CONTRIBUTIONS TO SAMPLE SPACINGS THEORY, II: TESTS OF THE PARAMETRIC GOODNESS OF FIT AND TWO-SAMPLE PROBLEMS[1]

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0. Summary. This paper is concerned with tests based on sample spacings for the two sample problem with nuisance location and scale parameters—Part A—Sections 1 through 4, and for the goodness of fit problem with nuisance scale and location parameters, Part B, Sections 5 through 7. Estimation of the nuisance scale parameter is also considered. The notation and methods of proof, as well as the basic results of the preceding paper, Blumenthal (1966) are used without restatement. That paper is referred to as Spacings I, throughout the sequel.

A. TWO SAMPLE PROBLEM

1. Introduction. The problem of testing whether two random samples have parent distributions which belong to the same family indexed by a location and a scale parameter will be considered here. Specifically let the samples be X_1, $X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ with parent distribution functions $F(x)$ and $G(x)$ respectively. The hypothesis to be tested is

$$H_0^1: F(x) = G((x - \mu)/\sigma) \tag{1.1}$$

where μ and $\sigma(>0)$ are two unspecified nuisance parameters. This is the "parametric" two sample hypothesis, and by setting $\mu = 0$, $\sigma = 1$, we obtain the usual two sample hypothesis,

$$H_0 : F(x) = G(x). \tag{1.2}$$

We shall consider here a family of possible tests of the parametric hypothesis H_0^1 all based on the sample successive differences (sample spacings) of the two random samples. The large sample properties of all the tests will be described in terms of limiting distributions for the test statistics and one particular test will be chosen for detailed analysis.

The test statistics are based on the quantities $S_n(r)$ whose construction is given in Section 1 of Spacings I. Using these $S_n(r)$ the symmetric statistic

$$T_n(r) = S_n(r)S_n(-r), \qquad 0 < r \leqq 1, \tag{1.3}$$

will be studied for testing the hypothesis H_0^1. The consistency of tests based on this statistic will be demonstrated and limiting distributions will be described.

Received 14 June 1965; revised 21 December 1965.

[1] Supported in part by the National Science Foundation under Contract No. NSF-GP-4933 at Rutgers—The State University and under Contract No. NSF-G-19126 at the University of Minnesota.

These distributions are found using the theory developed in Spacings I for the $S_n(r)$.

We find that when $0 < r < \frac{1}{2}$, the limiting distributions of $T_n(r)$ when H_0^1 is true depend on the function $G(x)$ so that tests of H_0^1 cannot be based on $T_n(r)$ for these values of r. When $\frac{1}{2} \leqq r \leqq 1$, however, the limiting distributions are independent of $G(x)$ and the statistic $T_n(r)$ can then be the basis of an approximately size α test for H_0^1. The distribution-freeness of $T_n(r)$ is only asymptotic. For finite sample sizes, the function $G(x)$ will influence the distribution of $T_n(r)$ even when H_0^1 is true. This phenomenon will be encountered with most test statistics for H_0^1 due to the nuisance parameters which must be estimated or eliminated in some manner, and to do this it is necessary to sacrifice the property of invariance under probability transformations enjoyed by most "non-parametric" test statistics.

We shall single out for further discussion the test based on $T_n(\frac{1}{2})$ since this statistic enjoys the desirable property of limiting normality. The large sample power of this test is shown to be less than that of the Kolmogorov-Smirnov test. For equivalent power the present test needs $(n \log n)$ observations where the Kolmogorov-Smirnov test needs only n. For comparison, it should be noted that the popular χ^2 test of fit needs $n^{5/4}$ observations where the Kolmogorov test needs only n. Thus the loss of power is not too great.

In Section 4, we discuss the use of the $S_n(r)$ to estimate σ when H_0^1 can be assumed true, and also the use of the $S_n(r)$ to test the variant of H_0^1 given by.

$$H_0^2: F(x) = G(x - \mu),$$
$$H_1^2: F(x) = G((x - \mu)/\sigma), \qquad (\sigma > 0, \sigma \neq 1)$$

where μ (real) is not specified.

2. Distribution and convergence properties. Before stating the convergence theorems, we give a result of Weiss (1957) which will be needed here and in Section 3.

LEMMA 2.1 (Weiss). *Let $F(x)$ and $G(x)$ be two distribution functions and u, v $(0 \leqq u < v \leqq 1)$ be two given numbers, and assume $F^{-1}(u)$, $F^{-1}(v)$, $G^{-1}(u)$, $G^{-1}(v)$ are all uniquely determined. Let $F(x)$ have a derivative $f(x)$ on $[F^{-1}(u), F^{-1}(v)]$ and $G(x)$ have a derivative $g(x)$ on $[G^{-1}(u), G^{-1}(v)]$. If $F(x) = G((x - \mu)/\sigma)$ for all x in $[F^{-1}(u), F^{-1}(v)]$ for some constants μ, $\sigma(\sigma > 0)$, then $f(F^{-1}(y)) = (1/\sigma)g(G^{-1}(y))$ for all y in $[u, v]$. If in addition, $f(x) > 0$ for x in $(F^{-1}(u), F^{-1}(v))$, the converse is true.*

The proof is given by Weiss (1957). Blumenthal (1962) gives an example showing the necessity of the extra condition for the converse.

The basic result of this section establishes the relation between the $T_n(r)$ and the U's and V's defined in Spacings I.

THEOREM 2.1 *Let $F(x) = G((x - \mu)/\sigma)$ and let $F(x)$ satisfy the conditions of Theorem 3.0, Spacings I. Then as n increases,*

(2.1a) $\log n[(1/n \log n)^2 T_n(1) - 1]$

$$\to_P (1/n) \sum ((U_i/V_i) + (V_i/U_i) - 2 \log n), \qquad r = 1;$$

(2.1b) $$(n^{(1-r)} \pi r/\sin \pi r)[(\sin \pi r/n\pi r)^2 T_n(r) - 1]$$
$$\to_P (1/n^r) \sum ((U_i/V_i)^r + (V_i/U_i)^r - 2(\pi r/\sin \pi r)), \quad \tfrac{1}{2} < r < 1;$$

(2.1c) $$(\pi n^{\frac{1}{2}}/2(\log n)^{\frac{1}{2}})[(2/\pi n)^2 T_n(\tfrac{1}{2}) - 1]$$
$$\to_P (1/n \log n)^{\frac{1}{2}} \sum ((U_i/V_i)^{\frac{1}{2}} + (V_i/U_i)^{\frac{1}{2}} - \pi), \qquad r = \tfrac{1}{2};$$

$$(\pi r/\sin \pi r) n^{\frac{1}{2}} [(\sin \pi r/n\pi r)^2 T_n(r) - 1]$$
(2.1d) $$\to_P (1/n)^{\frac{1}{2}} \sum [(U_i/V_i)^r + (V_i/U_i)^r - (2\pi r/\sin \pi r)$$
$$+ [\pi r^2 (U_i - V_i)/(\sin \pi r)(n - i + 1)] \textstyle\sum_{j=i+1}^{n} (h'(j)/h^2(j))]; \quad 0 < r < \tfrac{1}{2}.$$

PROOF. Due to the similarity of the results, only (2.1a) will be given in detail.

$$\log n[(1/n \log n)^2 T_n(1) - 1] = (\log^{-1} n)[(1/n\sigma) S_n(1)(\sigma/n) S_n(-1) - \log^2 n]$$
(2.2) $$= [(1/n\sigma) S_n(1) - \log n]$$
$$+ [(1/\sigma n \log n) S_n(1)][(\sigma/n) S_n(-1) - \log n].$$

Using (I-(3.4a)) and Lemma 2.1, we have

$$(1/n\sigma) S_n(1) \to_P \sum (U_i/V_i),$$
$$(\sigma/n) S_n(-1) \to_P \sum (V_i/U_i)$$

and using Theorem 4.1 of Spacings I, along with Lemma 2.1, $(1/\sigma n \log n) S_n(1) \to_P 1$. This completes the proof.

The right sides of (2.1) are sums of independent random variables whose distributions can be handled as in Theorem 3.1 of Spacings I. It is only necessary to observe that

$$P((U_i/V_i)^r + (V_i/U_i)^r < x)$$
(2.3) $$= 0, \qquad \text{for } x < 2,$$
$$= \{[x + (x^2 - 4)^{\frac{1}{2}}]^{1/r} - 2^{1/r}\}/\{[x + (x^2 - 4)^{\frac{1}{2}}]^{1/r} + 2^{1/r}\} \qquad \text{for } x > 2.$$

We summarize these results as:

THEOREM 2.2. *Let the assumptions of Theorem 2.1 be satisfied. Then as n increases,*

(2.4a) $$\mathcal{L}\{\log n[(1/n \log n)^2 T_n(1) - 1]\} \to S(1, 1, -2, \pi/2), \qquad r = 1;$$

(2.4b) $$\mathcal{L}\{(n^{(1-r)} \pi r/\sin \pi r)[(\sin \pi r/n\pi r)^2 T_n(r) - 1]\}$$
$$\to S[1/r, -1, 0, (-(2/r) M(1/r) \cos (\pi/2r))], \qquad \tfrac{1}{2} < r < 1;$$

(2.4c) $$\mathcal{L}\{(\pi n^{\frac{1}{2}}/2(2 \log n)^{\frac{1}{2}})[(2/\pi n)^2 T_n(\tfrac{1}{2}) - 1]\} \to \Phi(x), \qquad r = \tfrac{1}{2};$$

(2.4d) $$\mathcal{L}\{(\pi r/\sin \pi r)(n/\sigma_2^2(n, r))^{\frac{1}{2}}[(\sin \pi r/n\pi r)^2 T_n(r) - 1]\}$$
$$\to \Phi(x), \qquad 0 < r < \tfrac{1}{2}$$

where

$$\sigma_2^2(n, r) = (4\pi r/\sin 2\pi r - 4\pi^2 r^2/\sin^2 \pi r + 2) + (2\pi^2 r^4/\sin^2 \pi r)(1/n)\sum \{[1/(n - i + 1)]\sum_{j=i+1}^{n} (h'(j)/h^2(j))\}^2),$$

$\Phi(x)$ *is the standard normal distribution function, and* $S(a, b, c, d)$ *represents the stable distribution function whose characteristic function is specified by the four given parameters. The exact nature of this relation is spelled out in Spacings I.*

For the present purpose, it is sufficient to observe that that first index (a) gives the highest order moments which exist for the distribution. Thus, for $T_n(r)$, no moments of order greater than $(1/r)$ exist for the limiting distributions with $r > \frac{1}{2}$. Further implications of Theorem 2.2 will be discussed in Section 3.

It is also possible to use Theorems 3.0 and 4.1 of Spacings I, and the methods of Theorem 3.2 of that paper to establish a general version of Theorem 2.2. We state the result only for $r = \frac{1}{2}$ for reasons which will be explained in Section 3.

THEOREM 2.3. *Let the assumptions of Theorem* 3.0, *Spacings I be satisfied. Then as n increases,*

$$(2.5) \quad \mathcal{L}\{(\pi n^{\frac{1}{2}}/2\sigma_1(\log n)^{\frac{1}{2}})[(2/\pi n)^2 T_n(\tfrac{1}{2}) - (\textstyle\int_0^1 (g(G^{-1}(x))/f(F^{-1}(x)))^{\frac{1}{2}}\, dx) \cdot (\textstyle\int_0^1 (f(F^{-1}(x))/g(G^{-1}(x)))^{\frac{1}{2}}\, dx)]\} \to \Phi(x),$$

where

$$\sigma_1^2 = (\textstyle\int_0^1 (f(F^{-1}(x))/g(G^{-1}(x)))^{\frac{1}{2}}\, dx)^2(\textstyle\int_0^1 g(G^{-1}(x))/f(F^{-1}(x))\, dx) + (\textstyle\int_0^1 (g(G^{-1}(x))/f(F^{-1}(x)))^{\frac{1}{2}}\, dx)^2(\textstyle\int_0^1 f(F^{-1}(x))/g(G^{-1}(x))\, dx).$$

Finally, we note the stochastic convergence properties of $T_n(r)$.

THEOREM 2.4. *Let* $F(x)$ *and* $G(x)$ *satisfy the conditions of Theorem* 4.0, *Spacings I. Then as n increases,*

$$(2.6a) \quad (1/n \log n)^2 T_n(1) \to_P (\textstyle\int_0^1 (g(G^{-1}(x))/f(F^{-1}(x)))\, dx) \cdot (\textstyle\int_0^1 (f(F^{-1}(x))/g(G^{-1}(x)))\, dx), \quad r = 1;$$

$$(2.6b) \quad (\sin \pi r/n\pi r)^2 T_n(r) \to_P (\textstyle\int_0^1 (g(G^{-1}(x))/f(F^{-1}(x)))^r\, dx) \cdot (\textstyle\int_0^1 (f(F^{-1}(x))/g(G^{-1}(x)))^r\, dx), \quad 0 < r < 1.$$

3. Tests based on $T_n(r)$. Examination of Theorem 2.2 shows that as we decrease r, we increase the concentration of the limiting distribution as measured by the highest order moments which exist. For all $r \leqq \frac{1}{2}$, the limiting distribution is normal with moments of all orders existing. If we wish to test the hypothesis H_0^1 (see Equation (1.1)) using $T_n(r)$, we could presumably obtain (asymptotically) equivalent tests for all $r \leqq \frac{1}{2}$, except for one thing. A close look at $\sigma_2^2(n, r)$ in (2.4d) reveals that it depends on the distribution function $F(x)$ through $(h'(x)/h^2(x))$. Thus to find a critical region of size approximately α for large n, one must know $F(x)$ which we assume unknown. Therefore, no reasonable test can be based on $T_n(r)$ with $r < \frac{1}{2}$.

The choice then is among the $T_n(r)$, $\frac{1}{2} \leqq r \leqq 1$, since all are distribution free in the limit. Here the choice would seem to be with $T_n(\frac{1}{2})$ not only because of the fact that the normal distribution is more accessible than the other stable distributions, but also because power presumably is higher for tests based on statistics whose distributions have finite variances as opposed to tests based on statistics whose limiting distributions have infinite variance, especially if the size α is small.

Thus, $T_n(\frac{1}{2})$ would seem the most reasonable choice of test statistic, being asymptotically distribution free and normal. It might be noted in passing that the second moment of $T_n(\frac{1}{2})$ does not exist for any finite sample size n.

Before going further into the test based on $T_n(\frac{1}{2})$, we wish to point out that any of the statistics $T_n(r)$ ($\frac{1}{2} \leqq r \leqq 1$) will provide a consistent test of H_0^1 against essentially all alternatives. Using Lemma 2.1 and Theorem 2.4, we see that when H_0^1 is true,

$$(3.1) \qquad (1/n \log n)^2 T_n(1) \to_P 1, \qquad r = 1;$$

$$(\sin \pi r/n\pi r)^2 T_n(r) \to_P 1, \qquad \tfrac{1}{2} \leqq r < 1.$$

Also, it is easily seen that if a non-negative function $z(x)$ is not constant on the interval $[u, v]$, then

$$(3.2) \qquad 1/(v - u)^2 \int_u^v z(x)\, dx \int_u^v (1/z(x))\, dx > 1.$$

By the converse to Lemma 2.1 when H_0^1 is not true, $(f(F^{-1}(x))/g(G^{-1}(x)))$ will not be constant on $[0, 1]$. This establishes that any test which rejects H_0^1 when $T_n(r)$ is "too large" will be consistent against all alternatives satisfying the restrictions mentioned in the lemma.

Therefore, a consistent test of approximately size α for H_0^1 can be obtained by rejecting H_0^1 whenever $T_n(\frac{1}{2})$ exceeds $(n\pi/2)^2[(2(2 \log n)^{\frac{1}{2}} K_\alpha/\pi n^{\frac{1}{2}}) + 1]$ where $1 - \Phi(K_\alpha) = \alpha$.

To get an idea of the approximate power of this test, we need Theorem 2.3. Since $r = \frac{1}{2}$ is the only case for which we need the power, the reason for limiting of 2.3 to this case is apparent.

Denoting for convenience,

$$(3.3) \qquad I(F, G, r) = \int_0^1 (g(G^{-1}(x))/f(F^{-1}(x)))^r\, dx, \qquad 0 < |r| \leqq 1,$$

we can express the large sample power of a test based on $T_n(\frac{1}{2})$ as

$$(3.4) \qquad \Phi\{(I(F, G, \tfrac{1}{2}) I(F, G, -\tfrac{1}{2}) - 1)(\pi n^{\frac{1}{2}}/2\sigma_1(\log n)^{\frac{1}{2}}) - (2^{\frac{1}{2}} K_\alpha/\sigma_1)\}.$$

Using expression (3.4), we can investigate limiting power against sequences (F_n, G_n) such that $\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} G_n((x - \mu)/\sigma)$. It is common to choose a parametric family for which to study limiting power but the usual normal distribution cannot be used here since its only parameters are scale and location and the hypothesis and test statistic are invariant under changes of these parameters. We must go then to a three parameter family to study large sample power. A natural family is the Weibull family

(3.5) $$f(x) = (\beta(x-\mu)^{\beta-1}/\alpha)\exp[-((x-\mu)^{\beta}/\alpha)], \qquad x \geqq \mu,$$

where we let $\beta = 1$ for $f_n(x)$, and $\beta = \beta(n)$ for $g_n(x)$, and $\lim_{n\to\infty}\beta(n) = 1$.

This family is "natural" since one hazard rate $h_n(x) = \alpha^{-1}$, and the other hazard rate $q_n(x) = \alpha^{-1}\beta(n)(x-\mu)^{\beta(n)-1}$ which approaches α^{-1}.

In this situation, we need merely study the behavior of $I(F, G, \frac{1}{2})$, $I(F, G, -\frac{1}{2})$, and σ_1 (see (2.5) and (3.3)) as functions of $\beta(n)$ to determine for which sequences $\beta(n)$ the expression (3.4) has a non-trivial limit. It is straightforward to verify that such a limit will be obtained for

(3.6) $$\beta(n) = 1 + C(\log n/n)^{\frac{1}{4}}.$$

Using the facts that $I(F, G, r) = \int_{-\infty}^{\infty}[g(u)/f(F^{-1}(G(u)))]^r g(u)\,du$ and for the exponential $(f(x) = e^{-x})$, $f(F^{-1}(x)) = (1-x)$, it is easily checked that for this situation,

$$I(F, G, \tfrac{1}{2})I(F, G, -\tfrac{1}{2}) = \lambda\pi/\sin\lambda\pi$$

where $\lambda = (\frac{1}{2})(1-\beta(n)^{-1})$, and $\sigma_1^2 = (\lambda\Gamma(\lambda))^2\Gamma(1-2\lambda) + (\Gamma(1-\lambda))^2\cdot\Gamma(1+2\lambda)$, so that as n increases

$$(\pi/2\sigma_1)(n/\log n)^{\frac{1}{2}}(I(F, G, \tfrac{1}{2})I(F, G, -\tfrac{1}{2}) - 1) \to [\pi^3\lambda^2/12(2)^{\frac{1}{2}}](n/\log n)^{\frac{1}{2}}$$
$$\to C^2\pi^3/48(2)^{\frac{1}{2}},$$

and $\sigma_1^2 \to 2$.

Thus, the limiting power of the $T_n(\frac{1}{2})$ test is

(3.7) $$\Phi((C^2\pi^3/48(2)^{\frac{1}{2}}) - K_\alpha).$$

In order to have a basis for judging whether the proposed test is terribly inefficient or not we can make the following observations. It is known that the Kolmogorov-Smirnov test for the present problem will have a non-trivial limiting power if the distances $\sup_x |F_n(x) - G_n(x)|$ behave like $1/n^{\frac{1}{2}}$ (see for instance Kac, Kiefer, and Wolfowitz (1955)). A computation shows that it is then necessary to have

(3.8) $$\beta(n) = 1 + C/n^{\frac{1}{4}}.$$

The Pitman efficiency of the $T_n(\frac{1}{2})$ test relative to the Kolmogorov-Smirnov test would then be zero, but on the other hand, the important comparison is that for equivalent power, the $T_n(\frac{1}{2})$ test requires roughly $n \log n$ observations where the Kolmogorov-Smirnov test requires n observations. This may not be too high a price to pay for computational ease in many instances. It should be recalled that the popular χ^2 test of goodness of fit is even less efficient requiring $n^{5/4}$ observations for equivalent power to that achieved with n observations by the Kolmogorov test. Thus for the present problem, the proposed test is relatively more efficient than is the widely used χ^2 test for the goodness of fit problem. The proposed test has the added advantage of not requiring auxilliary estimates of the unknown scale and location parameters.

Another observation we can make is that if one wants to test whether $F(x) = G((x - \mu)/\sigma)$ except in the "tails" where by tails we mean $F(x) < u$, and $F(x) > 1 - v$ $(0 \leqq u < v \leqq 1)$. In this case, the truncated statistics $S_n(r, u, v)$ which are described in the next section can be used to define a truncated $T_n(r, u, v)$ in the obvious way and all of the previous results will apply. An advantage will be that no restriction need be made on the behavior of $F(x)$ in these "tails" in order for the theorems to apply. This is discussed more in Section 4.

4. Estimates and tests based on $S_n(r)$. Under the assumption that $H_0^1: F(x) = G((x - u)/\sigma)$ is true, we can now use the $S_n(r)$ quantities for estimating the "relative scale parameter" σ. We can in fact consider as possible estimates any of the quantities

$$(4.1) \qquad \begin{aligned} s_n(1) &= (1/n \log n)S_n(1), \\ s_n(r) &= [(\sin \pi r/\pi rn)S_n(r)]^{(1/r)}, \qquad 0 < r < 1. \end{aligned}$$

By virtue of Lemma 2.1 and Theorem 4.1 of Spacings I (setting $F(x) = G(x)$ in the theorem), any one of these estimates $s_n(r)$ $(0 < r \leqq 1)$ is consistent, provided that the distribution $G(x)$ is sufficiently regular in the tails. Not all of the $s_n(r)$ seem reasonable as estimators in view of Theorem 3.1 of Spacings I which shows that the limiting distributions will depend on the unknown $G(x)$ when $0 < r < \frac{1}{2}$. Thus for estimates whose limiting properties are independent of $G(x)$, we should restrict attention to the $s_n(r)$ for $\frac{1}{2} \leqq r \leqq 1$. Since the limiting distributions are known, any of these $s_n(r)$ could be used to derive confidence intervals for σ. In practice since the stable distributions are not tabulated, $s_n(\frac{1}{2})$ has a distinct advantage since the limiting distribution of $(s_n(\frac{1}{2}))^{\frac{1}{2}}$ is normal. In particular, using Theorem 3.1 of Spacings I along with Lemma 2.1 we have

$$(4.2) \qquad \mathfrak{L}\{(\pi n^{\frac{1}{2}}/(2\sigma \log n)^{\frac{1}{2}})[(s_n(\tfrac{1}{2}))^{\frac{1}{2}} - \sigma^{\frac{1}{2}}]\} \to \Phi(x).$$

From (4.2) it is easy to verify that an approximate $100(1 - \alpha)\%$ confidence interval for σ is:

$$(4.3) \qquad [\pi^2 a_n^2 s_n(\tfrac{1}{2})/(\pi a_n + 2^{\frac{1}{2}}k_{(\alpha/2)})^2] < \sigma < [\pi^2 a_n^2 s_n(\tfrac{1}{2})/(\pi a_n - 2^{\frac{1}{2}}K_{(\alpha/2)})^2]$$

where $a_n = (n/\log n)^{\frac{1}{2}}$ and $1 - \Phi(K_\beta) = \beta$.

It is apparent from the preceding paragraphs that under the assumption that $F(x) = G((x - u)/\sigma)$, the quantities $s_n(r)$ $(\frac{1}{2} \leqq r \leqq 1)$ can be used to produce approximately size α tests of the hypothesis that $\sigma = 1$, i.e. tests of

$$\begin{aligned} &H_0^2: F(x) = G(x - u) \quad \text{versus} \\ &H_1^2: F(x) = G((x - u)/\sigma), \qquad \sigma \neq 1 \ (\sigma > 0). \end{aligned}$$

Again being specific, an approximate size α test for this problem is given by the rule: "accept H_0 if and only if the confidence interval (4.3) includes unity."

In the discussion of this section, we have been limited by the assumption that $G(x)$ satisfies the tail limitations imposed by Theorem 3.0. If this assumption does not seem reasonable, truncated statistics can be used for estimating σ or for testing H_0^2 against H_1^2.

The truncated statistics are given as follows:

$$S_n(r, u, v) = \sum_{i=[nu]}^{[nv]} (DX_i/DY_i)^r, \qquad 0 < |r| \leqq 1, \tag{4.4}$$

where $0 < u < v < 1$ are given and $[x]$ is the greatest integer not exceeding x. By using $S_n(r, u, v)$, we need only put restrictions on the behavior of $G(x)$ in the interval $G^{-1}(u) < x < G^{-1}(v)$, and for conveniently chosen u, v regularity of $G(x)$ in this interval may be much more plausible than regularity on the entire real line. In the replacement of $S_n(r)$ by $S_n(r, u, v)$ one need only replace n by $n(v - u)$ in the normalizing factors in (4.1), (4.2), and (4.3), except that $\log n$ should be left unchanged. The stochastic convergence and limiting distribution arguments are as given in Spacings I for the untruncated statistics.

B. GOODNESS OF FIT PROBLEM

5. Introduction. Given a sample $X_1, X_2, \cdots, X_n$ of independent random variables having a common unknown distribution $F(x)$, we want to test the hypothesis

$$H_0 : F(x) = G((x - u)/\sigma), \tag{5.1}$$

where $G(x)$ is a specified distribution function with μ (real) and σ (positive) being unspecified parameters (nuisance parameters).

We are then testing whether $F(x)$ belongs to the class of distributions generated by varying μ and σ. A well known example is the test of normality obtained by taking $G(x)$ as the standard normal distribution. Another example of this hypothesis is the test of exponentiality encountered in life testing situations, where $G(x)$ is taken as the standard exponential distribution.

Use of sample spacings to test H_0 has been considered previously by Weiss (1957) who proposed a test which we shall describe below. Minimum distance methods for testing this version of H_0 were examined by Kac, Kiefer, and Wolfowitz (1955).

In further papers, Weiss (1961), (1963) considers a sample spacings estimate of σ in the situation where H_0 is true.

The parametric goodness of fit problem seems to have received more attention in the literature than the parametric two sample problem which we have considered above.

We shall introduce below a family of statistics all based on sample spacings. In the following section, we shall study the asymptotic distributions and stochastic convergence properties of these statistics, and in the last section, we shall apply these results to study tests of H_0 based on these statistics. In particular, it will be shown that a subclass of these statistics give tests of H_0 whose large sample power in discriminating exponential from Weibull distributions is comparable to that of the Kolmogorov test. Among this subclass, a best test is then found.

Using the notation previously adopted for the two-sample tests, we define a family of statistics which are analogous to the $S_n(r)$, namely

$$W_n(r) = \sum_{i=1}^{n-1} (ng(i)\, DX_i)^r, \qquad r \geqq -1, r \neq 0. \tag{5.2}$$

In terms of $W_n(r)$, the test proposed by Weiss (1957) is based on

$$Z_n = nW_n(2)/(W_n(1))^2, \tag{5.3}$$

with the difference that Weiss used truncated versions of $W_n(r)$ in order to avoid putting regularity restrictions on the tail of $G(x)$.

From the distribution results of Section 6, it can be seen that any $W_n(r)$ can be used to generate an estimate of σ if H_0 is true. Also, the $W_n(r)$ can be used to test

$$\begin{aligned} &H_0^1: F(x) = G(x - \mu) \quad \text{versus} \\ &H_1^1: F(x) = G((x - \mu)/\sigma), \qquad \sigma > 0,\ \sigma \neq 1,\ \mu \text{ (real) not specified.} \end{aligned} \tag{5.4}$$

The details of constructing such estimates and tests are similar to those of Section 4 and will not be carried out here. The test statistics for H_0 which we shall study are defined as

$$P_n(r) = W_n(r)W_n(-r), \qquad 0 < r \leqq 1. \tag{5.5}$$

In the following section on distribution theory, we shall make use of the methods used previously for the two-sample statistics $S_n(r)$ and $T_n(r)$. Although great similarities exist and are exploited, there are certain differences due to the symmetry of the distribution of $S_n(r)$ in r and $(-r)$ which does not exist for the distribution of $W_n(r)$.

6. Distribution and convergence results. First we relate the $W_n(r)$ to the analogous statistics based on the U_i in place of DX_i.

THEOREM 6.0. *Let $F(x)$ and $G(x)$ satisfy condition* (3.5) *of Theorem* 3.0, *Spacings I, except that it is not necessary to interchange $q(x)$ with $h(x)$, and/or r with $(-r)$. For this theorem, r ranges over $[-1, \infty)(r \neq 0)$, instead of $(0, 1]$. The definition of $a(\delta, r)$ becomes,*

$$a(\delta, r) = \max\{1, (-\log\delta)^{-2r-1}\} \tag{6.1a}$$

and that of $b(\delta, r)$ becomes

$$\begin{aligned} b(\delta, r) &= \delta^2(-\log\delta)^{\frac{3}{2}}, && r = -1, \\ &= \delta^{(-2r)}, && -1 < r < -\tfrac{1}{2}, \\ &= \delta(-\log\delta)^{-\frac{1}{2}}, && r = -\tfrac{1}{2}, \\ &= \delta, && -\tfrac{1}{2} < r, r \neq 0. \end{aligned} \tag{6.1b}$$

Then as n increases,

$$(1/n^r)W_n(r) \to_P (1/n^r)\sum (g(i)U_i/f(i))^r, \qquad -1 \leqq r < -\tfrac{1}{2}; \tag{6.2a}$$

(6.2b) $(1/n \ \log n)^{\frac{1}{2}} W_n(-\frac{1}{2}) \to_P (1/n \ \log n)^{\frac{1}{2}} \sum (g(i)U_i/f(i))^{-\frac{1}{2}}, \quad r = -\frac{1}{2};$

(6.2c) $(1/n)^{\frac{1}{2}} W_n(r) \to_P (1/n)^{\frac{1}{2}} \sum (g(i)U_i/f(i))^r$

$\cdot[-r(U_i - 1)/(n - i + 1)] \sum_{j=i}^{n-1} q^r(j)h'(j)/h^{2+r}(j), \quad r > -\frac{1}{2}, r \neq 0.$

Proof. From the expansions (2.6) and (2.7) of Spacings I, we obtain

$$\begin{aligned}\sum (ng(i)\, DX_i)^r = \sum \{&(g(i)/h(i))^r U_i^r - rU_i^r(H(\bar{X}_i) - H(i)) \\ &\cdot (q^r(i)h'(i)/h^{2+r}(i)) - rU_i^r(H(\bar{X}_i) - H(i))^2 q^r(i) \\ &\cdot [(h(\tilde{X}_i)h''(\tilde{X}_i) - (2 + r)(h'(\tilde{X}_i))^2)/h^{4+r}(i)]\}.\end{aligned} \tag{6.3}$$

The study of (6.3) is then done exactly as in Theorem 3.0 and Lemma 3.1 of Spacings I. Since $E(U_i^r)$ does not exist for $r \leqq -1$, U_i' variables are needed here also and are defined by

$$\begin{aligned} U_i' &= U_i & &\text{if } U_i > (n \log n)^{-1} \\ &= (n \log n)^{-1} & &\text{if } U_i \leqq (n \log n)^{-1}. \end{aligned} \tag{6.4}$$

In arriving at (6.1a, b) it is merely necessary to identify the normalizing constants and moment properties for $-1 \leqq r \leqq -\frac{1}{2}$ here with $\frac{1}{2} \leqq r \leqq 1$ previously, and $r > -\frac{1}{2}$, $r \neq 0$, here with $0 < r < \frac{1}{2}$ previously. This completes our outline of the proof.

Theorem 6.1. *If $F(x) = G((x - \mu)/\sigma)$, and $F(x)$ satisfies the conditions of Theorem 6.0, then as n increases:*

(6.5a) $\mathcal{L}\{\log n[(\sigma/n \ \log n)W_n(-1) - 1]\} \to S(1, 1, -\gamma, \pi/2), \quad r = -1;$

(6.5b) $\mathcal{L}\{(1/n^r)[(\sigma^r)W_n(r) - n\Gamma(1 + r)]\}$

$\to S[(-1/r, -1, 0, (1/r)M(-1/r) \cos(-\pi/2r)], \quad -1 < r < -\frac{1}{2};$

(6.5c) $\mathcal{L}\{(1/n \log n)^{\frac{1}{2}}[(\sigma^{\frac{1}{2}})W_n(-\frac{1}{2}) - n\pi^{\frac{1}{2}}]\} \to \Phi(x), \quad r = -\frac{1}{2};$

(6.5d) $\mathcal{L}\{(1/n\sigma_0^2(r))^{\frac{1}{2}}[(\sigma^{-r})W_n(r) - n\Gamma(1 + r)]\} \to \Phi(x), \quad -\frac{1}{2} < r, r \neq 0,$

where

$$\gamma = 0.57721 \cdots \ (\textit{Euler's constant}):$$

$$M(\mu) = \int_0^\infty ((e^y - 1 + y)/y^{1+\mu})\, dy;$$

$$\begin{aligned}\sigma_0^2(r) = \Gamma(2r + 1) &- \Gamma^2(r + 1) - 2r[\Gamma(r + 2) - \Gamma(r + 1)] \\ &\cdot \int_0^1 (1 - x)^{-1}[\int_0^1 (h'(F^{-1}(y))/h^2(F^{-1}(y)))\, dy]\, dx \\ &+ r^2 \int_0^1 (1 - x)^{-2}[\int_x^1 (h'(F^{-1}(y))/h^2(F^{-1}(y)))\, dy]^2\, dx\end{aligned}$$

(*taking $\mu = 0$, $\sigma = 1$ in computing $h(x)$*).

Proof. The multiplying factor of σ is a consequence of Lemma 2.1. The expressions (6.5) then follow from the same arguments used in Theorem 3.1 of Spacings I. This completes the proof.

For $r = 1$, with some manipulation, (6.5d) above agrees with Weiss's (1963) result. An important point to note is that $\sigma_0^2(r)$ depends on $G(x)$. If $G(x)$ is unknown as in the two sample problem, this lack of "distribution freeness" is a series drawback. In the present problem, as we shall see in the next section it does not cause any trouble.

Stochastic convergence of the $W_n(r)$ is treated next.

THEOREM 6.2. *Let $F(x)$ and $G(x)$ satisfy the conditions of Theorem* 4.0, *Spacings I, with the same exceptions as in Theorem* 6.0, *and with the redefinition*:

$$(6.6)\qquad \begin{aligned} a'(\delta, r) &= (-\log \delta)^{-1}, && r = -1,\\ &= \delta^{4(1+r)}(-\log \delta)^{-(2r+1)}, && -1 < r < -\tfrac{1}{2},\\ &= -\delta^2 \log \delta, && r = -\tfrac{1}{2},\\ &= \delta^2, && -\tfrac{1}{2} < r (r \neq 0). \end{aligned}$$

Then as n increases,

$$(6.7\text{a})\qquad (1/n \log n) W_n(-1) \to_P \int_0^1 (f(F^{-1}(x))/g(G^{-1}(x)))\, dx, \qquad r = -1;$$

$$(6.7\text{b})\qquad (1/n) W_n(r) \to_P \Gamma(1 + r) \int_0^1 (g(G^{-1}(x))/f(F^{-1}(x)))^r\, dx, \qquad r > -1.$$

The proof is omitted.

The next theorem deals with the products $P_n(r)$. A lack of symmetry will be noted when $r \geqq \frac{1}{2}$, since in these cases $W_n(r)$ is asymptotically normal but $W_n(-r)$ has a stable distribution which does not possess a second moment. As might be expected, the more diffuse variable determines the limiting distribution of the product.

THEOREM 6.3. *Let $F(x) = G((x - \mu)/\sigma)$ and satisfy the conditions of Theorem* 6.0. *Then as n increases*

$$(6.8\text{a})\qquad \mathcal{L}\{\log n[(1/n^2 \log n) P_n(1) - 1]\} \to S(1, 1, -\gamma, \pi/2), \qquad r = +1;$$

$$(6.8\text{b})\qquad \mathcal{L}\{n^{(1-r)}[(1/n^2 \Gamma(1 + r)) P_n(r) - \Gamma(1 - r)]\} \to S[1/r, -1, 0, (-1/r) M(1/r) \cos(\pi/2r)], \qquad \tfrac{1}{2} < r < 1;$$

$$(6.8\text{c})\qquad \mathcal{L}\{(n/\log n)^{\frac{1}{2}}[(2/n^2 \pi^{\frac{1}{2}}) P_n(\tfrac{1}{2}) - \pi^{\frac{1}{2}}]\} \to \Phi(x), \qquad r = \tfrac{1}{2};$$

$$(6.8\text{d})\qquad \mathcal{L}\{(n^{\frac{1}{2}}/\sigma_1)[(1/n^2 \Gamma(1 - r)\Gamma(1 + r)) P_n(r) - 1]\} \to \Phi(x), \qquad 0 < r < \tfrac{1}{2};$$

where γ, and $M(\mu)$ are as given in Theorem 6.1 *and*

$$\sigma_1^2 = (\Gamma(1-2r)/\Gamma^2(1-r)) + (\Gamma(1+2r)/\Gamma^2(1+r)) + (2/\Gamma(1-r)\Gamma(1+r)) - 4 + r^2[(\Gamma(1+r) - \Gamma(1-r))^2/(\Gamma(1+r)\Gamma(1-r))^2] \cdot \int_0^1 (1/(1-x)^2)[\int_x^1 (h'(F^{-1}(y))/h^2(F^{-1}(y)))\,dy]^2\,dx.$$

PROOF. We shall consider first (6.8a). Write the left side as

$$\log n((\sigma/n \log n)W_n(-1)(1/n\sigma)W_n(1) - 1) = \log n((1/n\sigma)W_n(1) - 1) + ((1/n\sigma)W_n(1)) \log n((\sigma/n\log n)W_n(-1) - 1).$$

By means of Theorem 6.0 it is easily verified that the first term, $\log n((1/n\sigma)W_n(1) - 1)$ approaches zero stochastically, while from Theorem 6.2 and Lemma 2.1 we see that $((1/n\sigma)S_n(1))$ approaches unity stochastically. It then follows from Slutsky's theorem (Cramér (1946), p. 254) that the expression in question has the same limiting distribution as $\log n((\sigma/n\log n)W_n(-1) - 1)$ which is given by Theorem 6.1.

We handle (6.8b) and (6.8c) in the same manner. For (6.8d), note that $n^{\frac{1}{2}}[(1/n^2\Gamma(1-r)\Gamma(1+r))P_n(r) - 1]$ has the same limiting distribution as $n^{-\frac{1}{2}}[(\sigma^r W_n(-r)/n\Gamma(1-r)) + (W_n(r)/n\sigma^r\Gamma(1+r)) - 2]$ which can then be handled by the approach of Theorems 6.0 and 6.1. This completes the proof.

In the next section, it will be necessary to consider distributions of the $P_n(r)$ for $0 < r \leqq \frac{1}{2}$ for general $F(x)$ and $G(x)$. The result needed is given next without proof.

THEOREM 6.4. *Let $F(x)$ and $G(x)$ satisfy the conditions of Theorem 6.0. Then as n increases,*

$$(6.9a) \quad \mathfrak{L}\{I(-\tfrac{1}{2})(n/I(-1)\log n)^{\frac{1}{2}}[(1/n^2 I(\tfrac{1}{2})I(-\tfrac{1}{2}))P_n(\tfrac{1}{2}) - 1]\} \to \Phi(x), \quad r = \tfrac{1}{2};$$

$$(6.9b) \quad \mathfrak{L}\{(n/\sigma_F^2)^{\frac{1}{2}}[(1/n^2 I(r)I(-r))P_n(r) - 1]\} \to \Phi(x), \quad 0 < r < \tfrac{1}{2};$$

where

$$(6.9c) \quad I(r) = \Gamma(1+r)\int_0^1 (g(G^{-1}(x))/f(F^{-1}(x)))^r\,dx, \qquad -1 < r;$$

$$(6.9d) \quad I(-1) = \int_0^1 (f(F^{-1}(x))/g(G^{-1}(x)))\,dx, \qquad r = -1;$$

and

$$\sigma_F^2 = (1/I^2(r))[I(2r) - (\Gamma^2(1+r)I(2r)/\Gamma(1+2r))] + (1/I^2(-r))[I(-2r) - (\Gamma^2(1-r)I(-2r)/\Gamma(1-2r))] + (2/I(r)I(-r))[1 - \Gamma(1+r)\Gamma(1-r)] + (r/I(r)I(-r))^2\int_0^1 (1-x)^{-2}$$

$$\cdot [I(r) \int_x^1 w^{-r}(y)v(y)\, dy - I(-r) \int_x^1 w^r(y)v(y)\, dy]^2\, dx$$
$$+ (2r/I(r)I(-r)) \int_0^1 [(\Gamma(2 + r) - \Gamma(1 + r))w^r(x)$$
$$+ (\Gamma(2 - r) - \Gamma(1 + r))w^{-r}(x)](1 - x)^{-1}$$
$$\cdot [I(r) \int_x^1 w^{-r}(y)v(y)\, dy - I(-r) \int_x^1 w^r(y)v(y)\, dy]\, dx,$$

defining

$$w(x) = g(G^{-1}(x))/f(F^{-1}(x));$$
$$v(x) = h'(F^{-1}(x))/h^2(F^{-1}(x)).$$

7. Tests based on $T_n(r)$. Using the stochastic convergence of $W_n(r)$ (Theorem 6.2), Lemma 2.1, and (3.2), it can be seen (as in Section 3) that any test which rejects H_0 whenever $P_n(r)$ is "too large" will be a consistent test. Further, using Theorem 6.3, an approximately size α test can be found based on any $P_n(r)$ statistic.

No test based on a $P_n(r)$ will be distribution free for finite sample sizes. Some, as Theorem 6.3 indicates, will be asymptotically distribution free allowing critical regions of approximately correct size to be established independently of the distribution $G(x)$. This property applies to the $P_n(r)$ for $\frac{1}{2} \leqq r \leqq 1$. If a $P_n(r)$ is used with $0 < r < \frac{1}{2}$, then the critical region will depend on the particular $G(x)$ which appears in H_0 through the variance σ_1^2. Having computed σ_1^2, the approximate critical region is easily found for these $P_n(r)$ using tables of the standard normal distribution. Thus a simple linear transformation makes these $P_n(r)$ essentially distribution free under H_0.

We now want to consider the choice of the value r. It will be noted that for $\frac{1}{2} < r \leqq 1$, the limiting stable distributions are not available either as formulas or in tables. This makes use of these $P_n(r)$ impossible at this time. Further, it is doubtful whether these statistics could provide very powerful tests of H_0 when compared to tests based on statistics whose limiting distributions have finite variances.

For r in the range $0 < r \leqq \frac{1}{2}$, any $P_n(r)$ can be used to define an approximately size α test of H_0. The critical regions are

(7.1a) $$P_n(\tfrac{1}{2}) > (\pi^{\frac{1}{2}}n^2/2)(\pi^{\frac{1}{2}} + K_\alpha(\log n)^{\frac{1}{2}}/n), \qquad r = \tfrac{1}{2};$$

(7.1b) $$P_n(r) > (n^2\Gamma(1 - r)\Gamma(1 + r))(1 + (\sigma_1 K_\alpha/n^{\frac{1}{2}})), \qquad 0 < r < \tfrac{1}{2};$$

(taking K_α so that $1 - \Phi(K_\alpha) = \alpha$).

To make a choice among these tests, we must use the limiting distributions in the general case. From Theorem 6.4 and Equations (7.1), the approximate large sample power of the tests based on $P_n(r)$ $(0 < r \leqq \frac{1}{2})$, is seen to be

(7.2a) $$\Phi\{(1/I(-1))^{\frac{1}{2}}[((2I(\tfrac{1}{2})I(-\tfrac{1}{2})/\pi) - 1)I(\tfrac{1}{2})(n/\log n)^{\frac{1}{2}} - K_\alpha]\}, \quad r = \tfrac{1}{2};$$

(7.2b) $$\Phi\{(\Gamma(1 + r)\Gamma(1 - r)\sigma_1/I(r)I(-r)\sigma_F)$$
$$\cdot [((I(r)I(-r)/\Gamma(1 + r)\Gamma(1 - r)) - 1)(1/\sigma_1)n^{\frac{1}{2}} - K_\alpha]\},$$
$$0 < r < \tfrac{1}{2}.$$

TABLE 1

r	.40	.30	.25	.20	.15	.13	.12	.11	.10
σ_1^2	.767	.143	.060	.022	.0065	.0036	.0025	.0019	.0014
(r^2/σ_1)	.183	.238	.256	.272	.278	.281	.287	.277	.270

The expressions (7.2) will give non-trivial values only if $F(x)$ and $G(x)$ are "close" in the sense that $(I(r)I(-r) - \Gamma(1 + r)\Gamma(1 - r))$ is of the order of $(\log n/n)^{\frac{1}{2}}$ (for $r = \frac{1}{2}$) or $(1/n)^{\frac{1}{2}}$ (for $0 < r < \frac{1}{2}$).

For the comparison of these tests we shall consider a special problem, namely that of testing for exponentiality with the Weibull distribution as the alternative. This gives a one dimensional set of alternatives indexed by the shape parameter β, just as in Section 3. Using (3.5), we see that H_0 corresponds to $\beta = 1$, and it is necessary that a sequence $\beta(n)$ approaching unity be found, so that (7.2) will have non-trivial limits.

Straightforward computations show that the sequences

$$\beta_n = 1 + C(\log n/n)^{\frac{1}{4}}, \qquad r = \tfrac{1}{2}; \tag{7.3a}$$

$$\beta_n = 1 + C(1/n)^{\frac{1}{4}}, \qquad 0 < r < \tfrac{1}{2}; \tag{7.3b}$$

lead to the following limiting powers:

$$\Phi((C^2\pi^{\frac{3}{2}}/24) - K_\alpha), \qquad r = \tfrac{1}{2}; \tag{7.4a}$$

$$\Phi((C^2\pi^2 r^2/6\sigma_1) - K_\alpha), \qquad 0 < r < \tfrac{1}{2}; \tag{7.4b}$$

where

$$\sigma_1^2 = (\Gamma(1 + 2r)/\Gamma^2(1 + r)) + (\Gamma(1 - 2r)/\Gamma^2(1 - r)) + (2/\Gamma(1 + r)\Gamma(1 - r)) - 4. \tag{7.5}$$

Note that (7.3a) is the same as (3.6), but (7.3b) drops the additional $(\log n)$ factor. In fact, in view of this fact and Equations (7.4), we see that the test based on $P_n(\frac{1}{2})$ has limiting efficiency of zero relative to any of the tests based on $P_n(r)$ $(0 < r < \frac{1}{2})$. Also, it is seen that the best choice of r (for this example) is that which maximizes (r^2/σ_1), noting from (7.5) that σ_1 is a function of r. Table 1 shows the behavior of (r^2/σ_1).

From Table 1, it is seen that the optimal r lies in the neighborhood of (but below) 0.12. A value of $\frac{1}{8}$ would not be too far from optimal and might prove convenient for computing. For $r = \frac{1}{8}$, (r^2/σ_1) is 0.284.

Note that for testing exponentiality, the statistic $W_n(r)$ takes on the simple form

$$W_n(r) = \textstyle\sum_{i=1}^{n-1} ((n - i)DX_i)^r. \tag{7.6}$$

In addition to the internal comparison above, we shall examine the limiting power of the Kolmogorov test adopted to this problem. Kac, Kiefer and Wolfowitz (1955) found that this test also is distribution free only in the limit. The limiting power of this test will be non trivial whenever

$$\sup_{-\infty<x<\infty} |F(x) - G(x)| = C/n^{\frac{1}{2}}. \quad (7.7)$$

In the present case, (7.7) is equivalent to (7.3b). Thus, the $P_n(r)$ $(0 < r < \frac{1}{2})$ tests will have the same order of limiting power as does the Kolmogorov test. It also follows that relative to the $P_n(r)$ tests, the χ^2 test (adopted to this problem) has zero efficiency. This is in contrast to the result of Section 3 and is seen to stem directly from being able to use the sequence (7.3b) instead of (7.3a). This is due to the fact that having the limiting variance depend on the common (or hypothesized) $F(x)$ is an inconvenience but not a block to using a test here, whereas in the previous problem it prevented the use of the $T_n(r)$ $(0 < r < \frac{1}{2})$ tests.

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