A SYSTEM OF DENUMERABLY MANY TRANSIENT MARKOV CHAINS[1]

By S. C. Port

The RAND Corporation

0. Summary. If P is a transient Markov chain having the invariant measure μ, and if at time 0 particles are distributed in the state space Ω according to the Poisson law, with mean $\mu(x)$ at x, and these particles are then allowed to move independently of the others according to the law P, the system maintains itself in macroscopic equilibrium. In this paper we investigate several phenomena connected with this system.

1. Introduction. Throughout this paper $P_n(x, y)$ will be the nth step transition matrix of an irreducible, transient Markov chain X_n, with the set Ω of nonnegative integers for its state space. We will assume that the chain has at least one positive invariant measure $\mu(x)$, which henceforth will be taken as fixed. Subsets $B \subset \Omega$ will always be finite and nonempty. We will use the following notation: $V_B = \min \{n > 0: X_n \varepsilon B\}$ ($= \infty$ if $X_n \not\varepsilon B$ for all $n > 0$) is the *hitting time* of B; $T_B = \min \{n \geqq 0: X_j \not\varepsilon B \text{ for all } j > n\}$ is the *time of last visit* to B; and $N_n(B) = \sum_{j=1}^{n} \delta_B(X_j)$, where $\delta_B(x) = 1$ for $x \varepsilon B$ and $= 0$ for $x \not\varepsilon B$, is the *occupation time* of B by time n. Since the chain is transient, $P_x(V_B = \infty) > 0$ for at least one point $x \varepsilon B$, and $N_n(B) \uparrow N(B) < \infty$ with probability one. The *dual* chain has transition matrix, $\hat{P}(x, y) = \mu(y)P(y, x)\mu(x)^{-1}$. Quantities which refer to the dual chain will be denoted by ^. Thus, e.g., $\hat{P}_x(V_B = \infty)$ is the quantity $P_x(V_B = \infty)$ computed for the dual chain.

The system we wish to investigate may be described as follows. At time 0 we put $A_0(x)$ particles at the point $x \varepsilon \Omega$, where the $A_0(x)$ are independent, Poisson distributed, random variables with means $\mu(x)$, respectively. We then allow each particle to move, independently of the others, according to the same transition law P. A system of this type was first investigated by Derman [1], where a more precise description can be found. The salient fact about this system [1], is that it maintains itself in macroscopic equilibrium, in the sense that at any time n, the number of particles in the various states, $A_n(x)$, are again independent Poisson variables with means $\mu(x)$.

Our purpose here is to establish the following facts about this system.

Theorem 1. *For $r \geqq 1$, let $M_n(B; r)$ denote the number of particles which have hit B exactly r times by time n. Then*

$$(1.1) \qquad P(\lim_{n\to\infty} (M_n(B; r)/n) = C_r(B)) = 1,$$

Received 5 August 1965; revised 5 October 1965.

[1] This research is sponsored by the United States Air Force under Project RAND—Contract No. AF 49(638)-700 monitored by the Directorate of Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

where

(1.2) $$C_r(B) = \sum_{x\varepsilon B} \mu(x)\hat{P}_x(V_B = \infty)P_x(N(B) = r - 1).$$

Moreover, for each fixed n and B, the $M_n(B; r)$ are mutually independent Poisson variates with means

(1.3) $$EM_n(B; r) = \sum_x \mu(x)P_x(N_n(B) = r),$$

and if $C_r(B) > 0$ then

(1.4) $$\lim_{n\to\infty} P\{[M_n(B; r) - EM_n(B; r)]/[nC_r(B)]^{\frac{1}{2}} \leqq u\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{u} e^{-t^2/2}\, dt.$$

Here and elsewhere we will use the convention that a Poisson variate with 0 mean is the distribution degenerate at 0.

THEOREM 2. *Let $S_n(B) = \sum_{j=1}^{n} A_j(B)$, where $A_j(B) = \sum_{x\varepsilon B} A_j(x)$. Then*

(1.5) $$P\{\lim_{n\to\infty} [S_n(B)/n] = \mu(B)\} = 1.$$

Moreover,

(1.6) $$\lim_{n\to\infty} P\{[S_n(B) - n\mu(B)]/[n\sigma^2(B)]^{\frac{1}{2}} \leqq u\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{u} e^{-t^2/2}\, dt,$$

where $\sigma^2(B) = \sum_{r=1}^{\infty} r^2 C_r(B)$.

THEOREM 3. *Let $J_n(B)$ denote the number of particles which are in B for a last time at time n. Then for fixed B the $J_n(B)$ are independent, Poisson distributed, random variables with a common mean $C(B) = \sum_{x\varepsilon B} \mu(x)P_x(V_B = \infty)$. Consequently, if $D_n(B) = J_1(B) + \cdots + J_n(B)$ then*

(1.7) $$P\{\lim_{n\to\infty} [D_n(B)/n] = C(B)\} = 1$$

and

(1.8) $$\lim_{n\to\infty} P\{[D_n(B) - nC(B)]/[nC(B)]^{\frac{1}{2}} \leqq u\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{u} e^{-t^2/2}\, dt.$$

REMARK. Theorem 1 is a refinement of Theorem 7.1 of [4] which asserts that if $L_n(B) = \sum_{r=1}^{\infty} M_n(B; r)$, then

(1.9) $$P\{\lim_{n\to\infty} [L_n(B)/n] = C(B)\} = 1$$

and

(1.10) $$\lim_{n\to\infty} P\{[L_n(B) - EL_n(B)]/[nC(B)]^{\frac{1}{2}} \leqq u\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{u} e^{-t^2/2}\, dt.$$

The constant $C(B)$ enters in potential theoretic studies on transient Markov chains and is called the *capacity* of B. (A discussion of capacities may be found in [3].) From (1.7) and (1.9) we see that the number of new particles which enter B per unit time equals the number of particles per unit time which leave B, never to return, and that this common number is the capacity of B. The result in (1.5) that $\mu(B)$ is the number of particles per unit time in B is, of course, intuitively very plausible.

2. Proofs.

PROOF OF THEOREM 1. Let V_B^j be the time of the jth visit to B. More precisely, $V_B^1 = V_B$ and if $V_B^j < \infty$ define $V_B^{j+1} = \min\{n > V_B^j: X_n \varepsilon B\}$ ($= \infty$ if $T_B = V_B^j$). If $V_B^j = \infty$ define $V_B^{j+1} = \infty$. Also, let $I_n(B;\ r)$ denote the number of particles which hit B for the rth time at time n.

LEMMA 1. *For fixed B and r, $I_n(B; r)$, $n \geqq 1$, are mutually independent, Poisson distributed, random variables with means* $EI_n(B;\ r) = \sum_x \mu(x)P_x(V_B^r = n)$.

PROOF. Let $I_{jx}(B;\ r)$ be the number of particles starting at x which hit B for the rth time at time j. Then for arbitrary $n \geqq 1$,

$$E(s_1^{I_1(B;r)} s_2^{I_2(B;r)} \cdots s_n^{I_n(B;r)} \mid A_0(0), A_0(1), \cdots)$$
$$= \prod_{x=0}^{\infty} E(s_1^{I_{1x}(B;r)} \cdots s_n^{I_{nx}(B;r)} \mid A_0(x))$$
$$= \prod_{x=0}^{\infty} [1 + \sum_{j=1}^{n} (s_j - 1)P_x(V_B^r = j)]^{A_0(x)}.$$

A simple computation now gives

$$E(\prod_{j=1}^{n} s_j^{I_j(B;r)}) = \exp[\sum_{j=1}^{n} (s_j - 1)\sum_x \mu(x)P_x(V_B^r = j)],$$

which establishes the lemma.

LEMMA 2. *Define $V_B^0 = 0$. Then for any $r \geqq 1$,*

$$\lim_{n\to\infty} EI_n(B;\ r) = \sum_{z\varepsilon B} \mu(z)P_z(V_B^{r-1} < \infty)\hat{P}_z(V_B = \infty),$$

and thus

$$\text{(2.1)} \quad \lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} EI_j(B;\ r) = \sum_{z\varepsilon B} \mu(z)P_z(V_B^{r-1} < \infty)\hat{P}_z(V_B = \infty).$$

PROOF. For $r \geqq 0$ we have

$$EI_n(B; r+1) = \sum_x \mu(x)\sum_{z\varepsilon B}\sum_{j=1}^{n} P_x(V_B = j;\ X_j = z)P_z(V_B^r = n - j)$$
$$= \sum_x \sum_{z\varepsilon B}\sum_{j=1}^{n} \hat{P}_z(V_B \geqq j;\ X_j = x)\mu(z)P_z(V_B^r = n - j)$$
$$= \sum_{z\varepsilon B}\sum_{j=1}^{n} \hat{P}_z(V_B \geqq j)\mu(z)P_z(V_B^r = n - j).$$

Using the fact that $\hat{P}_z(V_B \geqq n) \downarrow \hat{P}_z(V_B = \infty)$ and $\sum_{n=1}^{\infty} P_z(V_B^r = n) = P_z(V_B^r < \infty)$, we obtain, by a simple summability argument, that

$$\text{im}_{n\to\infty} \sum_{z\varepsilon B} \mu(z)\sum_{j=1}^{n} \hat{P}_z(V_B^r \geqq j)P_z(V_B^r = n - j)$$
$$= \sum_{z\varepsilon B} \mu(z)\hat{P}_z(V_B = \infty)P_z(V_B^r < \infty).$$

This establishes Lemma 2.

To complete the proof of Theorem 1 we now proceed as follows: From Lemma 2 we see that $\sup_n EI_n(B;\ r) \leqq \alpha(r) < \infty$, while from Lemma 1 we have $EI_n(B; r) = \text{var}\ I_n(B; r)$. Consequently, $\sum_{n=1}^{\infty} [\text{var}\ I_n(B; r)/n^2] < \infty$ and thus, by the strong law of large numbers,

$$\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} [I_j(B;\ r) - EI_j(B;\ r)] = 0, \text{ a.e.}$$

Hence from (2.1) we have

$\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} I_j(B;\ r) = \sum_{z\varepsilon B} \mu(z)\hat{P}_z(V_B = \infty)P_z(V_B^{r-1} < \infty)$, a.e. However, $\sum_{j=1}^{n} I_j(B;\ r) = \sum_{k=r}^{\infty} M_n(B;\ k)$ and $P_z(V_B^{r-1} < \infty) = P_z(N(B) \geqq r - 1)$. Consequently,

$$\lim_{n\to\infty} [M_n(B;r)/n] = \lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} [I_j(B;\ r) - I_j(B;\ r + 1)]$$
$$= \sum_{z\varepsilon B} \mu(z)\hat{P}_z(V_B = \infty)P_z(N(B) = r - 1) = C_r(B), \text{ a.e.}$$

This establishes (1.1). To establish the independence of the $M_n(B;r)$ we proceed as in the proof of Lemma 1. For any $r \geqq 1$ we have

$$E(s_1^{M_n(B;1)} \cdots s_r^{M_n(B;r)} \mid A_0(0), \cdots)$$
$$= \prod_{x=0}^{\infty} [1 + \sum_{j=1}^{r}(s_j - 1)P_x(N_n(B) = j)]^{A_0(x)},$$

and thus

$$E(\prod_{j=1}^{r} s_j^{M_n(B;j)}) = \exp [\sum_{j=1}^{r} (s_j - 1)\sum_x \mu(x)P_x(N_n(B) = j)],$$

which shows the independence. (This also shows $M_n(B;r)$ is Poisson distributed with the mean given in (1.3).) Now for any $r \geqq 1$ we have by (2.1),

$$\text{(2.2)} \quad \lim_{n\to\infty} [EM_n(B;r)/n]$$
$$= \lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} [EI_j(B;r) - EI_j(B;r + 1)] = C_r(B).$$

Since a normalized Poisson variable is asymptotically normally distributed when its mean becomes infinite, we see that (1.4) follows from (2.2). This completes the proof of Theorem 1.

PROOF OF THEOREM 2. Since $A_n(B) = \sum_{x\varepsilon B} A_n(x)$ and $\mu(B) = \sum_{x\varepsilon B} \mu(x)$, it suffices to establish (1.5) for all one point sets $\{x\}$. Let $x \varepsilon \Omega$ be arbitrary. Then a simple computation shows

$$\mu(x) = \sum_{r=1}^{\infty} r\mu(x)\hat{P}_x(V_{\{x\}} = \infty)P_x(N(\{x\}) = r - 1) = \sum_{r=1}^{\infty} rC_r(\{x\}).$$

However,

$$\text{(2.3)} \qquad S_n(\{x\}) = \sum_{r=1}^{\infty} rM_n(\{x\};r),$$

and thus by Theorem 1,

$$\text{(2.4)} \qquad P\{\mu(x) \leqq \liminf_{n\to\infty} [S_n(\{x\})/n]\} = 1.$$

But it is readily verified that the sequence $\{A_n(x)\}$ is strictly stationary. An appeal to the pointwise ergodic theorem then yields the result that there is a random variable S^* such that

$$\text{(2.5)} \qquad P\{\lim_{n\to\infty} [S_n(\{x\})/n] = S^*\} = 1,$$

and moreover,

$$\text{(2.6)} \qquad ES^* = \lim_{n\to\infty} [ES_n(\{x\})/n] = \mu(x).$$

From (2.4)–(2.6) we must then have $P(S^* = \mu(x)) = 1$. This establishes (1.5).

Since $S_n(B) = \sum_{j=1}^n A_j(B)$, and the $A_j(B)$ are stationary, it is clear that $ES_n(B) = n\mu(B)$. To establish (1.6) we may proceed as follows. From (2.3) we see that

$$\operatorname{var} S_n(B) = \sum_{r=1}^n r^2 \operatorname{var} M_n(B;r) = \sum_{r=1}^n r^2 EM_n(B;r).$$

But

$$EM_n(B;r) = \sum_x \mu(x)P_x(N_n(B) = r)$$
$$= \sum_{z\varepsilon B} \mu(z) \sum_{j=1}^n \hat{P}_z(V_B \geqq j)P_z(N_{n-j}(B) = r-1),$$

and thus

$$\sum_{r=1}^n EM_n(B;r)r^2 = \sum_{z\varepsilon B} \mu(z) \sum_{j=1}^n \hat{P}_z(V_B \geqq j)E_z(N_{n-j}(B) + 1)^2.$$

Since $E_z(N_n(B)+1)^2 \uparrow E_z(N(B)+1)^2$ and $\hat{P}_z(V_B \geqq n) \to \hat{P}_z(V_B = \infty)$ we obtain, by a simple summability argument, that

$$\sum_{r=1}^n EM_n(B;r)r^2 \sim n\sum_{z\varepsilon B} \mu(z)\hat{P}_z(V_B = \infty)E_z(N(B)+1)^2.$$

A simple computation now shows that

$$\sum_{z\varepsilon B} \mu(z)\hat{P}_z(V_B = \infty)E_z(N(B)+1)^2 = \sum_{r=1}^n r^2C_r(B) = \sigma^2(B).$$

Thus

(2.7) $$\lim_{n\to\infty} [\operatorname{var} S_n(B)/n\sigma^2(B)] = \lim_{n\to\infty} [\sum_{r=1}^n r^2EM_n(B;r)/n\sigma^2(B)] = 1.$$

Recall that if $\phi(\theta)$ is the characteristic function of an infinitely divisible law with finite variance, then the Kolmorgorov representation of $\phi(\theta)$ is (see [2], p. 307)

(2.8) $$\log \phi(\theta) = i\theta\gamma + \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x)x^{-2}G(dx).$$

For the random variable

$$Y_n(B;r) = r[M_n(B;r) - EM_n(B;r)](n\sigma^2(B))^{-\frac{1}{2}},$$

γ and G are respectively, $\gamma_{nr} = 0$ and

$$G_{nr}(x) = 0, \qquad x < r(n\sigma^2(B))^{-\frac{1}{2}},$$
$$= r^2EM_n(B;r)/n\sigma^2(B), \qquad x > r(n\sigma^2(B))^{-\frac{1}{2}}.$$

Hence for the random variable $\sum_{r=1}^n Y_n(B;r)$ we find that γ and G are respectively, $\gamma_n = 0$ and

(2.9) $$G_n(x) = \sum_{r=1}^n G_{nr}(x) = \sum_{r=1}^j [r^2EM_n(B;r)/n\sigma^2(B)],$$
$$\text{if } j(n\sigma^2(B))^{-\frac{1}{2}} \leqq x < (j+1)(n\sigma^2(B))^{-\frac{1}{2}}$$

clearly, $G_n(x) \leqq G_n(\infty) = \sum_{r=1}^n [r^2EM_n(B;r)/n\sigma^2(B)]$, and thus by (2.7),

$$\limsup_{n\to\infty} G_n(x) \leqq \lim_{n\to\infty} G_n(\infty) = 1.$$

On the other hand by (2.9) and (2.2) we have for any $x > 0$,

$$\liminf_{n\to\infty} G_n(x) \geqq \sum_{r=1}^{\infty} [r^2 C_r(B)/\sigma^2(B)] = 1.$$

Since $G_n(x) = 0$ for $x < 0$ we see from the above that

$$\begin{aligned} \lim_{n\to\infty} G_n(x) = G(x) &= 1, \qquad x > 0, \\ &= 0, \qquad x < 0. \end{aligned}$$

But this is precisely the G needed in (2.8) to represent the standard normal distribution, and thus by appeal to a well known convergence theorem in the theory of i.d. laws (see [2], p. 312) we see that (1.6) holds. This completes the proof.

PROOF OF THEOREM 3. We need to prove that $\{J_n(B)\}$ are independent, Poisson variates with the common mean $C(B)$. To see this, first of all observe that

$$\begin{aligned} EJ_n(B) = \sum_x \mu(x) P_x(T_B = n) &= \sum_x \sum_{y \in B} \mu(x) P_n(x, y) P_y(V_B = \infty) \\ &= \sum_{y \in B} \mu(y) P_y(V_B = \infty) = C(B). \end{aligned}$$

Now, proceeding as in the proof of Lemma 1, we readily obtain

$$\begin{aligned} &E\left(\prod_{i=1}^{n} s_i^{J_i(B)} \mid A_0(0), A_0(1), \cdots\right) \\ &\qquad = \prod_{x=0}^{\infty} \left[1 + \sum_{i=1}^{n} (s_i - 1) P_x(T_B = i)\right]^{A_0(x)}, \end{aligned}$$

and thus

$$E\left(\prod_{i=1}^{n} s_i^{J_i(B)}\right) = \exp \sum_{i=1}^{n} (s_i - 1) \sum_x \mu(x) P_x(T_B = i) = \prod_{i=1}^{n} e^{(s_i - 1) C(B)}.$$

REFERENCES

[1] DERMAN, C. (1955). Some contributions to the theory of denumerable Markov chains. *Trans. Amer. Math. Soc.* **79** 541–555.

[2] GNEDENKO, B. (1962). *Theory of Probability*. Chelsea, New York.

[3] KEMENY, J. G. and SNELL, J. L. (1961). Potentials for denumerable Markov chains. *J. Math. Anal. Appl.* **3** 196–260.

[4] PORT, S. C. (1965). Limit theorems involving capacities. To appear in *J. Math. Mech.*