k}^{a} being the hyperplane defined by $x_i = a_i$ for each i such that $i \varepsilon s_1$, $i \leqq m$. Combining (37), (38) and (39) the explicit definition of the set D_{N-k}^{a} is given below:

(41) $$x = (x_1, \cdots, x_N) \varepsilon D_{N-k}^{a}, \quad \text{if and only if}$$
$$x \varepsilon Q_{m-k}^{a^1} \quad \text{for some} \quad a^1 \varepsilon L_{N-m}^{a,s_1},$$

the set D_{N-k}^{a} being of infinite measure (μ_{N-k}). Here $0 \leqq k < m$. Now let $E_{N-k}^{a} \subset p_{N-k}^{a}$ be the set consisting of all those points $x \varepsilon p_{N-k}^{a}$ for which $h(s, x) \neq 0$ for at least one $s \varepsilon \bar{S}$, i.e. $x \varepsilon E_{N-k}^{a}$ if and only if, $x \varepsilon p_{N-k}^{a}$ and $h(s, x) \neq 0$ for at least one $s \varepsilon \bar{S}$. Then $D_{N-k}^{a} \subset E_{N-k}^{a}$ and hence the set E_{N-k}^{a} is also of infinite measure (μ_{N-k}). Now we again partition the set E_{N-k}^{a} into subsets by

(42) $$E_{N-k}^{a} = \bigcup_{s \varepsilon \bar{S}} L_{N-k}^{a,s},$$

the subset $L_{N-k}^{a,s}$ being defined for each specified $s \varepsilon \bar{S}$ by $x \varepsilon L_{N-k}^{a,s}$ if and only if $x \varepsilon E_{N-k}^{a}$ and $h(s, x) \neq 0$. Again at least one of the subsets in the right hand side of (42) must be non-null (μ_{N-k}). Further if there is only one non-null (μ_{N-k}) subset $L_{N-k}^{a,s}$ where s does not include each of the k units u_i, with $i \varepsilon s_1$, $i \leqq m$ we select it; if there are more than one such subsets we select one of them arbitrarily. Let L_{N-k}^{a,s_2} be the subset selected and let s_2 contain j $(0 \leqq j < k)$ out of the k units u_i, given by $i \varepsilon s_1$, and $i \leqq m$. Then again proceeding as from (37) to (42) we again obtain a set $D_{N-j}^{a} \subset p_{N-j}^{a}$ where the hyperplane p_{N-j}^{a} is defined by $x_i = a_i$, for all i satisfying $i \leqq m$, $i \varepsilon s_1$ and $i \varepsilon s_2$, the set D_{N-j}^{a} being of infinite measure (μ_{N-j}). Again denoting by E_{N-j}^{a} the set of all those points $x \varepsilon P_{N-j}^{a}$ for which $h(s, x) \neq 0$ for some $s \varepsilon \bar{S}$, i.e. $x \varepsilon E_{N-j}^{a}$ if and only if $x \varepsilon p_{N-j}^{a}$ and $h(s, x) \neq 0$ for some $s \varepsilon \bar{S}$, we have $D_{N-j}^{a} \subset E_{N-j}^{a}$, and hence the set E_{N-j}^{a} is also of infinite measure (μ_{N-j}).

Clearly the process can end only when we

(A) either reach a set $E_N \subset R_N$ such that E_N has infinite measure (μ_N) and for every $x \varepsilon E_N$, $h(s, x) \neq 0$ for some $s \varepsilon \bar{S}$; or

(B) we reach a hyperplane p_{N-j}^{a}, defined by some j $(0 < j < m)$, out of the first m variates having fixed values equal to the corresponding co-ordinates of the point a, i.e. $x_r = a_r$ for $r = i_1, i_2, \cdots, i_j$ where $i_1, i_2, \cdots, i_j$ all $\leqq m$, and a set $E_{N-j}^{a} \subset p_{N-j}^{a}$ such that E_{N-j}^{a} has infinite measure (μ_{N-j}), and for every $x \varepsilon E_{N-j}^{a}$, $h(s, x) \neq 0$ for some $s \varepsilon \bar{S}$, and further such that for any sample $s \varepsilon \bar{S}$ which does not include each of the j units $u_{i_1}, u_{i_2}, \cdots, u_{i_j}$, $h(s, x) = 0$ for almost all (μ_{N-j}), $x \varepsilon p_{N-j}^{a}$.

Now (A) leads to a contradiction because $E_N = E$ and E is a null (μ_N) set by Theorem 4.2. (B) also leads to a contradiction. For let $\bar{S}_j$ be the subset of $\bar{S}$, $\bar{S}_j \subset \bar{S}$, consisting of all those samples $s \varepsilon \bar{S}$, which include each of the units $u_{i_1}, u_{i_2}, \cdots, u_{i_j}$. Then for $s \varepsilon (\bar{S} - \bar{S}_j)$, $e'(s, x) = \hat{e}(s, x)$ for almost all (μ_{N-j}), $x \varepsilon p_{N-j}^{a}$, and hence from (2), for almost all (μ_{N-j}), $x \varepsilon p_{N-j}^{a}$,

$$\sum_{s \varepsilon \bar{S}_j} p(s)[e'(s, x) - T(x)]^2 \leqq \sum_{s \varepsilon \bar{S}_j} p(s)[\hat{e}(s, x) - T(x)]^2.$$

But then by Theorem 4.1 and 4.2 the set $E^a_{N-j} \subset p^a_{N-j}$ such that $h(s, x) \neq 0$ for some $s \varepsilon \bar{S}_j$ is a null (μ_{N-j}) set. But (B) requires E_{N-j} to be of infinite measure (μ_{N-j}). Thus neither (A) nor (B) is possible. Hence no point $a \varepsilon E$, exists such that $h(s, a) \neq 0$ for some $s \varepsilon \bar{S}$ and thus the set E is empty which was to be proved.

REMARK 5.1. Note the role of (39) in the development of this proof: What we want to show is that there exists a non-null set K (either in R_N or in R_{N-j}) on which $h(s, x) \neq 0$. The set is built up successively by increasing the number of its dimensions. Suppose that at the first stage we find a non-null (μ_3) set K in the 3-space of x_1, x_2, x_3. Suppose then that at the next stage we prove that for every point x belonging to K there exists a non-null (μ_3) set L in the space of x_1, x_5, x_6. Then from this we cannot deduce that the set K combined with the sets L defined for each point of K would together build up a non-null (μ_5) set in the space of x_1, x_2, x_3, x_4 and x_5. It seems necessary first to show that the set L entails a non-null (μ_2), set M in the space of x_4, x_5 which is orthogonal to the space of (x_1, x_2, x_3). Thus, if we first obtain the set Q^a_{N-m} and then for each point of Q^a_{N-m}, a non-null (μ_{N-m}) set $Q^{a^1}_{N-m}$, we cannot deduce that these would constitute a non-null (μ_{n-k}) set. It seems essential to first define the $(m - k)$-dimensional set $Q^{a^1}_{m-k}$ which is wholly orthogonal to Q^a_{N-m}. This is done by (39).

Combining Theorems 3.1, 4.2 and 5.1 the theorem may now be stated in its most general form as

THEOREM 5.2. *If*

(a) *the sampling design is one of fixed size m,* i.e. $p(s) = 0$ *whenever the sample size* $n(s) \neq$ *a fixed integer m, and*

(b) $\hat{e}(s, x)$ *is an estimate given by* $\hat{e}(s, x) = \sum_1^N b_r x_r$ *where the coefficients* b_r *satisfy*

(i) $b_r > 1$, $r = 1, 2, \cdots, N$; *and*

(ii) $\sum_1^N b_r^{-1} \geqq m$,

then the estimate $\hat{e}(s, x)$ *is strictly admissible for the population total* $T(x)$ *in the sense that there exists no regular estimate* $e'(s, x)$ *such that*

$$\sum_{s \varepsilon \bar{S}} p(s)[e'(s, x) - T(x)]^2 \leqq \sum_{s \varepsilon \bar{S}} p(s)[\hat{e}(s, x) - T(x)]^2 \tag{43}$$

for all $x \varepsilon R_N$, *the strict inequality holding for at least one point* $x \varepsilon R_N$. *Further if any estimate* $e'(s, x)$ *satisfies* (43), *then* $e'(s, x) = \hat{e}(s, x)$ *for all* $x \varepsilon R_N$.

REMARK 5.2. The conditions (i) $b_r \geqq 1$ and (ii) $\sum_{r=1}^N b_r^{-1} \geqq m$ (see Section 6) are not merely sufficient conditions for the theorem, but are also necessary in the sense that when they are not satisfied, the estimate $\hat{e}(s, x)$ is not always admissible. This is seen from the following simple, if artificial examples: The population U consists of only 2 units u_1, u_2; samples s_1 and s_2 consist respectively of units u_1 and u_2, and $p(s_1) = p(s_2) = \frac{1}{2}$. Then,

EXAMPLE 1. $b_1 = 3$, $b_2 = 3$, $e'(s, x) = 2x_1$, $e'(s_2, x) = 2x_2$.

EXAMPLE 2. $b_1 = 3$, $b_2 = \frac{1}{2}$, $e'(s_1, x) = 2x_1$, $e'(s_2, x) = \frac{3}{2}x_2$.

In Example 1 condition (ii) is broken while in Example 2 condition (i) is broken. Now Example 1,

$$\sum_{s\varepsilon\bar{S}} p(s)(\hat{e}(s, x) - T(x))^2 = \tfrac{1}{2}(2x_1 - x_2)^2 + \tfrac{1}{2}(2x_2 - x_1)^2$$
$$= \tfrac{1}{2}(x_1^2 + x_2^2) + 4(x_1 - x_2)^2 \geqq \sum_{s\varepsilon\bar{S}} p(s)(e'(s, x) - T(x))^2$$
$$= \tfrac{1}{2}(x_1 - x_2)^2 + \tfrac{1}{2}(x_1 - x_2)^2,$$

so that $\hat{e}(s, x)$ is inadmissible.

It is similarly seen to be inadmissible in Example 2.

6. Admissibility of the Horvitz-Thomson estimate. This estimate is given by

$$\bar{e}(s, x) = \sum_{r\varepsilon s} (x_r/\pi_r) \tag{44}$$

where π_r is the inclusion probability of the unit u_r, i.e. $\pi_r = \sum_{s\ni u_r} p(s)$. As is well known for a design of fixed sample size m, $\sum_{r=1}^{N} \pi_r = m$. Hence if in addition $\pi_r < 1$ for $r = 1, 2, \cdots, N$, both the conditions in clause (b) of Theorem 5.2 are satisfied and the admissibility of the estimate $\bar{e}(s, x)$ follows from the theorem.

But even if some of the inclusion probabilities $\pi_r = 1$, the estimate $\bar{e}(s, x)$ remains admissible. We shall show this by taking the more general case of the estimate $\hat{e}(s\ x) = \sum_{r\varepsilon s} b_r x_r$, for which some of the coefficients $b_r = 1$. Without loss of generality these may be taken to be the last k coefficients namely b_{N-k+1}, $b_{N-k+2}, \cdots, b_N$. We assume that the corresponding units u_{N-k+t}, $t = 1, 2, \cdots, k$; are included in each sample s for which $p(s) \neq 0$. Now suppose that for this estimate $\hat{e}(s, x)$ a uniformly superior regular estimate $e'(s, x)$ exists so that

$$\sum_{s\varepsilon\bar{S}} p(s)(e'(s, x) - T(x))^2 \leqq \sum_{s\varepsilon\bar{S}} p(s)(\hat{e}(s, x) - T(x))^2 \tag{45}$$

for all $x \varepsilon R_N$.

Now as in (20) let Q_{N-k}^{α} be the hyperplane given by $x_{N-k+t} = \alpha_{N-k+t}$, $t = 1, 2, \cdots, k$. Let U^* be the subpopulation $(u_1, u_2, \cdots, u_{N-k})$, $T^*(x)$ its population total $\sum_{r=1}^{N-k} x_r$, and s^* the sample obtained by omitting the last k variates from the sample s. Then for $x \varepsilon Q_{N-k}^{\alpha}$ the last k variates being fixed constants, we can define the estimate $e''(s^*, x)$ by

$$e''(s^*, x) = e'(s, x) - \sum_{r=N-k+1}^{N} \alpha_r. \tag{46}$$

Also clearly,

$$\hat{e}(s^*, x) = \hat{e}(s, x) - \sum_{r=N-k+1}^{N} \alpha_r \quad \text{and} \quad T^*(x) = T(x) - \sum_{r=N-k+1}^{N} \alpha_r. \tag{47}$$

Substituting (46) and (47) in (45) we have, for all $x \varepsilon Q_{N-k}^{\alpha}$,

$$\sum_{s^*\varepsilon\bar{S}^*} p(s^*)(e'(s^*, x) - T^*(x))^2 \leqq \sum_{s^*\varepsilon\bar{S}^*} p(s^*)(\hat{e}(s^*, x) - T^*(x))^2,$$

where $\bar{S}^*$ denotes the set of all samples s^* for which the sample $s \varepsilon \bar{S}$. But the probabilities $p(s^*)$ form a sampling design of fixed sample size $m - k$ and for this sample size the coefficients b_r of $\hat{e}(s^*, x)$ satisfy both conditions in clause (b) of Theorem 5.2. Hence by that theorem $e''(s^*, x) = \hat{e}(s^*, x)$ for all $s \varepsilon \bar{S}^{*\alpha}_{N-k}$ and $x \varepsilon Q_{N-K}^{\alpha}$. As this result holds for every hyperplane Q_{N-k}^{α}, it holds for all $x \varepsilon R_N$. We thus have, THEOREM 6.1. *Theorem 5.2 continues to hold if some of the coefficients* $b_r = 1$, *provided all the corresponding units* u_r *are included in every sample* s *for which* $p(s) \neq 0$.

Now for the Horvitz-Thomson estimate $e(s, x)$ defined by (44), if any $\pi_r = 1$, the corresponding unit u_r is included in every sample s for which $p(s) \neq 0$. Therefore as a corollary of Theorem 6.1 we have

COROLLARY 6.1. *The estimate* $\bar{e}(s, x)$ *defined in* (44) *is admissible in the entire class of regular estimates.*

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