A NOTE ON "A k-SAMPLE MODEL IN ORDER STATISTICS" BY W. J. CONOVER

By H. A. David

University of North Carolina at Chapel Hill

In the paper cited above Conover [1] considers the ordering of k mutually independent random samples of size n, each drawn from a parent distribution with absolutely continuous cdf $F(x)$, on the basis of the largest member in each sample. He defines Y_{ij} ($i = 1, 2, \cdots, n; j = 1, 2, \cdots, k$) to be the ith variate in order of magnitude in the sample whose largest member Y_{1j} has rank j among the k maxima $Y_{11}, Y_{12}, \cdots, Y_{1k}$.

For the distribution of Y_{ij} Conover obtains the result

$$
\begin{aligned}
F_{ij}(x) = \Pr(Y_{ij} < x) &= \sum_{\alpha=0}^{j-1} \binom{k}{\alpha}[1 - F^n(x)]^{\alpha}[F^n(x)]^{k-\alpha} \\
&+ \sum_{\alpha=0}^{j-1} \sum_{m=1}^{i-1} \sum_{\beta=0}^{m-1} j\binom{k}{j} m \binom{n}{m}\binom{j-1}{\alpha}\binom{m-1}{\beta}(-1)^{j+m-\beta-\alpha} \\
&\cdot([F(x)]^{n-1-\beta} - [F(x)]^{nk-n\alpha})/(nk - n\alpha + 1 - n + \beta)
\end{aligned} \tag{1}
$$

where the triple summation in (1) is zero for $i = 1$.

It is the purpose of this note to provide a greatly shortened proof of (1). To this end observe that

$$
\begin{aligned}
\Pr(Y_{ij} < x \mid Y_{1j} = y) &= 1, & x \geqq y, \\
&= \sum_{m=0}^{i-2} \binom{n-1}{m}[F(x)/F(y)]^{n-1-m} & \\
&\quad \cdot [1 - F(x)/F(y)]^m, & x < y,
\end{aligned} \tag{2}
$$

the last line following from the well-known fact (e.g. Rényi [2]) that conditionally on $Y_{1j} = y$, the variate Y_{ij} is distributed as an order statistic of rank $i - 1$ in a sample of $n - 1$ drawn from the truncated distribution with cdf $F(x)/F(y)$ $(-\infty < x < y)$. This line may also be written as

$$\sum_{m=1}^{i-1} \binom{n-1}{m-1}[F(x)]^{n-m}[F(y) - F(x)]^{m-1}/[F(y)]^{n-1}.$$

Since the probability element of Y_{1j} is given by

$$dF_{1j}(y) = j\binom{k}{j}[1 - F^n(y)]^{j-1}[F^n(y)]^{k-j} n[F(y)]^{n-1}\, dF(y),$$

we have on unconditionalizing (2)

$$
\begin{aligned}
\Pr(Y_{ij} < x) &= \int_{-\infty}^{x} dF_{1j}(y) + \int_x^{\infty} \sum_{m=1}^{i-1} \binom{n-1}{m-1}[F(x)]^{n-m}[F(y) - F(x)]^{m-1} \\
&\quad \cdot j\binom{k}{j}[1 - F^n(y)]^{j-1}[F(y)]^{n(k-j)} n\, dF(y) \\
&= \Pr(Y_{1j} < x) + \int_x^{\infty} \sum_{m=1}^{i-1} n\binom{n-1}{m-1}[F(x)]^{n-m}
\end{aligned}
$$

Received 23 August 1965.

$$\cdot \sum_{\beta=0}^{m-1} (-1)^{m-1-\beta} \binom{m-1}{\beta} [F(x)]^{m-1-\beta} [F(y)]^{\beta}$$

$$\cdot j\binom{k}{j} \sum_{\alpha=0}^{j-1} (-1)^{j-1-\alpha} \binom{j-1}{\alpha} [F(y)]^{n(j-1-\alpha)} [F(y)]^{n(k-j)}\, dF(y)$$

which immediately reduces to (1).

REFERENCES

[1] Conover, W. J. (1965). A k-sample model in order statistics. *Ann. Math. Statist.* **36** 1223–1235.

[2] Rényi, Alfréd (1953). On the theory of order statistics. *Acta Math. Acad. Sci. Hung.* **4** 191–231.