ON IDEMPOTENT MATRICES

By R. M. Loynes

University of Manchester

1. Introduction and summary. Banerjee [1] has recently given proofs of various properties of real symmetric idempotent matrices. The aim of this note is to give shorter proofs of some of these. Some of our proofs are suggested by the fact that idempotent matrices represent projection operators, and by the known properties of such operators (see e.g. Halmos [2]). The recent paper by Luther [4] overlaps this note in content; but has a different aim.

2. The theorems. We quote directly from Banerjee.

Given a collection of $n \times n$ *symmetric matrices* A_i $(i = 1, 2, \cdots, m)$, *where the rank of* A_i *is* p_i, $A = \sum A_i$, *the rank of* A *is* p, *and four conditions*: (a) *each* A_i *is idempotent*, (b) $A_iA_j = 0$ *for all* $i \neq j$, (c) A *is idempotent, and* (d) $p = \sum p_i$; *then*, (i): (a) *and* (c) *imply* (b); (ii): (b) *and* (c) *imply* (a); (iii): (a) *and* (b) *imply* (c); (iv): *any two of* (a), (b) *and* (c) *imply all four conditions*; (v): (c) *and* (d) *imply* (a) *and* (b); (vi): *if* A, A_i $(i = 1, 2, \cdots, m - 1)$ *are idempotent, and* A_m *is positive semi-definite, then* A_m *is idempotent.*

Throughout we assume all matrices real, and by an idempotent matrix A we mean a real symmetric matrix satisfying $A^2 = A$. The assumption of reality can of course be dropped, provided we suppose the matrices Hermitian rather than symmetric.

3. Proofs. We shall make use of the following lemma. Here, as elsewhere, we write $A \geqq B$ for two symmetric matrices to mean that $A - B$ is positive semi-definite, and I for the identity matrix.

Lemma. *If* A *is idempotent and* $P \geqq 0$, *then from* $I \geqq A + P$ *follows* $AP = PA = 0$.

From the lemma follows a result on the distribution of certain quadratic forms in normal variables. Suppose that the vector x is a sample from a normal distribution: then if A is idempotent, P is positive semi-definite, from $x'x \geqq x'Ax + x'Px$ follows the independence of $x'Ax$ and $x'Px$. This is an immediate consequence of Theorem 4.13 of [3]. (This appears to be new, but is of course not surprising in view of Cochran's theorem.)

For any given column vector x, let $y = Ax$. Then $Ay = A^2x = Ax = y$. Hence $y'y = y'Iy \geqq y'Ay + y'Py = y'y + y'Py \geqq y'y$, from which it follows that $y'Py = 0$. By reducing P to diagonal form, or otherwise, it is seen that $Py = 0$. Since x was arbitrary we have $PA = 0$, and by transposing $AP = 0$.

Now to prove (i) we note that, if $i \neq j$, $I \geqq A \geqq A_i + A_j$, the hypotheses of the lemma are satisfied and hence $A_iA_j = 0$.

To prove (ii) we use the fact that a symmetric matrix is idempotent if and

Received 24 December 1964.

only if all its eigenvalues are either zero or one. Let λ be an eigenvalue of A_1 with eigenvector x. Then $A_1x = \lambda x$. Either $\lambda = 0$, or we have $x = A_1x/\lambda$, so that $A_jx = A_jA_1x/\lambda = 0$, and consequently $Ax = A_1x = \lambda x$. Then $\lambda = 1$, and A_1 is idempotent.

The proofs of (iii) and (iv) previously given can hardly be improved.

To prove Part (v) we can argue as follows. The set of equations $Ax = x$, $A_2x = 0, A_3x = 0, \cdots A_mx = 0$ contains no more than $n - p + p_2 + p_3 + \cdots + p_m = n - p_1$ independent equations, and hence has at least p_1 independent solutions. These equations imply $A_1x = x$. It follows that the non-zero eigenvalues of A_1, which are p_1 in number, must all be equal to unity, so that A_1 is idempotent. The remaining conclusions are then consequences of the previous results.

Finally, the hypotheses of (vi) imply that if $i \neq j$ $I \geqq A_i + A_j$, and at least one of A_i and A_j is idempotent. Hence from the lemma $A_iA_j = 0$. Then we have $A = \sum A_i$, and on squaring, $A = \sum_1^{m-1} A_i + A_m^2$. It follows that $A_m^2 = A_m$ and A_m is idempotent.

REFERENCES

[1] BANERJEE, K. S. (1964). A note on idempotent matrices. *Ann. Math. Statist.* **35** 880–882.
[2] HALMOS, P. R. (1952). *An Introduction to Hilbert Space*. Chelsea, New York.
[3] LUKACS, E. and LAHA, R. G. (1964). *Applications of Characteristic Functions*. Griffin, London.
[4] LUTHER, NORMAN Y. (1965). Decomposition of symmetric matrices and distributions of quadratic forms. *Ann. Math. Statist.* **36** 683–690.