

MARKOVIAN SEQUENTIAL REPLACEMENT PROCESSES¹

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1. Introduction and summary. A sequential control process is a dynamic system which is observed periodically and classified into one of a number of possible states. After each observation one of a number of possible decisions is made. These decisions are the "control"; they determine the chance laws of the system. A *replacement* process is a control process with an additional special action, called replacement, which instantaneously returns the system to some initial state.

Let \mathbf{X} denote the state space of the system, assumed to be a Borel subset of a finite dimensional Euclidean space. The case where \mathbf{X} is finite has been treated by Derman [8], and thus \mathbf{X} is considered infinite here. Let \mathfrak{B} be the σ -algebra of Borel sets in \mathbf{X} .

Let $\{X_t; t = 0, 1, 2, \dots\}$ be the sequence of states and $\{\Delta_t; t = 0, 1, 2, \dots\}$ be the sequence of decisions. In a replacement problem it is assumed that there is a distinguished state $x_0 \in \mathbf{X}$ with $X_0 = x_0$ with probability one. For any time t let S_t be the history of states and decisions up to and including time t .

Let \mathbf{A} be the set of possible actions, excluding replacement, where:

A1° It is assumed that the action space \mathbf{A} is a finite set with $n(\mathbf{A})$ elements. Since \mathbf{A} is finite, assume $\mathbf{A} = \{1, 2, \dots, n(\mathbf{A})\}$. Let $k_0 \notin \mathbf{A}$ denote the replacement action. The action k_0 instantaneously returns the system to state x_0 , and it may be followed by some action $k \in \mathbf{A}$ which "acts on" the state x_0 . The pair (k_0, k) itself constitutes a possible action. A decision at time t is either a choice of an element $k \in \mathbf{A}$ or a choice of a pair (k_0, k) with $k \in \mathbf{A}$. Let \mathbf{A}_0 be the *total* action space, where:

$$\mathbf{A}_0 = \mathbf{A} \cup \{(k_0, k); k \in \mathbf{A}\}.$$

There are $2n(\mathbf{A})$ elements in \mathbf{A}_0 . Let $\Xi = \{\xi; \xi = \langle \xi_1, \dots, \xi_{2n(\mathbf{A})} \rangle, \xi_j \geq 0, \sum \xi_j = 1\}$ be the simplex of all probability distributions on \mathbf{A}_0 . A *sequential control rule* is a function $D(s_{t-1}, x) = \langle D_1(s_{t-1}, x), \dots, D_{2n(\mathbf{A})}(s_{t-1}, x) \rangle$ of histories s_{t-1} and present states x with values in Ξ . The interpretation is: At a history of $S_{t-1} = s_{t-1}$ and a present state $X_t = x$, decision $j \in \mathbf{A}_0$ is taken with probability $D_j(s_{t-1}, x)$. In order that the integrals later to be written have meaning it is necessary to restrict attention to control rules $D(s_{t-1}, x)$ which are Baire functions of their arguments. Let \mathbf{R} be the space of all such control rules.

A sequential control process is not specified until a "law of motion" is given.

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A2° It is assumed that for every $x \in \mathbf{X}$ and $k \in \mathbf{A}$ there exists a probability measure $Q(\cdot; x, k)$ on \mathcal{B} such that for some version $\Pr\{X_{t+1} \in B \mid S_{t-1}, X_t = x, \Delta_t = k\} = Q(B; x, k)$; for every $B \in \mathcal{B}$ and history S_{t-1} . For every $B \in \mathcal{B}$ and $k \in \mathbf{A}$, $Q(B; \cdot, k)$ is assumed to be a Baire function on \mathbf{X} . It is assumed that $Q(\cdot, x, k)$ is absolutely continuous with respect to some σ -finite measure μ on \mathcal{B} , and possessing a density $q(\cdot, x, k)$, also assumed to be a Baire function in x .

Since $X_0 = x_0$ a.s., once a rule $R \in \mathbf{R}$ is specified, the sequences $\{X_t, t = 0, 1, 2, \dots\}$ and $\{(X_t, \Delta_t); t = 0, 1, 2, \dots\}$ are stochastic processes. The previous Assumption A2° imposes a structure similar to that of a Markov process in that the law of motion does not depend on the past history, but only on the present state.

In a manner similar to Derman [9], the process $\{(X_t, \Delta_t); t = 0, 1, 2, \dots\}$ will be called a *Markovian sequential replacement process*. It is not true that $\{X_t; t = 0, 1, \dots\}$ nor even $\{(X_t, \Delta_t); t = 0, 1, \dots\}$ will always be Markov processes; whether they are or not will depend on the rule R .

Two assumptions particular to the development in this paper and insuring the ergodicity of the process are:

A3° For every $x \in \mathbf{X}$ and $k \in \mathbf{A}$ it is assumed that

$$\lim_{x' \rightarrow x} \int |q(y; x, k) - q(y; x', k)| \mu(dy) = 0.$$

A4° For every compact set $G \subset \mathbf{X}$ it is assumed that

$$\sup_{x \in G} \int_G q(y; x, k) \mu(dy) < 1$$

for all $k \in \mathbf{A}$. The last assumption, A4°, is stronger than needed, as may be seen in the examples in Section 4. However, it is easily verified and seems natural in many applications of the theory.

Let $w(x, k)$ be the immediate cost whenever the system is in state $x \in \mathbf{X}$ and decision $k \in \mathbf{A}$ is made. It often occurs that the cost in an actual situation is a random variable whose distribution is determined by knowledge of the state and decision. In such a case, with some loss in generality, attention is restricted to $w(x, k)$ representing the *expected* one stage cost under the appropriate distribution. Let $K(x)$ be the cost of replacing a system in state x . If $w_0(\cdot, \cdot)$ is the cost function defined on $\mathbf{X} \times \mathbf{A}_0$ then the relationship is:

$$w_0(x, k) = w(x, k) \qquad \text{for } k \neq k_0$$

and
$$w_0(x, (k_0, k)) = K(x) + w(x, k) \qquad \text{for } k \in \mathbf{A}.$$

A5° Assume that $K(\cdot)$ is bounded and continuous with $0 \leq K(x) \leq M$ for all $x \in \mathbf{X}$. For every $k \in \mathbf{A}$ assume that $w(\cdot, k)$ is a non-negative continuous function on \mathbf{X} with $\liminf_{x \rightarrow \infty} w(x, k) \gg 0$ (For the limiting operation here, a neighborhood of ∞ is the complement of a compact set.). The notation $a \gg 0$ means that a is much greater than zero, but not necessarily infinite. One needs $\liminf_{x \rightarrow \infty} w(x, k)$ large enough so that Lemmas 3.2, 3.3 and 3.4 will hold. Intuitively, one needs the cost of continuing sufficiently large for some states so as

to ensure that the expected time to a replacement action is finite. It should be noted that $\sup_{x \in X} \min_{a \in A_0} w_0(x, a) \leq M_0$ where $M_0 = M + \min_{k \in A} w(x_0, k)$.

Let

$$P_t(B, a \mid x, R) = \Pr\{X_t \in B, \Delta_t = a \mid X_0 = x, R\}$$

for $B \in \mathfrak{B}$, $x \in X$ and $a \in A_0$. Let the appropriate density be labeled $p_t(\cdot, \cdot \mid x, R)$ where

$$p_t(y, a \mid x, R)\mu(dy) = \Pr\{X_t \in dy, \Delta_t = a \mid X_0 = x, R\}.$$

Two common measures of effectiveness of a Markovian sequential decision process are the expected total discounted future cost and the average cost per unit time. The first, abbreviated to "discounted cost" assumes a discount factor $\alpha \in (0, 1)$, with the interpretation that a unit of value n periods hence has a present value of α^n . For a starting state of $X_0 = x_0$ the objective is to choose a rule R so as to minimize

$$\psi(x_0, \alpha, R) = \sum_{t=0}^{\infty} \alpha^t \int_{\mathbf{X}} \sum_{a \in A_0} w_0(x, a) p_t(x, a \mid x_0, R) \mu(dx).$$

The second criterion, abbreviated to "average cost" examines the function

$$\varphi(x_0, R) = \liminf_{T \rightarrow \infty} T^{-1} \sum_{t=0}^{T-1} \int_{\mathbf{X}} \sum_{a \in A_0} w_0(x, a) p_t(x, a \mid x_0, R) \mu(dx)$$

Section 2 presents the solution of the problem under the discounted cost measure. Building upon the work of Blackwell [4] and Karlin [12], Derman [9] has shown that an optimal non-randomized stationary rule exists for the case where \mathbf{X} is denumerable. Blackwell [5] recently has given a complete discussion of the general case. The rule is characterized by a functional equation of the dynamic programming type. Iterative methods for solving such functional equations are now almost commonplace.

Section 3 uses the known results in the discounted cost model: (a) to show the existence of a non-randomized stationary solution in the average cost case, (b) to show the existence of a functional equation characterizing the solution in the average cost case, and (c) to show that the average cost solution is the limit, in some sense, of the discounted cost solutions as the discount factor approaches unity.

Section 4 presents some applications of the theory. The attempt is to show how the work of several authors fits into this general theory of control of replacement processes. For example, while supporting one claim in a quality control paper by Girshick and Rubin [10], the theory also provides a counter example for another of their claims.

2. The existence of an optimal stationary non-randomized rule in the discounted cost case. Blackwell [5] has completely discussed the conditions under which for a given $\alpha \in (0, 1)$ there exists a rule $R_\alpha \in \mathbf{R}$ such that

$$\psi(x, \alpha, R_\alpha) = \min_{R \in \mathbf{R}} \psi(x, \alpha, R) \quad \text{for } x \in \mathbf{X}.$$

Since the replacement aspect of the problem is not important here, the notation may be simplified by using the total action space \mathbf{A}_0 together with the appropriate cost function $w_0(\cdot, \cdot)$. Throughout this section α remains a fixed discount factor with $0 < \alpha < 1$.

THEOREM 2.1. (See Blackwell [5] for proof). *If $w_0(\cdot, \cdot)$ is bounded and \mathbf{A}_0 is a finite set then there exists a rule $R_\alpha^* \in \mathbf{R}$ such that*

$$\psi(x, \alpha, R_\alpha^*) = \min_{R \in \mathbf{R}} \psi(x, \alpha, R) \quad \text{for all } x \in \mathbf{X}.$$

A sequential control rule $\{D(s_{t-1}, x)\}$ is said to be *stationary* if $D(s_{t-1}, x)$ is independent of s_{t-1} for every $x \in \mathbf{X}$. For a stationary rule one may write $D(s_{t-1}, x) = D(x)$. A *non-randomized* stationary rule is a rule $D(x) = \langle D_1(x), \dots, D_{2n(A)}(x) \rangle$ for which $D_j(x)$ is always either zero or one. Once the existence of at least one optimal rule is demonstrated one may show that a non-randomized stationary optimal rule exists. It is easily seen that any non-randomized stationary rule R may be stated as a partition $\{R(a); a \in \mathbf{A}_0\}$ of the state space \mathbf{X} with the interpretation that at time t , action a is taken if and only if $X_t \in R(a)$.

THEOREM 2.2. (See Blackwell [5] or Derman [9] for proof).

If $w_0(\cdot, \cdot)$ is bounded and \mathbf{A}_0 is a finite set then there exists a non-randomized stationary rule R_α which is optimal. One has the equation (which holds for R_α^ as well)*

$$\psi(x, \alpha, R_\alpha) = \min_{a \in \mathbf{A}_0} \{w_0(x, a) + \alpha \int \psi(y, \alpha, R_\alpha) q(y; x, a) \mu(dy)\}.$$

The rule R_α may be specified through the partition $\{R_\alpha(a); a \in \mathbf{A}_0\}$ of \mathbf{X} where $R_\alpha(a) = \{x; \psi(x, \alpha, R_\alpha) = w_0(x, a) + \alpha \int \psi(y, \alpha, R_\alpha) q(y; x, a) \mu(dy)\}$ and $x \notin R_\alpha(a')$ for $a' < a$.

It is easily seen from Theorem 2.2 that the boundedness condition on $w_0(\cdot, \cdot)$ is excessively restrictive and may be replaced by the weaker assumption implied by $A5^\circ$ that $\sup_{x \in \mathbf{X}} \min_{a \in \mathbf{A}_0} w_0(x, a) \leq M_0 < \infty$. For consider the cost function

$$\bar{w}_0(x, a) = \min\{w_0(x, a), M_0/(1 - \alpha)\}.$$

Since the cost function $\bar{w}_0(\cdot, \cdot)$ is bounded there exists an optimal solution \bar{R}_α , which minimizes the corresponding cost function $\bar{\psi}(x, \alpha, R)$. Further, if a rule R_α to the unbounded problem exists for which

$$\psi(x, \alpha, R_\alpha) = \bar{\psi}(x, \alpha, \bar{R}_\alpha)$$

then R_α is optimal in the unbounded problem, since in general $\psi(x, \alpha, R) \geq \bar{\psi}(x, \alpha, R) \geq \bar{\psi}(x, \alpha, \bar{R}_\alpha)$. Using the inequality

$$\begin{aligned} \bar{\psi}(x, \alpha, R) &= E\{\sum \alpha^t \bar{w}_0(X_t, \Delta_t)\} \\ &\leq M_0/(1 - \alpha) \end{aligned}$$

one may easily show that the same control rule \bar{R}_α optimizes both problems, since $\psi(x, \alpha, \bar{R}_\alpha) = \bar{\psi}(x, \alpha, \bar{R}_\alpha)$. The argument is that, when at state x , rule \bar{R}_α need never choose an action a for which $\bar{w}_0(x, a) = M_0/(1 - \alpha)$, for in this

case it is no more costly to replace and return to x_0 , as may be verified through the optimality equation.

Besides demonstrating the existence of an optimal non-randomized stationary rule, Theorem 2.2 furnishes a means for finding such a rule. Let $B(\mathbf{X})$ be the space of all bounded continuous real valued functions on \mathbf{X} . Under the supremum norm $\|g\| = \sup_{x \in \mathbf{X}} |g(x)|$, $B(\mathbf{X})$ is a Banach space. Let T_α be the operator in $B(\mathbf{X})$ defined by:

$$(T_\alpha g)(x) = \min_{a \in A_0} \{w_0(x, a) + \alpha \int g(y)q(y; x, a)\mu(dy)\}$$

for $g \in B(\mathbf{X})$ and $x \in \mathbf{X}$.

One has

$$|(T_\alpha g)(x) - (T_\alpha g)(x')| \leq \|g\| \max_a \int |q(y; x, a) - q(y; x', a)|\mu(dy) + \max_a |w_0(x, a) - w_0(x', a)|.$$

Hence, that $g \in B(\mathbf{X})$ implies that $T_\alpha g \in B(\mathbf{X})$ follows from Assumption A3°.

LEMMA 2.1. Let $g_0(\alpha, x) = 0$ for all $x \in \mathbf{X}$ and define $g_n(\alpha, \cdot) = T_\alpha g_{n-1}(\alpha, \cdot)$ for $n = 1, 2, \dots$. Then $\{g_n(\alpha, \cdot)\}$ converges to a function $g(\alpha, \cdot) \in B(\mathbf{X})$ for which

$$g(\alpha, \cdot) = T_\alpha g(\alpha, \cdot).$$

The solution to this equation is unique in $B(\mathbf{X})$. Hence

$$\lim_{n \rightarrow \infty} g_n(\alpha, \cdot) = \psi(\cdot, \alpha, R_\alpha).$$

PROOF. For any functions $g_i \in B(\mathbf{X})$; $i = 1, 2$ one has

$$\|T_\alpha g_1 - T_\alpha g_2\| \leq \alpha \|g_1 - g_2\|.$$

Thus T_α is a contraction operator in the Banach space $B(\mathbf{X})$. It follows that T_α possesses a unique fixed point $g(\alpha, \cdot)$ in $B(\mathbf{X})$ and that $g_n(\alpha, \cdot) \rightarrow g(\alpha, \cdot)$. By uniqueness one has

$$g(\alpha, \cdot) = \psi(\cdot, \alpha, R_\alpha). \qquad \text{Q.E.D.}$$

Lemma 2.1 is by no means new, being necessary to Bellman's "Principle of Optimality" development. Earlier proofs, in principle the same as given here, are given in [3].

3. The existence of an optimal stationary non-randomized rule under the average cost criterion. In the study of Markovian sequential decision processes under the average cost criterion it seems always necessary to make some assumption concerning the ergodicity of the process. For a replacement process, this assumption may take a simple form. If the time from immediately after one replacement action (k_0) to and including the next is called a *cycle* then one assumes, roughly speaking, that all rules "of interest" result in finite expected cycle lengths. Assumptions A4° and A5° more than suffice in the present case.

Given a replacement action (k_0, k) acting at time t on a state x , there is some

ambiguity as to the value of X_t . Should it be x or x_0 ? It is convenient to consider $X_t = x$ and to introduce the new variables \hat{X}_t and $\hat{\Delta}_t$ where

$$\begin{aligned} \hat{X}_t &= x_0 & \text{if } \Delta_t &= (k_0, k) \\ &= X_t & \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} \hat{\Delta}_t &= k & \text{if } \Delta_t &= (k_0, k) \\ &= \Delta_t & \text{otherwise.} \end{aligned}$$

Now let N_l be the length of and W_l be the total cost of the l th cycle. For every $l = 1, 2, \dots$ one has

$$\begin{aligned} W_1 &= \sum_{t=0}^{N_1-1} w(X_t, \Delta_t) + K(X_{N_1}) \\ W_l &= \sum_{t=N_{l-1}}^{N_l-1} w(\hat{X}_t, \hat{\Delta}_t) + K(X_{N_l}) \\ &= w(x_0, \hat{\Delta}_{N_{l-1}}) + \sum_{t=N_{l-1}+1}^{N_l-1} w(X_t, \Delta_t) + K(X_{N_l}). \end{aligned}$$

A semi-stationary control rule is a rule $D(s_{t-1}, x)$ which depends on the history s_{t-1} only since the most recent replacement action. Let $\mathbf{R}' \subset \mathbf{R}$ be the space of all semi-stationary control rules for which $E\{N_l\} < +\infty$. For any control rule R recall that $\varphi(x_0, R)$ is the associated average cost per period.

$$\varphi(x_0, R) = \liminf_{T \rightarrow \infty} T^{-1} \sum_{t=0}^{T-1} E\{w_0(X_t, \Delta_t)\}.$$

Under any rule $R' \in \mathbf{R}'$, $\{N_l\}$ and $\{W_l\}$ are sequences of independent and identically distributed random variables. Further $EN_l < \infty$ and $EW_l < \infty$ provided that $w(\cdot, \cdot)$ is bounded. It follows by the strong law of large numbers that

$$\varphi(x_0, R') = \lim_{l \rightarrow \infty} \{(W_1 + \dots + W_l) / (N_1 + \dots + N_l)\} = E^{R'}\{W_1\} / E^{R'}\{N_1\}$$

for any rule $R' \in \mathbf{R}'$. The average cost per period may now be studied to some extent by concentrating on the first cycle only.

THEOREM 3.1. *Let $f \in B(\mathbf{X})$ and $\gamma \in (-\infty, +\infty)$ be related by*

$$f(x) = \min\{K(x), w(x, k) - \gamma + \int f(y)q(y; x, k)\mu(dy)\}; k \in \mathbf{A}.$$

Let R^0 be a stationary rule given through a partitioning of \mathbf{X} by

$$\begin{aligned} R^0(0) &= \{x: f(x) = K(x)\} \\ R^0(k) &= \{x: f(x) = w(x, k) - \gamma + \int f(y)q(y; x, k)\mu(dy) \\ &\quad \text{and } x \notin R^0(j) \text{ for } j < k\}. \end{aligned}$$

The interpretation is: At time t

(i) *If $X_t \in R^0(0)$ then replace, followed by action k' where $x_0 \in R^0(k')$.*

(ii) *If $X_t \in R^0(k)$ for $k \neq 0$, take action k .*

Then, under any rule $R \in \mathbf{R}'$

$$E\{W_1 - \gamma N_1\} \geq f(x_0)$$

with equality at $R = R^0$, provided $R^0 \in \mathbf{R}'$.

PROOF. $Z_t = \sum_{i=1}^t \{f(X_i) - E[f(X_i) | S_{i-1}]\}$ is an $\{S_i\}$ martingale with mean zero. Since for all rules in R' , N_1 is an optional sampling for which $EN_1 < \infty$, one has $E\{Z_{N_1}\} = 0$. Adding and subtracting $\gamma + w(X_{t-1}, \Delta_{t-1})$ this becomes upon expanding

$$E\left\{\sum_{i=1}^{N_1} [f(X_i) - \{w(X_{i-1}, \Delta_{i-1}) - \gamma + E[f(X_i) | S_{i-1}]\} + w(X_{i-1}, \Delta_{i-1}) - \gamma]\right\} = 0.$$

But $E[f(X_t) | S_{t-1}] = \int f(y)q(y; x, k)\mu(dy)$ if $X_{t-1} = x$ and $\Delta_{t-1} = k$, and by assumption then

$$f(X_{t-1}) \leq w(X_{t-1}, \Delta_{t-1}) - \gamma + E[f(X_t) | S_{t-1}]$$

with equality only for rule R^0 .

Hence

$$E\left\{\sum_{i=1}^{N_1} [f(X_i) - f(X_{i-1}) + w(X_{i-1}, \Delta_{i-1}) - \gamma]\right\} \geq 0$$

or

$$E\{f(X_{N_1})\} - f(x_0) + E\sum_{i=0}^{N_1-1} [w(X_i, \Delta_i) - \gamma] \geq 0.$$

Again, $f(X_{N_1}) \leq K(X_{N_1})$ with equality under rule R^0 . Furthermore

$$W_1 = K(X_{N_1}) + \sum_{i=0}^{N_1-1} [w(X_i, \Delta_i)].$$

Hence $E\{W_1\} - \gamma E\{N_1\} \geq f(x_0)$ with equality for rule R^0 , provided $R^0 \in R'$.

Q.E.D.

For any real number γ let S_γ^α be the operator in $B(\mathbf{X})$ defined by

$$(S_\gamma^\alpha f)(x) = \min\{K(x), w(x, k) - \gamma + \alpha \int f(y)q(y; x, k)\mu(dy); k \in \mathbf{A}\}.$$

for $f \in B(\mathbf{X})$ and $x \in \mathbf{X}$. Write S_γ for S_γ^1 when $\alpha = 1$.

Let

$$\gamma_0 = \inf \{\varphi(x_0, R); R \in \mathbf{R}\}.$$

LEMMA 3.1. *If there exists a function $f \in B(\mathbf{X})$ with $f = S_{\gamma_0} f$ and $f(x_0) = 0$ and if the rule R^0 associated with f as in Theorem 3.1 has a finite expected cycle length, then $R^0 \in R'$ and R^0 is optimal with $\varphi(x_0, R^0) \leq \varphi(x_0, R)$ for any rule $R \in \mathbf{R}$ (not restricted to R').*

PROOF. R^0 is a stationary rule with finite expected cycle length, hence is in R' . By Theorem 3.1, since $f(x_0) = 0$ one has

$$E^{R^0}\{W_1\} - \gamma_0 E^{R^0}\{N_1\} = 0$$

or

$$E^{R^0}\{W_1\}/E^{R^0}\{N_1\} = \varphi(x_0, R^0) = \gamma_0 \leq \varphi(x_0, R)$$

for any rule $R \in \mathbf{R}$.

Q.E.D.

The next step is clear. A function f satisfying the conditions of Lemma 3.1 must be found. Define the family of functions $\{f_\alpha; \alpha \in (0, 1)\}$ by

$$f_\alpha(x) = g_\alpha(x) - g_\alpha(x_0)$$

where $g_\alpha(x) = \psi(x, \alpha, R_\alpha)$ is the unique solution to $g_\alpha = T_\alpha g_\alpha$. One has by simple manipulations

$$f_\alpha(x) = \min\{K(x), w(x, k) - \gamma_\alpha + \alpha \int f_\alpha(y)q(y; x, k)\mu(dy); k \in \mathbf{A}\}$$

where $\gamma_\alpha = (1 - \alpha)g_\alpha(x_0)$. Note that $0 \leq \gamma_\alpha \leq M_0$ for all $\alpha \in (0, 1)$.

LEMMA 3.2. *If for some constant M_1 , $\|f_\alpha\| \leq M_1$ for all $\alpha \in (0, 1)$ then (i) the family $\{f_\alpha\}$ is equicontinuous and (ii) there exists a compact set G such $f_\alpha(x) = K(x)$ for all $x \notin G$ and $\alpha \in (0, 1)$, provided that $\liminf_{x \rightarrow \infty} w(x, k) \gg 0$.*

PROOF.

$$\begin{aligned} \text{(i)} \quad |f_\alpha(x) - f_\alpha(x')| &\leq |[S_{\gamma_\alpha}^\alpha f_\alpha(\cdot)](x) - [S_{\gamma_\alpha}^\alpha f_\alpha(\cdot)](x')| \\ &\leq \max\{M_1 \max_k \int |q(y; x, k) - q(y; x', k)| \mu(dy), |K(x) - K(x')|\} \end{aligned}$$

and the righthand side converges to zero independently of α as $x' \rightarrow x$.

(ii) $f_\alpha(x) = K(x)$ wherever $w(x, k) - \gamma_\alpha + \alpha \int f_\alpha(y)q(y; x, k)\mu(dy) \geq K(x)$ for all $k \in \mathbf{A}$. Equivalently, where $w(x, k) \geq K(x) + \gamma_\alpha - \alpha \int f_\alpha(y)q(y; x, k)\mu(dy)$. But $\gamma_\alpha \geq 0$; $\liminf_{x \rightarrow \infty} w(x, k) \gg 0$ for all $k \in \mathbf{A}$ and $\|f_\alpha\| \leq M_1$. Hence, let G be a compact set such that for $x \notin G$

$$\min_k w(x, k) > \|K\| + M_1. \qquad \text{Q.E.D.}$$

It often is quite easy to show that the family $\{f_\alpha\}$ is equibounded as required for Lemma 3.2. In a typical case, x_0 is a preferred state in the sense that $g_\alpha(x) \geq g_\alpha(x_0)$ for all $x \in \mathbf{X}$ and $\alpha \in (0, 1)$, and this may be easy to prove. In this case one has $\|f_\alpha\| \leq \|K\|$. A general condition, adapted from Bather [2], which implies the result is:

A6° It is assumed that for every compact set G

$$\sup_{x \in G} \int^+ [q(y; x_0, k') - q(y; x, k)] \mu(dy) < 1$$

for every $k \in \mathbf{A}$, $k' \in \mathbf{A}$, where the integral is over the domain

$$\{y: q(y; x_0, k') > q(y; x, k)\}.$$

This condition is sufficient but not necessary, as will be seen in the examples.

LEMMA 3.3. *If $\liminf_{x \rightarrow \infty} w(x, k) \gg 0$ for all $k \in \mathbf{A}$ and Assumption A6° holds, then the family of functions $\{f_\alpha\}$ is bounded.*

PROOF. Easily $f_\alpha(x) \leq K(x)$ for all $x \in \mathbf{X}$ and the problem is to show that the functions are bounded from below. Letting $f_\alpha^{(n)}(x) = g_\alpha^{(n)}(x) - g_\alpha^{(n)}(x_0)$, where $g_\alpha^{(n)}(x)$ is defined as in Theorem 2.3 one may show

$$\begin{aligned} f_\alpha^{(n)}(x) &= \min\{K(x), [w(x, k) - w(x_0, k')]\} \\ &\quad + \alpha \int f_\alpha^{(n-1)}(z)[q(z; x, k) - q(z; x_0, k')]\mu(dz); k \in \mathbf{A} \end{aligned}$$

where k' is an action appropriate at x_0 . Choose A such that (a) $w(x, k) - w(x_0, k') > -A$ for all $k \in \mathbf{A}$ and (b) $K(x) \leq +A$, for all $x \in \mathbf{X}$. Let G be a compact set such that $x \notin G$ implies $w(x, k) - w(x_0, k') > A$, for $k \in \mathbf{A}$. Define

$$\beta = \sup_{x \in G} \max_k \int^+ [q(y; x, k) - q(y; x_0, k')]\mu(dy).$$

By $A\beta^0$ one has $\beta < 1$. Finally let B be such that

$$B > A\{(1 + \beta)/(1 - \beta)\} > A.$$

Then $f_\alpha^{(0)}(x) > -B$ for all $x \in \mathbf{X}$ and $\alpha \in (0, 1)$. Suppose $f_\alpha^{(n-1)}(x) > -B$ for all $x \in \mathbf{X}$ and $\alpha \in (0, 1)$.

CASE (i). $x \notin G$. One has $w(x, k) - w(x_0, k') > A$ for $x \notin G$. Hence

$$\begin{aligned} f_\alpha^{(n)}(x) &= \min\{K(x), [w(x, k) - w(x_0, k')] + \alpha \int f_\alpha^{(n-1)}(y)q(y; x, k)\mu(dy) \\ &\quad - \alpha \int f_\alpha^{(n-1)}(y)q(y; x_0, k')\mu(dy); k \in \mathbf{A}\} \\ &\geq \min\{0, A + \alpha(-B) - \alpha A\} \geq -B \end{aligned}$$

CASE (ii). $x \in G$.

$$\begin{aligned} w(x, k) - w(x_0, k') + \alpha \int f_\alpha^{(n-1)}(y)[q(y; x, k) - q(y; x_0, k')]\mu(dy) \\ \geq -A + \alpha \int f_\alpha^{(n-1)}(y)[q(y; x, k) - q(y; x_0, k')]^+\mu(dy) \\ \quad + \alpha \int f_\alpha^{(n-1)}(y)[q(y; x, k) - q(y; x_0, k')]^-\mu(dy) \\ \geq -A + \alpha(-B)\beta + \alpha(-A)\beta \\ \geq -A(1 + \beta) - \beta B \geq -B. \end{aligned}$$

Thus $f_\alpha^{(n)}(x) \geq -B$ for all $n = 1, 2, \dots$ and the family $\{f_\alpha\}$ is bounded. Q.E.D.

Before proceeding to the main theorems of this section, one more lemma will be presented.

LEMMA 3.4. *If $\liminf_{x \rightarrow \infty} w(x, k) \gg 0$ for $k \in \mathbf{A}$ and for every compact set G*

$$\sup_{x \in G} \int_G q(y; x, k)\mu(dy) < 1$$

then for every real number γ there is at most one function $f \in B(\mathbf{X})$ satisfying $f = S_\gamma f$. The rule defined by $f = S_\gamma f$ as in Theorem 3.1 has a finite expected cycle length.

PROOF. Suppose $f_i \in B(\mathbf{X})$, $i = 1, 2$ both satisfy $f_i = S_\gamma f_i$. Let $B = \max\{\|f_1\|, \|f_2\|\}$. Let G be a compact set such that $x \notin G$ implies $w(x, k) > \|K\| + B + |\gamma|$. Then for $x \notin G$ one has $f_1(x) = f_2(x) = K(x)$. But from $f_i = S_\gamma f_i$ one has

$$\begin{aligned} |f_1(x) - f_2(x)| &\leq \int |f_1(y) - f_2(y)| q(y; x, k)\mu(dy) \\ &\leq \|f_1 - f_2\| \int_G q(y; x, k)\mu(dy) \end{aligned}$$

and $\|f_1 - f_2\| \leq \|f_1 - f_2\| \cdot \beta$ with $\beta < 1$

where

$$\beta = \sup_{x \in G} \int_G q(y; x, k)\mu(dy).$$

Thus $f_1 = f_2$.

Under the rule defined by $f = S_\gamma f$ one repairs at time t if $X_t \notin G$. Since

$$\Pr\{X_{t+1} \in G \mid X_t = x \in G\} \leq \beta < 1$$

one has

$$\begin{aligned}
 E\{N_1\} &\leq 1 \cdot (1 - \beta) + 2(1 - \beta)\beta + 3(1 - \beta)\beta^2 + \dots \\
 &= (1 - \beta) \sum_{n=1}^{\infty} n\beta^{n-1} < \infty.
 \end{aligned}$$

Q.E.D.

The major theorem of this section is:

THEOREM 3.2. *Let $f_\alpha(x) = \psi(x, \alpha, R_\alpha) - \psi(x_0, \alpha, R_\alpha)$ and $\gamma_\alpha = (1 - \alpha) \cdot \psi(x_0, \alpha, R_\alpha)$. Under Assumptions A1° to A6°, (where A6° may be replaced by any assumption assuring that $\{f_\alpha\}$ is bounded):*

(i) *There exists a function $f \in B(\mathbf{X})$ and a number γ such that*

(a) *$f_\alpha \rightarrow f$ uniformly on compact sets as $\alpha \rightarrow 1$*

(b) *$\gamma_\alpha \rightarrow \gamma$ as $\alpha \rightarrow 1$*

(c) *$f = S_\gamma f$ and f is a unique solution in $B(\mathbf{X})$.*

If A5° does not hold then (a) is true only for some subsequence and f may not be unique in (c).

(ii) *Let $\gamma_0 = \inf \{\varphi(x_0, R); R \in \mathbf{R}\}$. Then*

(a) *$\gamma_0 = \gamma$*

(b) *The rule defined by $f = S_\gamma f$ as in Theorem 3.1 achieves the minimum average cost of γ_0 .*

PROOF. By Condition A6° and Lemma 3.3 the family $\{f_\alpha\}$ is bounded and thus, by Lemma 3.2, is equi-continuous. Let $\{\alpha_n\} \subset (0, 1)$ have $\lim \alpha_n = 1$. By the Ascoli-Arzelà theorem we may require as well that there exists a function f such that $f_{\alpha_n} \rightarrow f$ uniformly on compact sets. Since the set $\{\gamma_{\alpha_n}\}$ is a bounded set of real numbers, one may require that $\gamma_{\alpha_n} \rightarrow \gamma$ for some real number γ . By the bounded convergence theorem, $f = S_\gamma f$ and by Lemma 3.4, the rule defined by $f = S_\gamma f$ has a finite expected cycle length. Further, since $f_\alpha(x_0) = 0$ for all $\alpha \in (0, 1)$ one has $f(x_0) = 0$. Let R^* be the rule defined by $f = S_\gamma f$ as in Theorem 3.1. Then by Theorem 3.1, $\varphi(x_0, R^*) = \gamma$. If $\gamma = \gamma_0$, then by Lemma 3.1, R^* is optimal.

But for any fixed rule R

$$\liminf_{\alpha \rightarrow 1} (1 - \alpha)\psi(x_0, \alpha, R) = \varphi(x_0, R).$$

[see Hardy [10], Theorem 96, p. 155 for the underlying theorem used here.] And

$$\psi(x_0, \alpha, R_\alpha) \leq \psi(x_0, \alpha, R)$$

by definition of the rule R_α . Hence, if $\gamma_0 = \inf \{\varphi(x_0, R); R \in \mathbf{R}\}$ one has

$$\begin{aligned}
 \gamma_0 \leq \gamma &= \lim_{n \rightarrow \infty} \gamma_{\alpha_n} = \lim_{n \rightarrow \infty} (1 - \alpha_n)\psi(x_0, \alpha_n, R_{\alpha_n}) \\
 &\leq \liminf_{n \rightarrow \infty} (1 - \alpha_n)\psi(x_0, \alpha_n, R) = \varphi(x_0, R)
 \end{aligned}$$

for any rule $R \in \mathbf{R}$.

Thus $\gamma_0 = \gamma$ and γ is unique, in the sense that for any convergent sequence $\{f_{\alpha_n'}, \alpha_n'\}$ one has $\lim_{n \rightarrow \infty} \gamma_{\alpha_n'} = \gamma$. But then $\lim_{n \rightarrow \infty} f_{\alpha_n'} = f$ since by Lemma 3.4 and Condition A5° the solution to $f = S_\gamma f$ is unique. But in a complete metric space, a limit exists if and only if there exists a convergent subsequence and every convergent subsequence has the same limit. Hence $f_\alpha \rightarrow f$ and $\gamma_\alpha \rightarrow \gamma$.

Q.E.D.

Unfortunately this theorem does not state that the average cost per unit time under the optimal discounted cost rule R_α approaches the minimum average cost per unit time, $\gamma_0 = \inf \{ \varphi(x_0, R) ; R \in \mathbf{R} \}$. However this is the case as shown in:

THEOREM 3.3. *Under Assumptions A1° to A6°*

$$\lim_{\alpha \rightarrow 1} \varphi(x_0, R_\alpha) = \gamma_0 .$$

PROOF. Let $f_\alpha(x) = \psi(x, \alpha, R_\alpha) - \psi(x_0, \alpha, R_\alpha)$ and $\gamma_\alpha = (1 - \alpha)\psi(x_0, \alpha, R_\alpha)$. Theorem 3.2 has yielded that $\lim_{\alpha \rightarrow 1} \gamma_\alpha = \gamma_0$. Hence one need only show that $\varphi(x_0, R_\alpha) = \gamma_\alpha + O(1 - \alpha)$ where $O(1 - \alpha) \rightarrow 0$ as $\alpha \rightarrow 1$. But

$$f_\alpha(x) = \min \{ K(x) ; w(x, k) - \gamma_\alpha + \alpha \int f_\alpha(y) q(y ; x, k) \mu(dy), k \in \mathbf{A} \}$$

and R_α is the rule determined by this equation as in Theorem 3.1. Following the proof of Theorem 3.1 one can show

$$0 = E\{W_1\} - f_\alpha(x_0) - \gamma_\alpha E\{N_1\} - (1 - \alpha) E\{ \sum_{k=0}^{N_1-1} E f_\alpha(X_{k+1}) | X_k \}.$$

Since $f_\alpha(x_0) = 0$ and $\varphi(x_0, R_\alpha) = E\{W_1\}/E\{N_1\}$

$$\varphi(x_0, R_\alpha) = \gamma_\alpha + (1 - \alpha) E\{ \sum_{k=0}^{N_1-1} E f_\alpha(X_{k+1}) | X_k \} / E\{N_1\}.$$

The functions $\{f_\alpha\}$ are uniformly bounded and thus $\varphi(x_0, R_\alpha) = \gamma_\alpha + (1 - \alpha)\epsilon$ where $0 \leq |\epsilon| \leq \sup \|f_\alpha\| < \infty$. Q.E.D.

Thus, for a replacement problem both an average cost and a discounted cost solution exists. The optimal rules may be taken to be stationary and non-randomized. Finally, as the discount factor α approaches one, the optimal discounted cost rule approaches, in some sense, the optimal average cost rule.

There are two generalizations of the entire preceding theory which are easily made. The first is to allow the starting state x_0 to be chosen according to some fixed distribution, both initially and after each replacement. Secondly, replacement need not be considered instantaneous, but may take a period of T units during which no transitions are made. These generalizations affect the previously developed formulas in a straightforward manner.

4. Some applications.

4.1. *The Girshick and Rubin model of a production process.* Consider a simple production process which is assumed always to be in one of only two states, a good state and a bad state. Specifically, production begins in the good state and while there a chance event occurs before each item is produced so that the probability of remaining in the good state is $1 - \pi$ and the probability of a transition to the bad state is π . Once in the bad state, the process remains there until trouble is removed.

Associated with each item produced is a measurable characteristic or quality, denoted Y , assumed to be a random variable with a distribution depending on the unknown state of the machine. Let $p_0(\cdot)$ and $p_1(\cdot)$ be the density function for quality given that the machine is in the good and bad states, respectively.

A statistical control rule is a rule which specifies when the system is to be brought from production to repair, which has the effect of placing the process in

good state. Other than immediately after repair, the true process state is assumed unknown at all times. Hence a control rule must be based on the quality history of produced items. This history is adequately summarized in the posterior probability given the quality history that the next item will come from a machine in the bad state. This probability at time t is denoted X_t . Costs are associated with repairing the process and with the quality of each item produced, and the objective is to minimize the average cost per unit time.

Let $\mathbf{X} = [0, 1)$ be the state space of posterior probabilities. Let $x_0 = 0$ which assumes that a repaired machine is known to be in the good state. If the t th item has quality Y and $X_t = x$ then

(i) The posterior density for Y is

$$xp_1(y) + (1 - x)p_0(y)$$

and by Bayes' rule,

$$(ii) X_{t+1} = [xp_1(Y) + \pi(1 - x)p_0(Y)]/[xp_1(Y) + (1 - x)p_0(Y)]$$

assuming no repair.

If it costs $K_0(K_1)$ units to repair a good (bad) system and if an item of quality y costs $C(y)$ one has

$$K(x) = xK_1 + (1 - x)K_0$$

$$w(x) = x \int C(y)p_1(y)dy + (1 - x) \int C(y)p_0(y)dy.$$

where $w(x)$ rather than $w(x, k)$ is written since no action other than repair is allowed. Both are linear functions in x and assumed to be increasing, since $x = 1$ corresponds to a process in the bad state. Write $E_x(\cdot)$ for expectation under the distribution of X_{t+1} given that $X_t = x$ and no repair is made. Furthermore, the difference $w(x) - K(x)$ is assumed to be non-decreasing in x .

Let

$$g_\alpha^{(0)}(x) = 0$$

$$g_\alpha^{(n+1)}(x) = \min\{K(x) + w(0) + \alpha E_0 g_\alpha^{(n)}(X_{t+1}), w(x) + \alpha E_x g_\alpha^{(n)}(X_{t+1})\}.$$

It is important to show that if $g(\cdot)$ is any monotonically nondecreasing function of x then $E_x g(X_{t+1})$ is also.

Differentiability is assumed and an attempt is made to show that the derivative is positive. If $r(x) = xp_1(y) + (1 - x)p_0(y)$ and $\pi' = 1 - \pi$ then

$$(d/dx)\{X_{t+1}\} = \pi' p_1 p_0 / [r(x)]^2 \tag{and}$$

$$(d/dx)\{E_x g(X_{t+1})\} = \int g'(X_{t+1})[\pi' p_1(y)p_0(y)/r(x)]dy + \int g(X_{t+1})[p_1(y) - p_0(y)]dy.$$

The first term on the right is clearly positive since $g(\cdot)$ is assumed to be non-decreasing. For the second term let $A = \{y: p_1(y) > p_0(y)\}$. One has

$$\int g(X_{t+1})[p_1(y) - p_0(y)]dy \geq [\min_{y \in A} g(X_{t+1}) - \max_{y \notin A} g(X_{t+1})] \int_A [p_1(y) - p_0(y)]dy.$$

Thus since $g(\cdot)$ is monotonic nondecreasing, it suffices to show that if y_1 has $p_1(y) > p_0(y_1)$ and y_2 has $p_1(y_2) < p_0(y_2)$ then $X_{t+1}(y_1) > X_{t+1}(y_2)$ or

$$[xp_1(y_1) + \pi(1 - x)p_0(y_1)]/[xp_1(y_1) + (1 - x)p_0(y_1)] > [xp_1(y_2) + \pi(1 - x)p_0(y_2)]/[xp_1(y_2) + (1 - x)p_0(y_2)]$$

Letting $p_1(y_1)/p_0(y_1) = 1 + \delta_1$ and $p_1(y_2)/p_0(y_2) = 1 - \delta_2$ where both δ_1 and δ_2 are positive, this inequality reduces to

$$[x[1 + \delta_1] + \pi(1 - x)]/[x[1 + \delta_1] + (1 - x)] > [x[1 - \delta_2] + \pi(1 - x)]/[x[1 - \delta_2] + (1 - x)].$$

With further simplification one has the equivalent $1 - \delta_2 < 1 + \delta_1$ which is true by assumption. Thus, if $g(x)$ is a monotonic nondecreasing function of x then $E_x g(X_{t+1})$ is also.

It follows that for every $\alpha \in (0, 1)$ and $n = 1, 2, \dots$ that $g_\alpha^{(n)}(x)$ is non-decreasing in x and hence, $g_\alpha(x) = \lim g_\alpha^{(n)}(x)$ is also. One has $g_\alpha(0) = \min_x g_\alpha(x)$, which takes the place of Assumption A6°. Consequently, the form of the optimal rule is “Repair at time t if and only if $X_t > \xi^*$ ” for some critical value ξ^* . A rule of this form is optimal under both the discounted cost and the average cost criteria.

Girshick and Rubin [10] first discussed this model of a production process. In the average cost case they presented the above as being the form of the optimal rule, and they gave some integral equations describing the “steady state” behavior of the system as a function of the critical value ξ^* . Breiman [7] verified that the above rule form is optimal in the special case where inspection is by attributes, each item being classified as good or defective, and this simple case is adequate to show that the second proposal of Girshick and Rubin is not optimal.

This proposal concerns the non-100% inspection case, where inspection costs are allowed and where the rule must specify which items to inspect as well as when to repair. The optimal rule form is given by these authors as:

“At time t

- (a) If $X_t < \lambda_1$, then continue production but do *not* inspect the next item;
- (b) If $\lambda_1 \leq X_t < \lambda_2$ then continue production and inspect the next item produced;
- (c) If $\lambda_2 \leq X_t$ then stop and repair the machine.”

Let $p_0(p_1)$ be the fraction of good items produced when the machine is in the good (bad) state. For this example:

$$\begin{aligned} \pi &= \frac{1}{2} \\ p_0 &= 1 \\ p_1 &= \frac{1}{2}. \end{aligned}$$

It costs $K = 1$ unit to repair the machine, $c = 2$ units for each defective produced and ϵ units per item inspected. Let I be the inspection action and NI the non-inspection action. One has the cost function

$$w(x, NI) = c[xq_1 + (1 - x)q_0]$$

$$w(x, I) = \epsilon + c[xq_1 + (1 - x)q_0]$$

where $q_i = 1 - p_i$.

When no inspection takes place, $X_{t+1} = x + (1 - x)\pi$ if $X_t = x$, while under inspection X_{t+1} is defined through Bayes rule as before.

For the moment assume a zero inspection cost ϵ . The optimal procedure in this case is to always produce at least two items, inspecting the second one. If it is good then produce a third item, then repair; otherwise, repair immediately. The minimum average cost per unit time is $\gamma_0 = \frac{8}{11}$. Let $f^\epsilon(\cdot)$ be the solution to

$$f^\epsilon(x) = \min \{K, f_{NI}^\epsilon(x), f_I^\epsilon(x)\}$$

where

$$f_{NI}^\epsilon(x) = w(x, NI) - \gamma_0 + f^\epsilon(x + (1 - x)\pi)$$

$$f_I^\epsilon(x) = w(x, I) - \gamma_0 + E_x f^\epsilon(X_{t+1}).$$

For a zero inspection cost ϵ one has

$$f_I^0(x) = x + \frac{3}{11} \quad \text{for } \frac{5}{8} \leq x \leq 1$$

$$= \left(\frac{15}{11}\right)x + \frac{1}{22} \quad \text{for } \frac{2}{5} \leq x \leq \frac{5}{8}$$

$$= \left(\frac{65}{44}\right)x \quad \text{for } 0 \leq x \leq \frac{2}{5}$$

$$f_{NI}^0(x) = x + \frac{3}{11} \quad \text{for } \frac{5}{11} \leq x \leq 1$$

$$= \frac{3}{2}x + \frac{1}{22} \quad \text{for } \frac{1}{4} \leq x \leq \frac{5}{11}$$

$$= \left(\frac{37}{22}\right)x \quad \text{for } 0 \leq x \leq \frac{1}{4}.$$

The functions are graphed in Fig. 1. Now suppose ϵ is a small positive quantity. To a first order approximation f^ϵ is given by

$$f^\epsilon(x) \cong \min \{K, f_{NI}^0(x), f_I^0(x) + \epsilon\}.$$

This results in inspection being undesirable for posterior probabilities in the vicinity of 0 and .6 while in the vicinity of .5 inspections are desirable as may be seen in Fig. 2. Examining the physical situation makes the reason for this clear.

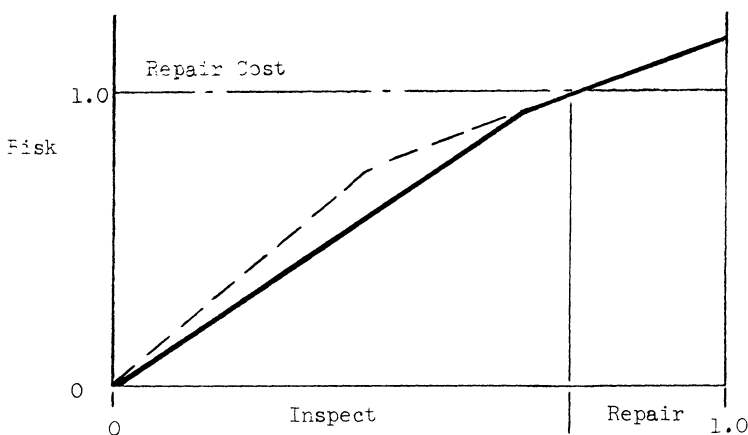


FIG. 1. Risk Function (Zero Inspection Cost)

The first item is not inspected since it is known to come from a good machine. The third item is not inspected (assuming it is produced), since it is always followed by repair. However the information gained by inspecting the second item is sufficient to offset a small inspection cost. Since the “no inspection” region consists of two disjoint intervals, the rule as given by Girshick and Rubin is not optimal. These intervals are not vacuous since

$$\begin{aligned}
 X_0 &= 0 \\
 X_1 &= \frac{1}{2} \\
 X_2 &= \frac{2}{3} && \text{If the second item produced is good} \\
 &= 1 && \text{If the second item is bad.}
 \end{aligned}$$

4.2 *A normally distributed random walk.* The average cost functional equation may be used to derive explicit solutions as this example due to Bather [2] shows. Suppose that the state space is the real line with $x_0 = 0$. Assume a model wherein:

(i) If no replacement is made then

$$X_{t+1} = X_t + \eta_{t+1}.$$

(ii) If a replacement is made then

$$X_{t+1} = \eta_{t+1}$$

where $\{\eta_t\}$ is a sequence of independent normally distributed random variables with zero mean and unit variance. Briefly, the process is a normal random walk which replacement restarts at zero. No actions other than replacement are allowed, and thus “ k ” will be dropped from all notation. Assume the simple cost functions

$$\begin{aligned}
 w(x) &= ax^2; && a > 0 \\
 K(x) &= K.
 \end{aligned}$$

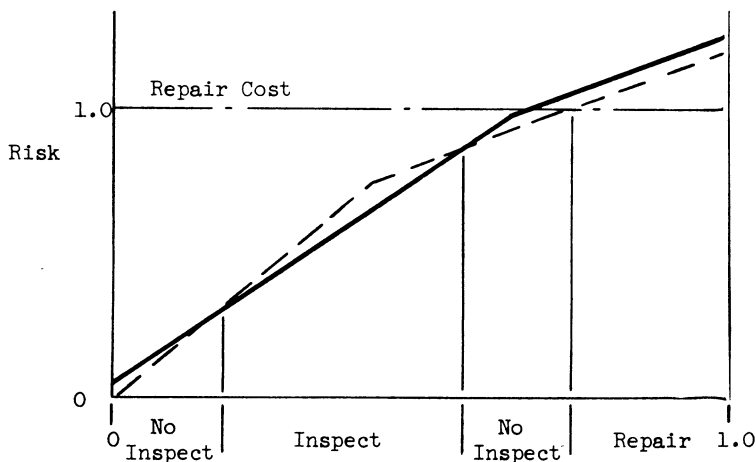


FIG. 2. Risk Function (Positive Inspection Cost)

The risk function given here is for the Girshick and Rubin model, non-100% inspection case. The abscissa is the probability $X_t = x$ that the machine has broken down at time t . The dashed line gives the no-inspect alternative while the solid line is associated with inspection. At $X_t = x$, the optimal decision corresponds to the lowest of these two lines and the repair cost line at 1.0.

All of the necessary assumptions are satisfied in this particular case. Letting

$$g_\alpha^{(n)}(x) = 0 \quad \text{and}$$

$$g_\alpha^{(n+1)}(x) = \min \{K + \alpha E g_\alpha^{(n)}(\eta); w(x) + \alpha E g_\alpha^{(n)}(x + \eta)\}$$

where the expectation considers η as a normal zero-one random variable, one sees that

$$\min_x g_\alpha^{(n)}(x) = g_\alpha^{(n)}(0).$$

According to the earlier remarks one need not verify Assumption A6°. Further, $g_\alpha^{(n)}(x)$ is symmetric about the origin and increasing as $|x|$ increases. It follows that under both the average cost and the discounted cost criterion, the optimal rule is of the form:

“Repair at time t if and only if $X_t > \lambda$.”

for some critical value λ .

To find the optimal critical value λ^* , Bather assumes that the variance of the process is small compared to the length of time between repairs so that a continuous time Wiener process approximation is valid. He shows the existence of the analogous average cost functional equation:

$$f(x) = \min \{K, (\alpha x^2 - \gamma)(\Delta t) + E f(x + \Delta x)\}$$

where Δx is normally distributed with zero mean and variance (Δt) . Expanding in a Taylors series,

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2}(\Delta x)^2 f''(x) + \dots$$

and taking expectations

$$Ef(x + \Delta x) = f(x) + \frac{1}{2}(\Delta t)f''(x) + \dots$$

one arrives at the differential equation

$$f(x) = (ax^2 - \gamma)(\Delta t) + f(x) + \frac{1}{2}(\Delta t)f''(x) + \dots \quad \text{for } -\lambda \leq x \leq +\lambda.$$

As $\Delta t \rightarrow 0$ one has

$$f''(x) = 2(\gamma - ax^2)$$

or

$$f(x) = \gamma x^2 - \frac{1}{6}ax^4 + C_1x + C_2.$$

The side conditions

$$f(\lambda) = K, \quad f'(0) = 0$$

imply that $C_1 = C_2 = 0$ and thus one has

$$\gamma = \frac{1}{6}a\lambda^2 + K/\lambda^2.$$

To minimize the average cost per unit time γ , one takes derivatives with respect to λ^2 yielding

$$d\gamma/d(\lambda^2) = \frac{1}{6}a - K/\lambda^4.$$

Equating to zero gives the optimal critical value λ^*

$$\lambda^* = (6K/a)^{\frac{1}{2}}.$$

Box and Jenkins [6] and Antelman and Savage [1] arrive at this same answer by different paths and all three authors treat much more general problems than the example shown here.

REFERENCES

- [1] ANTELMAN, G. R. and I. RICHARD SAVAGE (1964). Surveillance problems: Weiner processes. Technical Report No. 34, Univ. of Minnesota, Dept. of Statistics.
- [2] BATHER, J. A. (1963). Control charts and the minimization of costs. *J. Roy. Statist. Soc. Ser. B* **25** 49-80.
- [3] BELLMAN, R. (1957). *Dynamic Programming*. Princeton Univ. Press, Princeton.
- [4] BLACKWELL, DAVID (1962). Discrete dynamic programming. *Ann. Math. Statist.* **33** 719-726.
- [5] BLACKWELL, DAVID (1965). Discounted dynamic programming. *Ann. Math. Statist.* **36** 226-235.
- [6] BOX, G. E. P. and JENKINS, G. (1963). Simultaneous estimation of dynamics: Non-zero costs (Section 6-Application to the "Machine Tool" control problem). Proc. 34th Session, I. S. I., Ottawa, Canada. 943-974.
- [7] BREIMAN, LEO (1960). A model for quality control. Inst. of Engr. Res. Ser. No. 60, Issue No. 438, Electronics Research Laboratory, Univ. of California, Berkeley.
- [8] DERMAN, CYRUS (1963). On optimal replacement rules when changes of state are

- Markovian. *Mathematical Optimization Techniques*, (R. Bellman, ed.). Project Rand Report, April 1963.
- [9] DERMAN, CYRUS (1965). Markovian sequential control processes—denumerable state space. *J. Math. Anal.* **10** 295–302.
- [10] GIRSHICK, M. A. and RUBIN, HERMAN (1952). A Bayes approach to a quality control model. *Ann. Math. Statist.* **23** 114–125.
- [11] HARDY, G. H. (1949). *Divergent Series*. Oxford Univ. Press, London.
- [12] KARLIN, SAMUEL (1955). Structure of dynamic programming. *Naval Res. Logist. Quart.* **2** 285–294.