WEIGHING DESIGNS WHEN n IS ODD[1]

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1. Summary and introduction. This paper attempts to add to the existing understanding of weighing designs (WD). Let us suppose that n objects are to be weighed in n weighings with a chemical balance having no bias. In the previous papers [14], [15], for odd n, optimum weighing designs are given for some particular cases subject to the conditions, viz., (i) variances of the estimated weights are equal and (ii) the estimated weights are equally correlated. Assuming these conditions we find some more optimum designs when n is odd. Throughout this paper n is assumed to be odd, except in Section 4, where n may take any value.

Let

$$\begin{aligned} x_{ij} &= 1 && \text{if the } j\text{th object is placed in the left pan in the } i\text{th weighing} \\ &= -1 && \text{if the } j\text{th object is placed in the right pan in the } i\text{th weighing} \\ &= 0 && \text{if the } j\text{th object is not weighed in the } i\text{th weighing.} \end{aligned}$$

The nth order matrix $X = ((x_{ij}))$ is known as the design matrix. Also let y_i be the result recorded in the ith weighing, ϵ_i is the error in the result, w_j the true weight of the jth object, so that we have n equations

$$x_{i1}w_1 + x_{i2}w_2 + \cdots + x_{in}w_n = y_i + \epsilon_i, \quad i = 1, 2, \cdots, n.$$

We assume X to be a non-singular matrix. The method of least squares or the theory of linear estimation gives the estimated weights $\hat{w}$ by the equation $\hat{w} = (X'X)^{-1}X'y$ where $\hat{w}$ and y are the column vectors of the estimated weights and the observations respectively.

If σ^2 is the variance of each weighing, then

$$V(\hat{w}) = (X'X)^{-1}\sigma^2 = ((c_{ij}))\sigma^2$$

where $((c_{ij}))$ is the inverse of $X'X$.

The design will be called optimum in the sense of Mood, if det $((c_{ij}))$ (det stands for determinant) is minimum and this is the case when det $X'X$ is maximum. The efficiency of the weighing design can be measured, in the sense of Mood, by det $X'X$/max det $X'X$ [12].

If $\lambda_{\min}$ be the minimum characteristic root of $X'X$, then the efficiency of the design can be measured, in the sense of Ehrenfeld, by $\lambda_{\min}/\max \lambda_{\min}$ [6].

According to Kishen, the efficiency can be measured by $1/\sum c_{ii}$ [11].

Received 16 June 1965; revised 7 January 1966.

[1] This work was financially supported by a Research Training Scholarship of the Government of India.

The first two definitions of efficiency were first introduced by Wald in [16]. Kiefer gave some optimality criteria for experimental designs in which the three criteria mentioned above were included and discussed them fully in [9] and [10]. He named these three definitions as D-optimum, E-optimum and A-optimum. We use Kiefer's notations in Section 6 for the definitions of efficiency of a weighing design given above.

2. Fundamental necessary condition. We shall confine ourselves to the case when (i) the variances of the estimated weights are equal and (ii) the estimated weights are equally correlated. In this case we get a design X with the parameters n, s, λ where n is its size, s the number of zeros in any column and

$$\lambda = \sum_{i=1}^{n} x_{ij}x_{ij'}, \quad j, j', j \neq j' = 1, 2, \cdots, n. \tag{2.1}$$

Thus we get

$$X'X = (n - s - \lambda)I_n + \lambda E_{nn}, \tag{2.2}$$

where I_n is the identity matrix of order n and E_{nm} is the $n \times m$ order matrix with positive unit elements everywhere. Hence

$$\begin{aligned} \det X &= \pm(\det X'X)^{\frac{1}{2}} \\ &= \pm[n - s + (n-1)\lambda]^{\frac{1}{2}}(n - s - \lambda)^{(n-1)/2}. \end{aligned} \tag{2.3}$$

Since $\det X$ is real integral value, $n - s + (n-1)\lambda$ is a perfect square.

THEOREM 2.1. *A necessary condition for the existence of X is that $n - s + (n-1)\lambda$ is a perfect square.*

We have the design with the parameters n, s and λ. Denote this weighing design by $[n, s, \lambda]$. Let

$$n - s + (n-1)\lambda = d^2 \tag{2.4}$$

where d is some integer. Since we are considering odd n, it takes the values either 1 mod 4 or 3 mod 4. Let

$$\begin{aligned} n &= 4t + 1 \quad \text{if} \quad n \text{ is } 1 \bmod 4 \\ &= 4t + 3 \quad \text{if} \quad n \text{ is } 3 \bmod 4 \end{aligned} \tag{2.5}$$

where t is non-negative integer. Let s take one of the values $4t'$, $4t' + 1$, $4t' + 2$, $4t' + 3$ where $t'(<t)$ is also some non-negative integer.

CASE (i). $n = 4t + 1$. For $n - s + (n-1)\lambda$ to be a perfect square, s should be 0, 1 mod 4. Hence,

REMARK 2.1. The weighing designs $[4t + 1, 4t' + 2, \lambda]$ and $[4t + 1, 4t' + 3, \lambda]$ do not exist.

CASE (ii). $n = 4t + 3$. When $s - 2\lambda = 0$, 1 mod 4, $n - s + (n-1)\lambda$ is not a perfect square.

REMARK 2.2. The weighing designs $[4t + 3, 4t', \lambda]$, $[4t + 3, 4t' + 1, \lambda]$ where λ is even, and the designs $[4t + 3, 4t' + 2, \lambda]$, $[4t + 3, 4t' + 3, \lambda]$ where λ is odd, do not exist.

3. On the impossibilities of the $[n, s, \lambda]$ **when** $\lambda \neq 0$. We use here Legendre symbols, Hilbert norm residues and the Hasse-Minkowski invariants for showing the non-existence of some designs. The following lemma, given by Ogawa [13], will be of use for our purpose.

LEMMA 3.1. *If* $A = eI_n + fE_{nn}$ *where* e *and* f *are non-zero rationals, then*

$$(3.1) \quad C_p(A) = (-1, -1)_p(-1, e)_p^{n(n-1)/2}(-1, g)_p(n, g)_p(g, e)_p^{n-1}(n, e)_p$$

where $g = e + nf$ *and* p *is any odd prime.*

Since we are considering non-singular weighing design, its inverse exists and is also a matrix with rational elements. Thus $I_n = (X^{-1})'(X'X)X^{-1}$. We have that I_n and $X'X$ are rationally congruent and they can be written $X'X \sim I_n$. Hence

$$(3.2) \qquad C_p(X'X) = C_p(I_n) = (-1, -1)_p .$$

But $X'X = (n - s - \lambda)I_n + \lambda E_{nn}$. From the lemma given above, we see that

$$(3.3) \quad C_p(X'X) = (-1, -1)_p(-1, n - s - \lambda)_p^{n(n-1)/2} \cdot(-1, g)_p(n, g)_p(n, n - s - \lambda)_p(n - s - \lambda, g)_p^{n-1}$$

where $g = n - s + (n - 1)\lambda = d^2$. Hence

$$(3.4) \quad C_p(X'X) = (-1, -1)_p(-1, n - s - \lambda)_p^{n(n-1)/2}(n, n - s - \lambda)_p .$$

On equating the right hand sides of (3.2) and (3.4), we get for all primes that

$$(3.5) \qquad (-1, n - s - \lambda)_p^{n(n-1)/2}(n, n - s - \lambda)_p = 1.$$

It follows from (3.5) that

$$(n, n - s - \lambda)_p = 1 \quad \text{for} \quad n = 4t + 1;$$

$$(-n, n - s - \lambda)_p = 1 \quad \text{for} \quad n = 4t + 3.$$

These results can be stated in the form of the following theorem:

THEOREM 3.1. *A necessary condition for the existence of the* $[n, s, \lambda]$ *with* $\lambda \neq 0$, *is that* $(n, n - s - \lambda)_p = 1$ *when* $n = 1 \bmod 4$ *and* $(-n, n - s - \lambda)_p = 1$ *when* $n = 3 \bmod 4$ *for all primes.*

Examples for some non-existing designs:

n	15	19	27	27	29	31	31	43
s	4	12	4	15	13	1	10	0
λ	−1	1	13	2	2	19	2	3

4. Structure of the design $[n, s, \lambda]$ **with** $\lambda \neq 0$. The distribution of the elements $+1$, 0 and -1 in this matrix is of particular interest. Let the first row of this matrix contain r positive units and t zeros. We bring them in the columns $1, 2, \cdots, r$ and in the last t columns, respectively. Then we construct the matrix X_1 of $n \times (n - 1)$ in which the first row vanishes and it is given as

$$(4.1) \qquad X_1 = (x_i - x_{1i}x_1), \qquad i = 2, 3, \cdots, n,$$

where x_i is the ith column vector of X. After deleting the first row in X_1 there remains a matrix X_2 which gives that

$$\det X_2'X_2 = \det X'X \tag{4.2}$$

and

$$X_2'\,X_2 = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}' & X_{22} & X_{23} \\ X_{13}' & X_{23}' & X_{33} \end{bmatrix}, \tag{4.3}$$

where

$$\begin{aligned} X_{11} &= (n - s - \lambda)(I_{r-1} + E_{r-1\,r-1}); \\ X_{12} &= -(n - s - \lambda)E_{r-1\,n-t-r}; \\ X_{13} &= 0_{r-1\,t} \quad \text{(null matrix)}; \\ X_{22} &= (n - s - \lambda)I_{n-t-r} + (n - s + 3\lambda)E_{n-t-r\,n-t-r}; \\ X_{23} &= 2\lambda E_{n-t-r\,t}; \\ X_{33} &= (n - s - \lambda)I_t + \lambda E_{tt}. \end{aligned} \tag{4.4}$$

Hence from (4.2), (4.3) and (4.4) we get that

$$r = (n - t)/2 \pm (d/2\lambda)[\lambda(s + \lambda - t)]^{\frac{1}{2}} \tag{4.5}$$

where $d^2 = n - s + (n - 1)\lambda$. Since the design is non-singular, we know that λ can not be -1 for $s \geqq 1$. Thus from (4.5) we see that, because r takes integral values, t is of the form $s - (i^2 - 1)\lambda$ where i takes some of the values $0, 1, 2, \cdots, q$ and $s - (q^2 - 1)\lambda \geqq 0$. Further i takes different values only when it takes the values including zero. This is due to $\sum_{j=1}^{n} t_j = ns$ where t_j is the number of zeros in the jth row of X. When i does not take the value zero, then all t_j^s are equal ($= s$). Same is the case with $s < 3\lambda$. Hence

LEMMA 4.1. *A necessary condition for the existence of* $[n, s, \lambda]$ *having rows with different number of zeros, where* $s > 0$ *and* $\lambda \neq 0$, *is that* $n - s - \lambda$ *is even or* $s \geqq 3\lambda$.

Let $X_0 = [n, s, 0]$ which gives

$$X_0'X_0 = (n - s)I_n = X_0X_0'. \tag{4.6}$$

Hence

LEMMA 4.2. *The designs* $[n, s, 0]$ *having rows with different number of zeros are non-existent.*

Now consider the designs having every row s zeros and $\lambda \neq 0$. Then (4.5) becomes

$$r = (n - s \pm d)/2 \tag{4.7}$$

and (4.7) shows that every row of X contains either $(n - s + d)/2$ or $(n - s - d)/2$ positive units. Let n_1 be the number of rows of X where each row contains $(n - s + d)/2$ positive units and let n_2 be the number of rows of X where each row contains $(n - s - d)/2$ positive units and $n_1 + n_2 = n$.

Write

$$X = \begin{bmatrix} X_{n_1 n} \\ X_{n_2 n} \end{bmatrix} \tag{4.8}$$

where $X_{n_i n}$ is $n_i \times n$ $(i = 1, 2)$ matrix such that

$$X_{n_1 n} E_{n1} = dE_{n_1 1} \quad \text{and} \quad X_{n_2 n} E_{n1} = -dE_{n_2 1}\,. \tag{4.9}$$

Let

$$X^* = \begin{bmatrix} X_{n_1 n} \\ -X_{n_2 n} \end{bmatrix}. \tag{4.10}$$

Consequently,

$$X'X = X^{*\prime}X^* \quad \text{and} \quad X^* E_{n1} = dE_{n1}\,. \tag{4.11}$$

Hence, we get

$$(X^*)'X^* = X^*(X^*)' = (n - s - \lambda)I_n + \lambda E_{nn} \tag{4.12}$$

and also that every row and column of X^* has the same number of positive units. Since X with $\lambda \neq 0$ implies X^*, we use X^* for $[n, s, \lambda]$ and X_0 for $[n, s, 0]$ here afterwards when X has every row with s zeros.

4a. *Some non-existing designs.* Let N be a matrix obtained from X^* (or X_0) by changing the negative units to zeros. Let M be a matrix obtained from X^* (or X_0) by changing positive units and negative units to zeros and zeros of X^* (or X_0) to positive units. Hence

$$X^* \text{ (or } X_0) = 2N + M - E_{nn}\,. \tag{4.13}$$

Let

$$NN' = ((\lambda_{ij})),\ MM' = ((\mu_{ij})),\ MN' + NM' = ((\nu_{ij})). \tag{4.14}$$

We can deduce from (4.12), (4.13) and (4.14) that

$$4((\lambda_{ij})) + 2((\nu_{ij})) + ((\mu_{ij})) = (n - s - \lambda)I_n + (2s + \lambda - n)E_{nn} + 2((r_i + r_j)) \tag{4.15}$$

where r_i is the number of positive units in the ith row of X^* (or X_0). All r_i' are equal in the case of design X^*.

Let $s = 0$, then $M = 0_{nn}$ (null matrix). Hence (4.15) gives

LEMMA 4a.1. *A necessary condition for the existence of X^* with $s = 0$ or for the existence of $[n, 0, 0]$ except $n = 2$, is that $n - \lambda = 0 \bmod 4$.*

Again, considering (4.15), when n is even and λ is odd or when n is odd and λ is even, we have that μ_{ij} is odd for $i, j\ (i \neq j) = 1, 2, \cdots, n$. From the matrix MM' we get that

$$ns^2 = ns + \textstyle\sum_{i \neq j = 1, \cdots, n} \mu_{ij}\,. \tag{4.16}$$

When all μ_{ij}^s are odd, it follows from (4.16) that

$$s(s-1)+1 \geqq n \tag{4.17}$$

which gives

LEMMA 4a.2. *A necessary condition for the existence of X^* (or X_0), when n is even and λ is odd or when n is odd and λ is even, is that $n \leqq s(s-1)+1$.*

Let

$$((\delta_{ij})) = (E_{nn}-M)(E_{nn}-M)' = MM' + (n-2s)E_{nn}. \tag{4.18}$$

From (4.15) and (4.18) we get that

$$4((\lambda_{ij})) + 2((\nu_{ij})) + ((\delta_{ij})) = (n-s-\lambda)I_n + \lambda E_{nn} + 2((r_i + r_j)). \tag{4.19}$$

It follows from (4.19) that δ_{ij}^s are odd when λ is odd. Hence

LEMMA 4a.3. *When λ is odd, a necessary condition for the existence of X^* is that*

$$n \leqq (n-s)(n-s-1)+1. \tag{4.20}$$

Some examples of non-existing designs:

n	s	λ	Reference	n	s	λ	Reference
11	2	0	Lemmas 4.2, 4a.2	21	17	3	Lemmas 4.1, 4a.3
11	2	4	Lemmas 4.1, 4a.2	23	3	2	Lemmas 4.1, 4a.2
19	3	0	Lemmas 4.2, 4a.2	27	2	0	Lemmas 4.2, 4a.2
19	3	10	Lemmas 4.1, 4a.2	29	4	2	Lemmas 4.1, 4a.2

Now we can easily show that the existence of $[n, 0, \lambda]$ with $\lambda \neq 0$ implies the existence of symmetrical balanced incomplete block design (SBIBD) with the parameters $v^* = n = b^*, r^*, r^* = (n-d)/2 = k^*$ and $\lambda^* = (n-2d+\lambda)/4$; and conversely if a SBIBD exists with the above parameters, we get $[n, 0, \lambda]$.

5. The non-existence of the designs $[n, 1, 1]$ **and** $[n, 1, 3]$. By the Lemma 4.1 we see that each row (and each column) of these designs contains one zero. Hence on transforming to X^* we get that every row (and every column) contains the same number of positive units. Let this number be r. Hence $r = (n-1+d)/2$.

Consider the design $[n, 1, 1]$. Here $M = I_n$ and

$$X^* = 2N + I_n - E_{nn}. \tag{5.1}$$

Also

$$\begin{aligned} X^*(X^*)' &= 4NN' + 2(N+N') + (n-4r-2)E_{nn} + I_n \\ &= (n-2)I_n + E_{nn}. \end{aligned} \tag{5.2}$$

Hence

$$2NN' + (N+N') = [(n-3)/2]I_n + (2r-(n-3)/2)E_{nn}. \tag{5.3}$$

Let $N = ((n_{ij}))$ where $n_{ij} = 1$ or 0 for all $i, j = 1, 2, \cdots, n$. (5.3) gives that

$$(5.4) \quad 2\lambda_{ij} + n_{ij} + n_{ji} = 2r - (n - 3)/2 \quad (i, j\ (i \neq j) = 1, 2, \cdots, n).$$

Let $2r - (n - 3)/2$ be odd. Hence, N should be skew symmetric which is impossible due to the fact that every row and column of X^* contains $(n - 1 + d)/2$ positive units and $(n - 1 - d)/2$ negative units. Thus it follows that N is symmetric and $n = 3 \bmod 4$. Let $n = 4t + 3$. Then

$$(5.5) \qquad NN' = rI_n + (r - t - 1)N + (r - t)(E_{nn} - I_n - N).$$

Easily we can show [2] that N is symmetrical partially balanced incomplete block design (SPBIBD) with the following parameters: $v = b = n$, $r = k = (n - 1 + d)/2$, $n_1 = r$, $n_2 = n - r - 1$, $\lambda_1 = r - t - 1$, $\lambda_2 = r - t$ and

$$(5.6) \qquad P_1 = \begin{bmatrix} r - t - 1 & t \\ & n - r - t - 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} r - t & t \\ & n - r - t - 2 \end{bmatrix}.$$

For the existence of this design we must have Δ be a perfect square and η be an integer where

$$(5.7) \quad \Delta = \nu^2 + 2\beta + 1; \qquad \nu = p_{12}^2 - p_{12}^1; \qquad \beta = p_{12}^2 + p_{12}^1;$$

$$\eta = [(v - 1)(1 - \nu) - 2n_1]/2\Delta^{\frac{1}{2}}.$$

This result is due to Connor and Clatworthy [4]. For the parameters given in (5.6), $\Delta = 4t + 1$ and $\eta = (n - 1 - 2r)/2(4t + 1)^{\frac{1}{2}}$. On substituting the value of r we get $\eta = -[(2t + 1)/(4t + 1)]^{\frac{1}{2}}$ which is not an integer except for $t = 0$. Hence it follows that SPBIBD with the parameters given in (5.6) does not exist; this shows that the design $[n, 1, 1]$ is impossible for $n > 3$. Similarly, we can prove that the design $[n, 1, 3]$ is also non-existent for $n > 5$.

6. Some optimum designs. Raghavarao [14], [15] showed that P_n matrices $[n, 0, 1]$ are A-optimum, D-optimum and E-optimum. Ehilich [5] showed that these matrices give maximum determinant. Hence in the D-optimum sense the efficiency of the designs is one. But the existence of these matrices are limited to some numbers with $n = 1 \bmod 4$ where $2n - 1$ is a perfect square. In fact this is a necessary condition for the existence of P_n matrices. In this section we find some more optimum designs. Let the matrices $[n, 0, -1]$, $[n, 0, 3]$, $[n, 0, 5]$ be denoted by Σ_n^*, Q_n, R_n respectively. By Lemma 4a.1 we have that $n = 3 \bmod 4$ and $n = 1 \bmod 4$ are the necessary conditions for the existence of Σ_n^*, Q_n and R_n respectively.

Case (i). Let $n = 4t + 3$. With the help of the sections given above we can show that the design sets $[n, 0, 0]$, $[n, 0, 1]$, $[n, 0, 2]$, $[n, 1, 0]$, $[n, 2, 0]$ and $[n, 1, 1]$ do not exist. The efficiency of the design $[n, s, \lambda]$ in the A-optimum sense is given by [14]

$$(6.1)\qquad f(n-s,\lambda) = (n-s-\lambda)[n-s+(n-1)\lambda]/n[n-s+(n-2)\lambda];$$

$$(6.2)\qquad f(n,3) - f(n-s,\lambda) = (n-3)(4n-3)/n(4n-6) - (n-s-\lambda)[n-s+(n-1)\lambda]/n[n-s+(n-2)\lambda] \quad n-s, \lambda \text{ are positive}, \; n > s+\lambda \text{ or } s = 0, \lambda = -1$$

$$(6.3)\qquad = (n-s')[n(4s'-9) - 6s' + 9] + \lambda[(4s'-13)n^2 + (4-s')6n - 9]/n(4n-6)[n-s+(n-2)\lambda]$$

where $s' = s + \lambda$.

For Q_n to be efficient, (6.3) should be positive. Evidently this is positive when $\lambda = -1$ and $n > 3$ and also it is positive when $s' > 3$. When $s' = 3$, (6.3) becomes $(n-3)^2(3-\lambda)/n(4n-6)[n-3+(n-1)\lambda]$ which is non-negative since $\lambda \leqq 3$. Hence (6.3) is positive for $s' \geqq 3$. Also we know that the design $[n, s, \lambda]$ does not exist for $0 \leqq s < 3$. Thus

THEOREM 6.1. *For $n = 3 \bmod 4$ and $n > 3$ the weighing design Q_n is A-optimum and E-optimum.*

Consider the difference of the determinants Q_nQ_n' and $X'X$ where X is any $[n, s, \lambda]$:

$$(6.4)\qquad \det Q_nQ_n' - \det XX' = (4n-3)(n-3)^{n-1} - [n-s+(n-1)\lambda](n-s-\lambda)^{n-1}$$

$$(6.5)\qquad = n(n-3)^{n-1}\{(4-3/n) - (\lambda+1-s/n)'[1-(s'-3)/(n-3)]^{n-1}\}$$

where $s' = s + \lambda$. For large values of n the expression in the braces of (6.5) tends to $4 - (\lambda+1)e^{-s'+3}$. Thus we have

$$(6.6)\qquad 4 - (\lambda+1)e^{-s'+3} \geqq 4 - (s'+1)e^{-s'+3} \geqq 0 \qquad \text{for} \qquad s' \geqq 3.$$

It follows from (6.5) and (6.6) the difference of the determinants is positive for $n > 3$, $s' \geqq 3$ and $\lambda \geqq 0$. We know that $[n, s, \lambda]$ does not exist for $s' < 3$ except for s' $(s = 0, \lambda = -1) = -1$. From (6.5)

$$(6.7)\qquad \det Q_nQ_n' - \det \Sigma_n^*\Sigma_n^{*\prime} = (n-3)^{n-1}\{(4n-3) - [1+4/(n-3)]^{n-1}\}$$

and $4n - 3 - [1 + 4/(n-3)]^{n-1}$ tends to $4n - 3 - e^4$ for large values of n. Hence, we have that the difference (6.7) of the determinants is positive for $n > 14$ and Q_{15} does not exist. Thus

THEOREM 6.2. *For* $n > 15$, Q_n *is D-optimum and* $\Sigma_n{}^*$ *is D-optimum for* $n \leqq 15$. The maximum determinants for $n = 3, 7, 11$ are given as 4, $2^6 \cdot 9$, $2^{10} \cdot 264$, respectively. Hence the efficiencies of $\Sigma_n{}^*$ for $n = 3, 7, 11$ are respectively 1, .79, .84.

CASE (ii). Let $n = 4t + 1$. The above sections enable us to show that the design sets $[n, 0, \lambda]$, $\lambda = -1, 0, 2, 3, 4$; $[n, 1, \lambda]$, $\lambda = 0, 1, 2, 3, 4$; $[n, 2, \lambda]$, $[n, 3, \lambda]$ and $[n, 4, 0]$ $(n > 13)$ do not exist. Also $[9, 4, 0]$ does not exist. For $n = 13$, P_n exists and it is the optimum weighing design.

Consider the difference

$$f(n, 5) - f(n - s, \lambda) = (n - 5)(6n - 5)/n(6n - 10) - (n - s - \lambda)[n - s + (n - 1)\lambda]/ n[n - s + (n - 2)\lambda] \tag{6.8}$$

$$= \{(n - s')[(6n - 10)s' - 25(n - 1)] + \lambda[n(6n - 10)s' - (31n^2 - 60n + 25)]\}/ n(6n - 10)[n - s + (n - 2)\lambda] \tag{6.9}$$

where $s' = s + \lambda$. As in the case (i), (6.9) can be shown positive for $s' \geqq 5$. Hence we have the following theorem:

THEOREM 6.3. *For* $n = 1 \bmod 4$ *and* $n > 5$ *also when* P_n *does not exist*, R_n *is A-optimum and E-optimum.*

Consider the difference of the determinants $R_n R_n{}'$ and XX' where X is any $[n, s, \lambda]$:

$$\det R_n R_n{}' - \det XX' = (6n - 5)(n - 5)^{n-1} - [n - s + (n - 1)\lambda](n - s - \lambda)^{n-1}, \tag{6.10}$$

$$= n(n - 5)^{n-1}\{(6 - 5/n) - (\lambda + 1 - s'/n)[1 - (s' - 5)/(n - 5)]^{n-1}\} \tag{6.11}$$

where $s' = s + \lambda$. For large values of n, the expression in the larger braces of (6.11) tends to $6 - (\lambda + 1)e^{-s'+5}$ which is greater than $6 - (s' + 1)e^{-s'+5}$. Hence,

$$6 - (s' + 1)^{-s'+5} \geqq 0 \quad \text{for} \quad s' \geqq 5. \tag{6.12}$$

It follows from (6.11) and (6.12) that for $n > 5$ and $s' \geqq 5$ the difference of the determinants is positive. Also we know that $[n, s, \lambda]$ does not exist for $s' < 5$ except for $s = 0$ and $\lambda = 1$. Hence,

THEOREM 6.4. *For* $n > 5$ *and* $n = 1 \bmod 4$, *when* P_n *does not exist*, R_n *is D-optimum.*

We know that the existence of SBIBD with the parameters $v^* = b^* = n$,

TABLE 6.1

SBIBD		Corresponding	
v^*	r^*	λ^*	WD
n	$(n - d_1)/2$	$(n - 2d_1 + 3)/4$	Q_n
n	$(n - 1)/2$	$(n - 3)/4$	Σ_n^*
n	$(n - d_2)/2$	$(n - 2d_2 + 5)/4$	R_n

TABLE 6.1.1

v^*	r^*	λ^*	WD
7	3	1	Σ_7^*
7	6	5	Q_7
11	5	2	Σ_{11}^*
15	7	3	Σ_{15}^*
21	5	1	R_{21}
31	10	3	Q_{31}

$r^* = k^* = (n - d)/2, \lambda^* = (n - 2d + \lambda)/4$ when $\lambda \neq 0$ implies the existence of $[n, 0, \lambda]$. Let $d_1^2 = 4n - 3$ and $d_2^2 = 6n - 5$ where d_1 and d_2 are integral values. Thus we have Table 6.1.

We also give some examples from the Table 11.3, pages 469–470 of [3].

Acknowledgment. My sincere thanks are due to Professor M. C. Chakrabarti for his kind guidance in preparing this paper. I am also thankful to the referee for his suggestions on the original manuscript of my paper.

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