SPECTRAL ESTIMATES USING NONLINEAR FUNCTIONS[1]

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0. Summary. The main result proved here (Theorem 3) is a generalization of a formula of Goldstein [2], who showed that if the estimate $\hat{S}(\omega)$ for the spectral density is computed by the use of the function $y(x) = \text{sgn}(x)$, and the spectrum is flat, then the dominant term in the variance of $\hat{S}(\omega)$ is $\frac{1}{2}\pi^2 K/N$. Theorem 3 evaluates this term for nonflat spectra and for more general functions $y(x)$.

This analysis shows that the loss in accuracy caused by working with $y(x)$ instead of x itself can be decreased considerably by using for $y(x)$ a step function with more than two values. Some results on Gaussian process, interesting in their own right, are proved along the way.

1. Introduction. The spectral density $S(\omega)$, $|\omega| \leqq \pi$, of a discrete stationary Gaussian process $\{x_k\}$, $-\infty < k < \infty$, of mean zero, can be expressed in terms of the correlations $R_x(k) = E(x_n x_{n+k})$ by

$$S(\omega) = \sum_{k=-\infty}^{\infty} e^{ik\omega} R_x(k) = R_x(0) + 2 \sum_{k=1}^{\infty} \cos k\omega R_x(k).$$

An estimate of $S(\omega)$ can be obtained from observations of $\{x_k\}$ by truncating the series and replacing the quantities $R_x(k)$ by appropriate estimates $\hat{R}_x(k)$. Assume that $R_x(0) = E(x_k^2)$ is known. Then the x_k's can be normalized so that $R_x(0) = 1$, and the estimate for $S(\omega)$ is

$$\hat{S}(\omega) = 1 + 2 \sum_{k=1}^{K} \cos k\omega \hat{R}_x(k). \tag{1}$$

A simple choice of $\hat{R}_x(k)$ is

$$\hat{R}_x(k) = N^{-1} \sum_{n=1}^{N} x_n x_{n+k}, \tag{2}$$

where N is large. Each term in the sum has mean $R_x(k)$ and the variance of this expression approaches zero as $N \to \infty$. Hence for large N, it is close to $R_x(k)$ with high probability. However, for large N the evaluation of the sum can be quite time consuming.

It has been observed [2] that if

$$\begin{aligned} y_i &= +1, \qquad x_i \geqq 0, \\ &= -1, \qquad x_i < 0, \end{aligned}$$

then $R_y(k) = E(y_n y_{n+k})$ satisfies the relation

$$R_x(k) = \sin[(\pi/2)R_y(k)].$$

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This suggests putting

$$(3)\qquad \hat{R}_y(k) = N^{-1} \sum_{n=1}^{N} y_n y_{n+k}, \qquad \hat{R}_x(k) = \sin[(\pi/2)\hat{R}_y(k)].$$

For large N, and K small compared with N, this formula for $\hat{R}_x(k)$ can be evaluated much more rapidly than (2) [2]. The problem arises of estimating the mean and variance of $\hat{S}(\omega)$, when evaluated by (1) and (3). It is hoped that the mean is close to $S(\omega)$ and the variance is small.

A more general method is considered here. We take $y_i = y(x_i)$, where $y(x)$ is any odd, bounded, nondecreasing function, normalized so that $E(y_i^2) = 1$. It is shown (Lemma 3) that there is a function $F(t)$ such that $R_x(k) = F[R_y(k)]$. In general $F(t)$ is not an entire function, but is analytic in a region of the complex plane including the open interval $-1 < t < 1$. For the present purposes, $F(t)$ may be extended in any way to a continuous function on $-\infty < t < \infty$. We take

$$\hat{R}_y(k) = N^{-1} \sum_{n=1}^{N} y_n y_{n+k}, \qquad \hat{R}_x(k) = F[\hat{R}_y(k)].$$

With this definition, it is shown that except for a term of order N^{-1}, $E\{\hat{S}(\omega)\}$ is

$$S_K(\omega) = 1 + 2 \sum_{k=1}^{K} \cos k\omega R_x(k),$$

and $E\{\hat{S}(\omega)^2\}$ is approximately $S_K(\omega)^2$, with a leading error term of order K/N, which is given explicitly, and another error term $o(K/N)$, whose exact order depends on the degree of regularity assumed for $S(\omega)$ (Theorem 3). This result is obtained for a large class of summation methods (or windows) in place of (1), including Césaro sums.

The hypothesis on $S(\omega)$ in Theorem 3 is satisfied if $S(\omega)$ is a periodic function of bounded variation which satisfies a Lipshitz condition of order α, $\alpha > 0$ ([3], p. 136).

2. Estimates for the moments of $\hat{R}_y(k)$.

LEMMA 1. *Let $f(z)$, $z = (z_1, \cdots, z_n)$ be analytic in a convex region D containing the origin, with $|f(z)| \leqq M$. Let $\{\pi_k(z)\}$ be a set of products of the z_j's such that in the power series expansion of $f(z)$ at $(0, \cdots, 0)$, every term is divisible by one of the π_k's. If ζ is a point whose δ-neighborhood $|z_j - \zeta_j| < \delta$, $j = 1, \cdots, n$, is in D, then $|f(\zeta)| \leqq 2^n M \sum_k (3/\delta)^{d_k} |\pi_k(\zeta)|$, where d_k is the degree of π_k.*

PROOF. Suppose p of the ζ_j's have absolute value less than $\delta/3$. By renaming the coordinates, these may be taken to be $\zeta_1, \zeta_2, \cdots, \zeta_p$. Then if

$$(4)\qquad |z_j| < 2\delta/3, \quad j = 1, \cdots, p, \qquad |z_j - \zeta_j| < \delta, \quad j = p+1, \cdots, n,$$

z is in D. It follows that the power series expansion

$$(5)\qquad f(z) = \sum_{k_1, \cdots, k_p = 0}^{\infty} g_{k_1 \cdots k_p}(z_{p+1}, \cdots, z_n) z_1^{k_1} \cdots z_p^{k_p}$$

converges in the region (4), and

$$|g_{k_1\cdots k_p}(z_1, \cdots, z_p)| \leqq M(3/2\delta)^{k_1+\cdots+k_p}.$$

By the convexity of D, there is a sequence of overlapping regions of this type converging to zero, in which $f(z)$ is analytic and (5) is valid. Hence (5) is also valid in a neighborhood of the origin. There we have

$$(6) \qquad g_{k_1\cdots k_p}(z_{p+1}, \cdots, z_n) = \sum_{k_{p+1},\cdots,k_n=0}^{\infty} a_{k_1\cdots k_n} z_{p+1}^{k_{p+1}}, \cdots, z_n^{k_n},$$
$$f(z) = \sum_{k_1,\cdots,k_n=0}^{\infty} a_{k_1\cdots k_n} z_1^{k_1} \cdots z_n^{k_n}.$$

Let $\pi_k'(z)$ be obtained from $\pi_k(z)$ by omitting the factors z_j with $j > p$. Then each term in (6) must be divisible by one of the π_k''s. It follows that the same is true in (5). The sum of the absolute values of all terms in (5) which are divisible by one of the products $\pi_k'(z)$ is dominated by

$$M(3/2\delta)^{d_k'} |\pi_k'(z)| \prod_{j=1}^{p} (1 - 3/2\delta\, |z_j|)^{-1},$$

where d_k' is the degree of $\pi_k'(z)$. At $z = \zeta$, this is at most

$$M(3/2\delta)^{d_k'} |\pi_k'(\zeta)| (1 - \tfrac{1}{2})^{-p} = 2^p M(3/2\delta)^{d_k'} |\pi_k'(\zeta)|$$
$$\leqq 2^n M(3/\delta)^{d_k'} |\pi_k'(\zeta)| \leqq 2^n M(3/\delta)^{d_k} |\pi_k(\zeta)|,$$

since in the last step, only factors of the form $3/\delta\, |z_j|$, $j \geqq p + 1$, were added. Summing over the π_k's accounts for each term in (5) at least once. Hence the result follows.

LEMMA 2. *Let $\{x_i\}$ be a stationary Gaussian process with mean zero and variance 1, and a spectral density $S(\omega)$, $|\omega| \leqq \pi$, which is integrable.*

Then for any positive m, there is a constant $\mu_m > 0$ such that any $m \times m$ covariance matrix $[R_x(p_j - p_k)]_{j,k=1,\cdots,m}$, $p_1 < p_2 < \cdots < p_m$, has its eigenvalues $\geqq \mu_m$.

PROOF. We will use induction on m. Let μ_m be the infimum of eigenvalues for matrices of rank m. It will be shown that $\mu_m > 0$, and $\mu_m \leqq \mu_{m-1}$ for $m \geqq 2$.

(i) For $m = 1$, the matrix is the identity. Hence $\mu_1 = 1$.

(ii) Suppose we have determined that $\mu_1, \cdots, \mu_{m-1}$ are positive.

We have

$$\mu_m = \inf_{L_1(m)} \sum_{j,k=1}^{m} a_j a_k R_x(p_j - p_k),$$

where $L_1(m) = \{a_i, p_i \mid a_1^2 + \cdots + a_{m=1}^2, p_1 < p_2 < \cdots < p_m\}$, μ_{m-1} can be obtained from this expression by putting $a_m = 0$. Hence $\mu_m \leqq \mu_{m-1}$. If $\mu_m = \mu_{m-1}$, it is positive. Hence we may assume $\mu_m < \mu_{m-1}$. Expressing $R_x(p_j - p_k)$ in terms of $S(\omega)$,

$$\mu_m = \inf\, (2\pi)^{-1} \int_{-\pi}^{\pi} d\omega S(\omega) \,|\sum_{k=1}^{m} a_k e^{ip_k\omega}|^2.$$

By the Riemann-Lebesgue lemma $\lim_{n\to\infty} (2\pi)^{-1} \int_{-\pi}^{\pi} d\omega S(\omega) e^{in\omega} = 0$. Hence there is a number n_0 such that if $n > n_0$, this integral has absolute value less than $(\mu_{m-1} - \mu_m)/4m$. If one of the differences $p_{l+1} - p_l$, $1 \leqq l \leqq m - 1$, is greater than n_0,

$$
\begin{aligned}
(2\pi)^{-1} &\int_{-\pi}^{\pi} d\omega S(\omega) \left| \sum_{k=1}^{m} a_k e^{ip_k\omega} \right|^2 \\
&= (2\pi)^{-1} \int_{-\pi}^{\pi} d\omega S(\omega) \{ \left| \sum_{k=1}^{l} a_k e^{ip_k\omega} \right|^2 + \left| \sum_{k=l+1}^{m} a_k e^{ip_k\omega} \right|^2 \} \\
&\quad + 2 \sum_{j=1}^{l} \sum_{k=l+1}^{m} a_j a_k \cdot (2\pi)^{-1} \int_{-\pi}^{\pi} d\omega S(\omega) e^{i\omega(p_k - p_j)} \\
&\geqq \mu_l \sum_{k=1}^{l} a_k^2 + \mu_{m-l} \sum_{k=l+1}^{m} a_k^2 - 2 \sum_{j,k=1}^{m} |a_j|\, |a_k| (\mu_{m-1} - \mu_m)/4m \\
&\geqq \mu_{m-1} - 2m \cdot (\mu_{m-1} - \mu_m)/4m = (\mu_{m-1} + \mu_m)/2 > \mu_m .
\end{aligned}
$$

Hence such sets of p_j's need not be considered in finding μ_m, and

$$\mu_m = \inf_{L_2(m)} (2\pi)^{-1} \int_{-\pi}^{\pi} d\omega S(\omega) \left| \sum_{k=1}^{m} a_k e^{ip_k\omega} \right|^2$$

where $L_2(m) = \{a_i, p_i \mid a_1^2 + \cdots + a_m^2 = 1, 0 \leqq p_1 < \cdots < p_m < mn_0\}$,

or $$\mu_m = \min_{0 \leqq p_1 < \cdots < p_m \leqq mn_0} \mu(m; p_1, \cdots, p_m),$$

where

$$\mu(m; p_1, \cdots, p_m) = \min_{a_1^2 + \cdots + a_m^2 = 1} (2\pi)^{-1} \int_{-\pi}^{\pi} d\omega S(\omega) \left| \sum_{k=1}^{m} a_k e^{ip_k\omega} \right|^2.$$

This is the minimum eigenvalue of a certain matrix. It is the value of the integral when $(a_1, \cdots, a_m)$ is the corresponding eigenvector. It is positive, since $S(\omega) > 0$ on a set of positive measure, and the other factor in the integrand has only a finite number of zeros. Hence $\mu_m > 0$.

REMARK. If $S(\omega)$ is bounded below by a positive constant $\underline{S}$, we may take $\mu_m = \underline{S}$ for all m.

LEMMA 3. *Let $y(x)$ be an odd, monotonic increasing function of x with $y(x) = O(x^n)$ as $x \to \infty$ for some power n. Let x_1 and x_2 be random variables with mean 0 and variance 1, and a bivariate Gaussian distribution. Let $y_i = y(x_i)$, $i = 1, 2$, and assume $E(y_i^2) = 1$. There is a function $f(z)$ of the complex variable z and an inverse function $F(z)$, depending only on the function $y(x)$, such that $E(y_1y_2) = f\{E(x_1x_2)\}$, $E(x_1x_2) = F\{E(y_1y_2)\}$. f and F are odd functions, analytic in a region of the complex plane containing the open interval $-1 < z < 1$. $f(\pm 1) = \pm 1$, $F(\pm 1) = \pm 1$, and $f(z)$ and $F(z)$ are continuous, increasing functions of real z on $-1 \leqq z \leqq 1$.*

PROOF. Define

$$f(z) = [2\pi(1 - z^2)^{\frac{1}{2}}]^{-1} \int dx_1\, dx_2 y(x_1) y(x_2) \exp[-(x_1^2 + x_2^2 - 2zx_1x_2)/2(1 - z^2)]$$

for $-1 < \operatorname{Re} z < 1$, taking the branch of the square root which is positive for z real. The integral converges uniformly in any compact subset of this strip, hence it defines an analytic function there. Define $f(1) = 1$, $f(-1) = -1$.

Differentiating,

$$
\begin{aligned}
f'(z) &= [2\pi(1 - z^2)^{\frac{1}{2}}]^{-1} \int dx_1\, dx_2 y(x_1) y(x_2) \\
&\quad \cdot [z/(1 - z^2) + (x_1 - zx_2)(x_2 - zx_1)/(1 - z^2)^2] \qquad (7) \\
&\quad \cdot \exp[-(x_1^2 + x_2^2 - 2zx_1x_2)/2(1 - z^2)],
\end{aligned}
$$

and by integration by parts

$$f'(z) = [2\pi(1 - z^2)^{\frac{1}{2}}]^{-1} \int dy(x_1)\, dy(x_2) \exp[-(x_1^2 + x_2^2 - 2zx_1x_2)/2(1 - z^2)],$$

which is positive for $0 < z < 1$. Hence $f(z)$ has an inverse $F(z)$ analytic in a neighborhood of the image under f of $\{-1 < z < 1\}$.

We need to show $\lim_{z\to\pm1} f(z) = \pm1$, taking the limit through real z with $|z| < 1$. In (7), put $x_2 = zx_1 + t(1 - z^2)^{\frac{1}{2}}$:

$$f(z) = (2\pi)^{-1} \int dx_1 y(x_1) \exp[-\tfrac{1}{2}x_1^2] \int dt \exp[-\tfrac{1}{2}t^2] y(zx_1 + t(1 - z^2)^{\frac{1}{2}}).$$

Since $y[zx_1 + t(1 - z^2)^{\frac{1}{2}}] = O(1 + |x_1|^n + |t|^n)$, by the dominated convergence theorem,

$$\lim_{z\to1-} f(z) = (2\pi)^{\frac{1}{2}} \int dx_1 y(x_1) \exp[-\tfrac{1}{2}x_1^2] \int dt \exp[-\tfrac{1}{2}t^2] y(x_1) = 1.$$

Since $f(z)$ is clearly an odd function, $\lim_{z\to-1+} f(z) = -1$. Hence the analytic inverse function $F(z)$ defined above is defined in a region which intersects the real axis on $-1 < z < 1$. Define $F(\pm1) = \pm1$. Then all the conclusions of the lemma follow.

The following hypothesis will be used in several lemmas:

HYPOTHESIS A. (i) $\{x_i\}$ is a Gaussian process with $E(x_i) = 0$, $E(x_i^2) = 1$ for all i, such that for any set of distinct integers $i_1, i_2, \cdots, i_m$ the covariance matrix $[R_x(i_j, i_k)]_{j,k=1,\cdots,m}$ is positive definite with its minimum eigenvalue at least μ_m, a positive constant.

(ii) $y(x)$ is an odd, bounded, nondecreasing function on $-\infty < x < \infty$ with $E[y(x_i)^2] = 1$. The random process $\{y_i\}$ is defined by $y_i = y(x_i)$.

In some of the lemmas, the full strength of the hypothesis is not necessary. For example, in the next lemma, $y(x)$ may be any bounded measurable function.

LEMMA 4. *Assume Hypothesis A, with* $|y(x)| \leqq Y$. *Let* $n_1, \cdots, n_p$ *be integers, of which only* $n_{i_1}, \cdots, n_{i_q}$ *are distinct. Then there is an analytic function* $F_{n_1\cdots n_p}(\{z_{kj}\}_{1\leqq k\leqq j\leqq q})$ *of* $\frac{1}{2}q(q-1)$ *complex variables such that*

$$E(y_{n_1} \cdots y_{n_p}) = F_{n_1\cdots n_p}[\{R_x(n_{i_k}, n_{i_j})\}_{i\leqq k<j\leqq q}]. \tag{8}$$

$F_{n_1\cdots n_p}$ *depends only on the function* $y(x)$ *and the coincidences in the sequence* $n_1, \cdots, n_p$. *It is analytic in the convex region* D_p *formed by the union on the regions*

$$|z_{kj} - \rho_{kj}| < \mu[(\rho_{mn})]/4p, \qquad 1 \leqq k < j \leqq q, \tag{9}$$

where (ρ_{mn}) *is any positive definite symmetric* $q \times q$ *matrix with* 1's *along the diagonal, and* $\mu[(\rho_{mn})]$ *is its minimum eigenvalue. In* D_p,

$$|F_{n_1\cdots n_p}(\{z_{kj}\})| < (2^{\frac{1}{2}}Y)^p. \tag{10}$$

In particular, this inequality is valid if

$$|z_{kj} - R_x(n_{i_k}, n_{i_j})| < \mu_p/4p, \qquad 1 \leqq k < j \leqq q.$$

PROOF. Define the $q \times q$ matrix M by $M_{ii} = 1, i = 1, \cdots, q; M_{ij} = M_{ji} = z_{ij}$, $1 \leqq i < j \leqq q$. Let

$$(11)\quad F_{n_1 \cdots n_p}\{z_{kj}\} = [(2\pi)^{q/2}(\det M)^{\frac{1}{2}}]^{-1} \cdot \int dt_1 \cdots dt_q y(t_1)^{l_1} \cdots y(t_q)^{l_q} \exp\left[-\tfrac{1}{2} \textstyle\sum_{i,j=1}^{q} t_i t_j (M^{-1})_{ij}\right],$$

where l_j is the number of appearances of n_{i_j} among $n_1, \cdots, n_p$. Let $\{\rho_{kj}\}$ be a value of $\{z_{kj}\}$ at which M is positive definite. It will be shown that in the region (9), det $(M) \neq 0$, and the integral in (11) converges uniformly. Since the union of all these regions is a convex set, there is a unique continuation of det $(M)^{\frac{1}{2}}$ throughout D_p, subject to the condition det $(M)^{\frac{1}{2}} > 0$ if M is positive definite. The bound (10) will be established in (9).

Let M_0 be the value of M when $\{z_{kj}\} = \{\rho_{kj}\}$. Then if we set $M = M_0 + M_1$, the elements of M_1 have absolute value less than $\mu/4p$ in (9), where $\mu = \mu[(\rho_{kj})]$. M_0 has a positive definite square root $M_0^{\frac{1}{2}}$. In terms of the norm $\|u\| = (\sum_{j=1}^{q} |u_j|^2)^{\frac{1}{2}}$ of a complex q-vector u, we have $\|M_0^{-\frac{1}{2}}u\| \leqq \mu^{-\frac{1}{2}}\|u\|$, and in (9) $\|M_1 u\| \leqq [\sum_{j=1}^{q} ((\mu/4p) \sum_{k=1}^{q} |u_k|)^2]^{\frac{1}{2}} \leqq (\mu/4)\|u\|$. Hence $\|M_0^{-\frac{1}{2}} M_1 M_0^{-\frac{1}{2}} u\| \leqq \frac{1}{4}\|u\|$. This shows that the expansion

$$M^{-1} = M_0^{-\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n (M_0^{-\frac{1}{2}} M_1 M_0^{-\frac{1}{2}})^n M_0^{-\frac{1}{2}}$$

converges, and

$$\|M_0^{\frac{1}{2}}(M^{-1} - M_0^{-1})M_0^{\frac{1}{2}}u\| = \|\textstyle\sum_{n=1}^{\infty} (-1)^n (M_0^{-\frac{1}{2}} M_1 M_0^{-\frac{1}{2}})^n u\| \leqq \textstyle\sum_{n=1}^{\infty} 4^{-n} \|u\| = \frac{1}{3}\|u\|.$$

Two consequences of this are

$$(12)\quad \operatorname{Re}\{\textstyle\sum_{i,j=1}^{q} u_i (M_0^{\frac{1}{2}} M^{-1} M_0^{\frac{1}{2}})_{ij} u_j\} \geqq \frac{2}{3} \sum_{j=1}^{q} u_j^2$$

for real u, and $\|M_0^{\frac{1}{2}} M_1^{-1} M_0^{\frac{1}{2}} u\| \leqq \frac{4}{3}\|u\|$. By the last inequality,

$$(13)\quad |\det(M_0^{\frac{1}{2}} M_1^{-1} M_0^{\frac{1}{2}})| \leqq (\tfrac{4}{3})^q.$$

In (11), put $t = M_0^{\frac{1}{2}}u$. We have

$$|F_{n_1 \cdots n_p}| \leqq [|\det M^{-1}|^{\frac{1}{2}}/(2\pi)^{q/2}]Y^p \int dt_1 \cdots dt_q \exp\left[-\tfrac{1}{2} \operatorname{Re} \textstyle\sum_{i,j=1}^{q} t_i (M^{-1})_{ij} t_j\right]$$
$$= [|\det(M_0^{\frac{1}{2}} M_1^{-1} M_0^{\frac{1}{2}})|^{\frac{1}{2}}/(2\pi)^{q/2}]Y^p \cdot \int du_1 \cdots du_q \exp\left[-\tfrac{1}{2} \operatorname{Re} \textstyle\sum_{i,j=1}^{q} u_i (M_0^{\frac{1}{2}} M_1^{-1} M_0^{\frac{1}{2}})_{ij} u_j\right].$$

By (12) and (13), the integral converges uniformly in (9), and

$$|F_{n_1 \cdots n_p}| \leqq [(\tfrac{4}{3})^{q/2}/(2\pi)^{q/2}]Y^p \int du_1 \cdots du_q \exp(-\tfrac{1}{3} \textstyle\sum_{i=1}^{q} u_i^2) = 2^{q/2} Y^p \leqq (2^{\frac{1}{2}} Y)^p.$$

LEMMA 5. *Assume Hypothesis A. Let p be a fixed positive integer, and let n_j, m_j, $j = 1, \cdots, p$, be integers, with $m_j > n_j$. Then*

$$E\{\textstyle\prod_{j=1}^{p} [y_{n_j} y_{m_j} - E(y_{n_j} y_{m_j})]\} = O[\sum |\pi^{(p)}|]$$

where the sum is over all products $\pi^{(p)}$ *of correlations* $R_x(l_1, l_2)$ *with* l_1, l_2 *taken from* $\{n_1, \cdots, m_p\}$ *and* $l_1 \neq l_2$, *which have the following properties, and are minimal in this respect:*

(i) *each of the* $2p$ *letters appears an odd number of times;*

(ii) *for each pair* (n_j, m_j), *there is an even number of factors, at least two, with exactly one index in the pair* (n_j, m_j).

PROOF. First assume that $n_1, \cdots, m_p$ are all distinct. Expand the product $P = \prod [y_{n_j}y_{m_j} - E(y_{n_j}y_{m_j})]$ into a sum of products of y_k's and expectations. Taking the expected value term by term, by Lemma 4, $E(P)$ is the value of a function $G_{n_1\cdots m_p}(\{z_{kj}\})$ which is analytic in the region D_{2p}, with $|G_{n_1\cdots m_p}| < 2^p(2^{\frac{1}{2}}Y)^{2p}$. Lemma 1 will be applied to this function. A product which is minimal with respect to (i) and (ii) has degree less than $3\binom{2p}{2}$, since no correlation may appear more than three times. Hence it suffices to show that in the expansion of $G_{n_1\cdots m_p}$ about the origin, every term has properties (i) and (ii). For simplicity, we may consider the corresponding expansion of $E(P)$, which is valid if all the correlations are small.

Replacing any one of the variables y_{n_j} (or y_{m_j}) by its negative changes the sign of $E(P)$. This may be accomplished by replacing x_{n_j} by $-x_{n_j}$, which changes the sign of $R_x(n_j, l)$ for $l \neq n_j$. Hence $E(P)$ is odd in these correlations, which shows that each term has the property (i).

Now, for a given $j \leqq p$ consider the correlations $R_x(n_j, l)$, $R_x(m_j, l)$ for $l \neq n_j$ or m_j. Replacing x_{n_j} and x_{m_j} by $-x_{n_j}$ and $-x_{m_j}$ changes the signs of these correlations, and leaves $E(P)$ unchanged. Hence $E(P)$ is an even function of them. Also, if all of these correlations were zero, we would have

$$E(P) = E[y_{n_j}y_{m_j} - E(y_{n_j}y_{m_j})]E\{\textstyle\prod_{k\neq j} [y_{n_k}y_{m_k} - E(y_{n_k}y_{m_k})]\}$$
$$= 0.$$

Hence each term is of positive degree in $\{R_x(n_j, l), R_x(m_j, l), l \neq n_j \text{ or } m_j\}$, so that (ii) is satisfied.

If the values of $n_1, \cdots, m_p$ are not all distinct, the above procedure shows that

$$E(P) = O[\textstyle\sum |(\pi')^{(p)}|],$$

where, instead of (i) and (ii), each product satisfies the following properties:

(i′) Any number which is the value of an odd (even) number of subscripts appears an odd (even) number of times.

(ii′) Each pair (n_j, m_j) which is distinct from the other subscripts satisfies (ii).

In one of the factors $R_x(q_1, q_2)$ of one of the products $(\pi')^{(p)}$, there is no unique way of assigning one of the letters $n_1, \cdots, m_p$ to q_1, unless q_1 is the value of only one letter. Suppose that this assignment is made in any permissible way. Then $(\pi')^{(p)}$ can be modified, without changing its value, so that it satisfies (i) and (ii).

First (i) will be satisfied. Let $l_1, \cdots, l_m$ be a set of equal letters. For $1 \leqq j \leqq m - 1$, insert into $(\pi')^{(p)}$ the factor $R_x(l_j, l_m)(= 1)$ if necessary to make l_j

appear an odd number of times. Then by (i′), l_m appears an odd number of times. Applying this procedure to each set of equal letters, (i) is satisfied.

(i) implies that for each j, $(\pi')^{(p)}$ contains an even number of the factors $R_x(n_j, l)$, $R_x(m_k, l)$, with $l \neq n_j$ or m_j. If this number is zero, then by (ii′) one of the letters, e.g., n_j, is equal to another letter l_1. Insert into $(\pi')^{(p)}$ the factor $R_x(n_j, l_1)^2$. If this is done for each j, (ii) is satisfied.

LEMMA 6. *Assume Hypothesis A. Let* $\{x_i\}$ *be stationary, with a spectral density* $S(\omega)$, $|\omega| \leqq \pi$, *which is in* L^2. *Let*

$$R_y(k) = E(y_n y_{n+k}),$$
$$\hat{R}_y(k) = N^{-1} \textstyle\sum_{n=1}^N y_n y_{n+k}.$$

Let p be a fixed positive integer. Then for large N and arbitrary positive $k_1, \cdots, k_p$,

$$E\{\textstyle\prod_{j=1}^p [\hat{R}_y(k_j) - R_y(k_j)]\} = O(N^{-p/2}).$$

PROOF. We have $\hat{R}_y(k_j) - R_y(k_j) = N^{-1} \sum_{n=1}^N [y_n y_{n+k_j} - E(y_n y_{n+k_j})]$. Hence

$$E\{\textstyle\prod_{j=1}^p [\hat{R}_y(k_j) - R_y(k_j)]\}$$
$$= N^{-p} \textstyle\sum_{n_1, \cdots, n_p=1}^N E\{\prod_{j=1}^p [y_{n_j} y_{n_j+k_j} - E(y_{n_j} y_{n_j+k_j})]\}.$$

Applying Lemma 5, with $m_j = n_j + k_j$, this is $O(\sum_{\pi^{(p)}} N^{-p} \sum_{n_1, \cdots, n_p=1}^N |\pi^{(p)}|)$. The number of products $\pi^{(p)}$ depends only on p. Hence it suffices to show $N^{-p} \sum_{n_1, \cdots, n_p=1}^N |\pi^{(p)}| = O(N^{-p/2})$. Since $S(\omega)$ is in L^2, $\sigma = \sum_{-\infty}^{\infty} |R_x(k)|^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} S(\omega)^2 \, d\omega < \infty$. For $1 \leqq j \leqq p$, if l_1, l_3 are selected from n_j, m_j, and l_2, l_4 from the other subscripts, by Schwarz' inequality

$$(14) \quad \textstyle\sum_{n_j=1}^N |R_x(l_1 - l_2) R_x(l_3 - l_4)| \leqq [\sum_{n_j=1}^N R_x(l_1 - l_2)^2 \sum_{n_j=1}^N R_x(l_3 - l_4)^2]^{\frac{1}{2}} \leqq \sigma,$$

and similarly

$$(15) \quad \textstyle\sum_{n_j=1}^N |R_x(l_1 - l_2)| \leqq \sigma^{\frac{1}{2}} N^{\frac{1}{2}}.$$

Let some of the factors of $\pi^{(p)}$ be removed, if necessary, to make it a product π' which is a minimal product with the property that for each j, the total number of appearances of n_j and m_j is at least two, and they do not appear in the combination $R_x(n_j - m_j)$. Note that the degree of π' is at least p.

Consider $\sum_{n_1, \cdots, n_p=1}^N |\pi'|$. For any j, if n_j or m_j occur in exactly two factors of π' we may estimate the sum from above by summing over n_j and applying (14). If there is only one such factor, (15) may be applied. This gives a sum over less indices of a product of lower degree. Apply this procedure repeatedly, until we arrive at a product in which, for any j such that n_j or m_j appears, they appear at least three times. By the minimal property of π', there are no factors remaining.

Let ν_2 be the number of times (14) was applied, ν_1 the number of times (15) was applied. Then

$$\textstyle\sum_{n_1, \cdots, n_p=1}^N |\pi'| \leqq \sigma^{\nu_2} (\sigma N)^{\frac{1}{2}\nu_1} N^{p - \nu_1 - \nu_2} = \sigma^{\frac{1}{2}(\nu_2 + \nu_1)} N^{p - \frac{1}{2}(\nu_2 + \nu_1)}.$$

The degree of π' is $\nu_1 + 2\nu_2 \geqq p$. Also, $\nu_1 + \nu_2 \leqq p$. Hence

$$N^{-p} \sum_{n_1,\cdots,n_p=1}^{N} |\pi^{(p)}| \leqq N^{-p} \sum_{n_1,\cdots,n_p=1}^{N} |\pi'|$$
$$\leqq \sigma^{\frac{1}{2}(\nu_2+\nu_1)} N^{-\frac{1}{2}(\nu_2+\nu_1)} \leqq \sigma^p N^{-p/2}.$$

LEMMA 7. *In Lemma* 6, *if we add the condition*

$$\sum_{-\infty}^{\infty} |R_x(k)| < \infty,$$

then

$$E\{\prod_{j=1}^{p} [\hat{R}_y(k_j) - R_y(k_j)]\} = O\{N^{-[(p+1)/2]}\},$$

where $[(p + 1)/2]$ *denotes the integral part of* $(p + 1)/2$.

PROOF. In the proof of Lemma 6, whenever a factor $N^{\frac{1}{2}}$ was introduced into the estimate by the use of (15), instead, $\sum_{n_j=1}^{N} |R_x(l_1 - l_2)| = O(1)$ could be used. Thus, in the estimate $O(N^{-p/2})$, if $p/2$ is not an integer, we may replace it by the next larger integer $[(p + 1)/2]$.

LEMMA 8. *Under the hypotheses of Lemma* 6, *for fixed* m *and* δ, *and arbitrary* $k > 0$,

$$\Pr [|\hat{R}_y(k) - R_y(k)| > \delta] = O(N^{-m}).$$

PROOF. Take $p = 2m$ in Lemma 6, and $k_1, \cdots, k_p = k$. We obtain

$$E[|\hat{R}_y(k) - R_y(k)|^{2m}] = O(N^{-m}).$$

This expected value is at least as large as $\delta^{2m} \Pr [|\hat{R}_y(k) - R_y(k)| > \delta]$. Hence the result follows.

THEOREM 1. *Let* $G(z_1, \cdots, z_m)$ *be analytic in a domain of complex m-space which contains the set* $-1 < z_j < 1, j = 1, \cdots, m$, *and let* $G(z_1, \cdots, z_m)$ *be defined and bounded when all the* z_j*'s are real.*

Let $\{x_i\}$ *be a stationary Gaussian process with mean* 0 *and variance* 1, *and a spectral density* $S(\omega) \varepsilon L^2$.

Let $y(x)$ *be an odd, bounded, nondecreasing function with* $E[y(x_i)^2] = 1$. *Put*

$$y_i = y(x_i),$$
$$R_y(k) = E(y_n y_{n+k}),$$
$$\hat{R}_y(k) = N^{-1} \sum_{n=1}^{N} y_n y_{n+k}.$$

Then for fixed p, large N, and arbitrary $k_1, \cdots, k_m > 0$,

$$E\{G[\hat{R}_y(k_1), \cdots, \hat{R}_y(k_m)]\} = G[R_y(k_1), \cdots, R_y(k_m)] + \sum_{2 \leqq q_1+\cdots+q_m \leqq p} a_{q_1\cdots q_m} E\{\prod_{j=1}^{m} [\hat{R}_y(k_j) - R_y(k_j)]^{q_j}\} + O[N^{-(p+1)/2}], \tag{16}$$

where

$$a_{q_1\cdots q_m} = (q_1 ! \cdots q_m !)^{-1} (\partial/\partial z_1)^{q_1} \cdots (\partial/\partial z_m)^{q_m} G(z_1, \cdots, z_m) \,|_{z_j=R_y(k_j), j=1,\cdots,m}.$$

PROOF. Let μ_2 be the quantity of Lemma 2. Then for $k > 0$, and real t_1, t_2,

$$t_1^2 + t_2^2 + 2t_1t_2R_x(k) \geqq \mu_2(t_1^2 + t_2^2).$$

This implies $|R_x(k)| \leqq 1 - \mu_2$. By Lemma 3, $|R_y(k)| \leqq f(1 - \mu_2) < 1$.

For sufficiently small ρ, the set of $(z_1, \cdots, z_m)$ such that $|z_j - \zeta_j| < \rho$, $j = 1, \cdots, m$, for some $\zeta_1, \cdots, \zeta_m$ with $-f(1 - \mu_2) \leqq \zeta_j \leqq f(1 - \mu_2)$, $j = 1, \cdots, m$, lies in the region of analyticity of $G(z_1, \cdots, z_m)$. Then for any $k_1, \cdots, k_m$, the set $|z_j - R_y(k_j)| < \rho$, $j = 1, \cdots, m$, lies in this region.

Suppose

$$|z_j - R_y(k_j)| \leqq \rho/2, \qquad j = 1, \cdots, m. \tag{17}$$

Then

$$G(z_1, \cdots, z_m) = G[R_y(k_1), \cdots, R_y(k_m)]$$
$$+ \sum_{1 \leqq q_1 + \cdots + q_m \leqq p} a_{q_1 \cdots q_m} \prod_{j=1}^{m} [z_j - R_y(k_j)]^{q_j} + O[\sum_{j=1}^{m} |z_j - R_y(k_j)|^{p+1}].$$

By Lemma 8, the probability that (17) is not satisfied by $z_j = \hat{R}_y(k_j)$ for all j is $O[N^{-(p+1)/2}]$. Hence

$$\begin{aligned} E\{G[\hat{R}_y(k_1), \cdots, \hat{R}_y(k_m)] - G[R_y(k_1), \cdots, R_y(k_m)] \\ - \sum_{1 \leqq q_1 + \cdots + q_m \leqq p} a_{q_1 \cdots q_m} \prod_{j=1}^{m} [\hat{R}_y(k_j) - R_y(k_j)]^{q_j}\} \\ = O\{N^{-(p+1)/2} + \sum_{j=1}^{m} E[|\hat{R}_y(k_j) - R_y(k_j)|^{p+1}]\}. \end{aligned} \tag{18}$$

By Schwarz's inequality and Lemma 6,

$$\begin{aligned} E[|\hat{R}_y(k_j) - R_y(k_y)|^{p+1}] &\leqq (E\{[\hat{R}_y(k_j) - R_y(k_j)]^{2p+2}\})^{\frac{1}{2}} \\ &= O[N^{-(p+1)/2}]. \end{aligned}$$

Thus, taking the expected value term by term on the left in (18) yields the desired result, for the terms with $\sum q_j = 1$ have expectation 0.

THEOREM 1a. *If in addition to the hypotheses of Theorem* 1, *we have*

$$\sum_{-\infty}^{\infty} |R_x(k)| < \infty,$$

the error term in (16) *is* $O[N^{-[p/2]-1}]$.

PROOF. As in Theorem 1, using Lemma 7 instead of Lemma 6.

3. The main theorem.

LEMMA 9. *Assume Hypothesis A.*

(i) $E(y_{n_1} y_{n_2} y_{n_3} y_{n_4}) = R_y(n_1, n_2) R_y(n_3, n_4) + R_y(n_1, n_3) R_y(n_2, n_4) + R_y(n_1, n_4) R_y(n_2, n_3) + O(\sum |\pi_3|)$,

where the sum is over all products π_3 *of three distinct correlations* $R_x(n_j - n_{j'})$ *with* $j \neq j'$ *such that each index* $n_1, \cdots, n_4$ *occurs in the product.*

(ii) *Let* $v_j = y_{n_j} y_{m_j} - E(y_{n_j} y_{m_j})$, $j = 1, \cdots, 4$. *Then*

$$E(v_1 v_2 v_3 v_4) = E(v_1 v_2) E(v_3 v_4 + E(v_1 v_3) E(v_2 v_4) + E(v_1 v_4) E(v_2 v_3) + O(\sum |\pi_3^*|),$$

where the sum is over all products π_3^* *of three distinct correlations* $R_x(q_j, q_{j'})$ *with* $q_j = n_j$ *or* m_j, $q_{j'} = n_{j'}$ *or* $m_{j'}$, *such that the three pairs* (j, j') *include* 1, 2, 3 *and* 4.

PROOF. (i) Assume first that $n_1, \cdots, n_4$ are distinct. Consider the expansion of $E(y_1y_2y_3y_4)$ in a power series when the correlations are all small. Since $y(x)$ is odd, each term involves every subscript. Hence the only terms not divisible by one of the products π_3 are those which contain only two distinct correlations. There are three possible choices for these correlations, namely $R_x(n_1, n_2)$ and $R_x(n_3, n_4)$, and the two other pairs obtained by permutation of the indices. The same applies to the quantity

$$Q = E(y_{n_1}y_{n_2}y_{n_3}y_{n_4}) - E(y_{n_1}y_{n_2})E(y_{n_3}y_{n_4}) - E(y_{n_1}y_{n_3})E(y_{n_2}y_{n_4}) - E(y_{n_1}y_{n_4})E(y_{n_2}y_{n_3}).$$

But if all correlations are zero except $R_x(n_1, n_2)$ and $R_x(n_3, n_4)$, or one of the similar pairs, this is zero. Hence (i) is true by Lemma 1.

If $n_1 = n_2 \neq n_3, n_2 \neq n_4 \neq n_3$,

$$Q = E(y_{n_1}^2y_{n_3}y_{n_4}) - E(y_{n_3}y_{n_4}) - 2E(y_{n_1}y_{n_3})E(y_{n_1}y_{n_4}).$$

For small values of $R_x(n_1, n_3)$, $R_x(n_1, n_4)$, $R_x(n_3, n_4)$, Q may be expanded in a power series in these correlations. Consider first the terms which involve only one of them. The terms involving only $R_x(n_1, n_3)$ give the value of Q when $R_x(n_1, n_4) = R_x(n_3, n_4) = 0$, which is zero. Hence there are no such terms. Similarly, there are no terms which do not contain at least two distinct correlations. Thus by Lemma 1

$$Q = O[|R_x(n_1, n_3)R_x(n_1, n_4)| + |R_x(n_1, n_3)R_x(n_3, n_4)| + |R_x(n_1, n_4)R_x(n_3, n_4)|].$$

Inserting the factor $R_x(n_1, n_2)(=1)$ in each term gives $O(\sum|\pi_3|)$.

The cases with more than one pair of equal subscripts may be handled similarly.

(ii) The proof of (ii) is analogous to that of (i). Only the case of distinct subscripts will be considered.

In the expansion of

$$Q^* = E(v_1v_2v_3v_4) - E(v_1v_2)E(v_3v_4) - E(v_1v_3)E(v_2v_4) - E(v_1v_4)E(v_2v_3),$$

every term which is not divisible by one of the products π_3^* contains only correlations which may be put into two classes: after a permutation of subscripts, one class of correlations depends on n_1, m_1, n_2, m_2, the other class depends on n_3, m_3, n_4, m_4. But setting all correlations zero which are not in one of these classes makes $Q^* = 0$, since then $E(v_1v_2v_3v_4) = E(v_1v_2)E(v_3v_4)$, $E(v_1v_3) = E(v_1v_4) = 0$. Hence there are no such terms. The result follows by Lemma 1.

LEMMA 10. *Assume Hypothesis A, with $\{x_i\}$ stationary and $\sum_{-\infty}^{\infty} |R_x(k)| < \infty$. Let*

$$R_y(k) = E(y_ny_{n+k}),$$
$$\hat{R}_y(k) = N^{-1} \sum_{n=1}^{N} y_ny_{n+k}.$$

Then for arbitrary N, $K > 0$,

$$(19)\quad \sum_{k,l=1}^{K} E\{[\hat{R}_y(k) - R_y(k)]^3[\hat{R}_y(l) - R_y(l)]\} = O(K/N^2 + K^2/N^3),$$

for $m = 3$ *or* 4 *and* $0 \leqq t \leqq m$,

$$(20)\quad \sum_{k,l=1}^{K} [|R_y(k)| + |R_y(l)|]E\{[\hat{R}_y(k) - R_y(k)]^t[\hat{R}_y(l) - R_y(l)]^{m-t}\} = O(K/N^2),$$

if $|a_k|$, $|b_k| \leqq 1$, $k = 1, \cdots, K$,

$$(21)\quad \begin{aligned} &\sum_{k,l=1}^{K} a_k b_l E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\} \\ &= \sum_{k,l=1}^{K} a_k b_l N^{-2} \sum_{n_1,n_2=1}^{N} [R_y(n_1 - n_2)R_y(n_1 - n_2 + k - l) \\ &\quad + R_y(n_1 - n_2 + k)R_y(n_1 - n_2 - l)] + O(N^{-1}), \end{aligned}$$

$$(22)\quad \sum_{k,l=1}^{K} [|R_y(k)| + |R_y(l)|]E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\} = O(N^{-1}),$$

$$(23)\quad \sum_{k,l=1}^{K} |R_y(k)R_y(l)|E\{[\hat{R}_y(k) - R_y(k)]^2\} = O(N^{-1}),$$

and

$$(24)\quad \sum_{k=1}^{K} |R_y(k)|E\{[\hat{R}_y(k) - R_y(k)]^2\} = O(N^{-1}).$$

Proof. By Lemma 9, (ii), with $m_j = n_j + k, j = 1, 2, 3, m_4 = n_4 + l$,

$$\begin{aligned} &E\{[\hat{R}_y(k) - R_y(k)]^3[\hat{R}_y(l) - R_y(l)]\} \\ &= N^{-4} \sum_{n_1,\cdots,n_4=1}^{N} E\{\prod_{j=1}^{4} [y_{n_j}y_{m_j} - E(y_{n_j}y_{m_j})]\} \\ &= 3E\{[\hat{R}_y(k) - R_y(k)]^2\}E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\} \\ &\quad + N^{-4} \sum_{n_1,\cdots,n_4=1}^{N} O(\sum |\pi_3^*|), \end{aligned}$$

where each product π_3^* is a product of three distinct correlations involving different n_j's, such that $n_1, \cdots, n_4$ all occur. By summing in the proper order, using estimates such as

$$(25)\quad \sum_{n_1=1}^{N} |R_x(n_1 - n_2)| \leqq \sum_{-\infty}^{\infty} |R_x(n_1)| = O(1),$$

we find that for any such product, $N^{-4} \sum_{n_1,\cdots,n_4=1}^{N} |\pi_3^*| = O(N^{-3})$. By the application of Lemma 7,

$$(26)\quad \begin{aligned} &E\{[\hat{R}_y(k) - R_y(k)]^3[\hat{R}_y(l) - R_y(l)]\} \\ &= O(N^{-3} + N^{-1}|E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\}|). \end{aligned}$$

We have

$$\begin{aligned} E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\} = N^{-2} \sum_{n_1,n_2=1}^{N} &[E(y_{n_1}y_{n_1+k}y_{n_2}y_{n_2+l}) \\ &- E(y_{n_1}y_{n_1+k})E(y_{n_2}y_{n_2+l})]. \end{aligned}$$

Hence by Lemma 9, (i),

$$(27)\quad \begin{aligned} &\sum_{k,l=1}^{K} |E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\}| \\ &\leqq N^{-2} \sum_{k,l=1}^{K} \sum_{n_1,n_2=1}^{N} \{|R_y(n_1 - n_2)R_y(n_1 - n_2 + k - l)| \\ &\quad + |R_y(n_1 - n_2 + k)R_y(n_1 - n_2 - l)| + O(\sum |\pi_3|)\}, \end{aligned}$$

where each product π_3 is the product of three distinct correlations of pairs of the variables x_{n_1}, x_{n_1+k}, x_{n_2}, x_{n_2+l}, such that all subscripts occur. By summing in the proper manner, using estimates such as (25),

$$N^{-2} \sum_{k,l=1}^{K} \sum_{n_1,n_2=1}^{N} |\pi_3| = O(N^{-1}).$$

The contribution of the first two terms in the sum on the right in (27) is $O(KN^{-1})$, since $R_y(j) = O[|R_x(j)|]$. Hence

$$\sum_{k,l=1}^{K} |E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\}| = O(K/N),$$

and summing (26) yields (19). Similarly, an equation analogous to (27) shows that (21) is true.

For the left side of (22), we have instead of (27),

$$\sum_{k,l=1}^{K} [|R_y(k)| + |R_y(l)|]|E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\}|$$
$$= O\{N^{-2} \sum_{k,l=1}^{K} \sum_{n_1,n_2=1}^{N} [|R_y(k)| + |R_y(l)|][|R_y(n_1 - n_2)R_y(n_1 - n_2 + k - l)|$$
$$+ |R_y(n_1 - n_2 + k)R_y(n_1 - n_2 - l)| + \sum |\pi_3|]\}.$$

Since $R_y(j) = O[|R_x(j)|]$, each R_y on the right may be replaced by R_x. The term $\sum |\pi_3|$ contributes $O(N^{-1})$ as before. Multiplying out the remaining factors, we get a sum of terms of the type π_3. Hence (22) is true.

The remaining relations, (20), (23), and (24) depend only on Lemma 7:

$$E\{[\hat{R}_y(k) - R_y(k)]^t[\hat{R}_y(l) - R_y(l)]^{m-t}\} = O(N^{-2}), \qquad m = 3, 4,$$
$$E\{[\hat{R}_y(k) - R_y(k)]^2\} = O(N^{-1}),$$

and

$$\sum_{-\infty}^{\infty} |R_y(k)| < \infty.$$

Theorem 2. *Let $G(z)$ and $H(z)$ be odd functions, analytic in a region including the interval $-1 < z < 1$ and defined and bounded for all real z. Let a_k, b_k, $k = 1, 2, \cdots$, be numbers of absolute value at most* 1.

Let $\{x_i\}$ be a stationary Gaussian process with mean 0 *and variance* 1, *with $\sum_{-\infty}^{\infty} |R_x(k)| < \infty$.*

Let $y(x)$ be an odd, bounded, nondecreasing function with $E\{y(x_i)^2\} = 1$. Let

$$y_i = y(x_i),$$
$$R_y(k) = E(y_n y_{n+k}),$$
$$\hat{R}_y(k) = N^{-1} \sum_{n=1}^{N} y_n y_{n+k}.$$

Then for arbitrary positive integers K, N,

$$E\{\sum_{k=1}^{K} a_k G[\hat{R}_y(k)]\} = \sum_{k=1}^{K} a_k G[R_y(k)] + O(N^{-1} + KN^{-2}), \tag{28}$$

$$E\{\sum_{k,l=1}^{K} a_k b_l G[\hat{R}_y(k)] H[\hat{R}_y(l)]\} = \sum_{k,l=1}^{K} a_k b_l G[R_y(k)] H[R_y(l)]$$
$$+ G'(0)H'(0) \sum_{k,l=1}^{K} a_k b_l N^{-2} \sum_{n_1,n_2=1}^{N} [R_y(n_1 - n_2)R_y(n_1 - n_2 + k - l) \tag{29}$$
$$+ R_y(n_1 - n_2 + k)R_y(n_1 - n_2 - l)] + O(N^{-1} + K^2N^{-3}).$$

PROOF. Apply Theorem 1a to the function $G(z_1)H(z_2)$. For $p = 4$ we have

$$\begin{aligned} & E\{G[\hat{R}_y(k)]H[\hat{R}_y(l)]\} \\ (30) \quad & = G[R_y(k)]H[R_y(l)] + \textstyle\sum_{m=2}^{4}\sum_{t=0}^{m} \{G^{(t)}[R_y(k)]H^{(m-t)}[R_y(l)]/t!\,(m-t)!\} \\ & \cdot E\{[\hat{R}_y(k) - R_y(k)]^t[\hat{R}_y(l) - R_y(l)]^{m-t}\} + O(N^{-3}). \end{aligned}$$

Similarly,

$$(31) \quad E\{G[\hat{R}_y(k)]\} = G[R_y(k)] + \tfrac{1}{2}G''[R_y(k)]E\{[\hat{R}_y(k) - R_y(k)]^2\} + O(N^{-2}).$$

The expressions on the left in (28) and (29) may be expressed in terms of (30) and (31). The O-terms in these equations contribute $O(K^2N^{-3})$ to (29) and $O(KN^{-2})$ to (28). The contributions of the other terms will now be investigated.

For any derivative of G (or H) which occurs, we have

$$\begin{aligned} G^{(t)}[R_y(k)] &= G^{(t)}(0) + O[|R_y(k)|], && t \text{ odd}, \\ &= O[|R_y(k)|], && t \text{ even}, \end{aligned}$$

since $G(z)$ is odd. Eliminate the derivatives in (30) in this way, multiply by $a_k b_l$, and sum over k and l. By Lemma 10, Equations (19)–(23), the result is (29).

Similarly, (28) follows from (31) by the use of (24).

LEMMA 11. *Let $g(\omega)$, $|\omega| \leqq \pi$, have the Fourier series*

$$g(\omega) = \textstyle\sum_{k=-\infty}^{\infty} a_k e^{ik\omega}.$$

Let $f(z)$ be analytic in $|z| < \rho$, with $f(0) = f'(0) = f''(0) = 0$. Let b_k, $-\infty < k < \infty$, be such that $b_k = f(a_k)$ if $|a_k| < \rho$. If $g(\omega)$ has bounded variation, and $\sum_{-\infty}^{\infty} |a_k| < \infty$, then the function $h(\omega) = \sum_{k=-\infty}^{\infty} b_k e^{ik\omega}$ has a continuous second derivative.

PROOF. Choose r between 0 and ρ. There are at most a finite number of values of k for which $|a_k| > r$, and removing the corresponding terms from the series for $g(\omega)$ and $h(\omega)$ cannot affect the conclusion. Hence we may assume that $|a_k| \leqq r$ for all k. Then $b_k = O[|a_k|^3]$.

For $k \neq 0$,

$$|a_k| = |(2\pi)^{-1} \textstyle\int_{-\pi}^{\pi} e^{ik\omega} g(\omega)\, d\omega| = |(2\pi k)^{-1} \int_{-\pi}^{\pi} e^{ik\omega}\, dg(\omega)| \leqq (2\pi k)^{-1} \int_{-\pi}^{\pi} |dg(\omega)|.$$

Hence $a_k = O(k^{-1})$, and $k^2 b_k = O(k^2|a_k|^3) = O(|a_k|)$. This shows that $-\sum k^2 b_k e^{ik\omega}$ is a continuous function, for the series converges uniformly. $h(\omega)$ is obtained by integrating twice.

THEOREM 3. *Let $\{x_i\}$ be a stationary Gaussian process with $E(x_i) = 0$, $E(x_i^2) = 1$,*

$$\textstyle\sum_{-\infty}^{\infty} |R_x(k)| < \infty,$$

and a spectral density $S(\omega)$, $|\omega| \leqq \pi$, which is a function of bounded variation.

Let $y(x)$ be an odd, bounded, nondecreasing function on $-\infty < x < \infty$ such that $E[y(x_i)^2] = 1$. Define $y_i = y(x_i)$ and

$$\hat{R}_y(k) = N^{-1} \sum_{n=1}^{N} y_n y_{n+k} .$$

Let $S_y(\omega)$ be the spectral density of $\{y_i\}$.

Let $F(z)$ be the function of Lemma 3, with its definition extended to all real z so that $F(z)$ is bounded for z real.

Let $\{c_k, k = 1, \cdots, K\}$ *be a nonincreasing sequence of numbers with* $c_1 \leqq 1$, $c_K \geqq 0$.

Define

$$\hat{S}(\omega) = 1 + 2 \sum_{k=1}^{K} c_k \cos k\omega F[\hat{R}_y(k)],$$

$$S_K(\omega) = 1 + 2 \sum_{k=1}^{K} c_k \cos k\omega R_x(k).$$

Then for any positive integers K, N with $K \leqq N$,

$$E[\hat{S}(\omega)] = S_K(\omega) + O(N^{-1}) \tag{32}$$

and for $0 < \omega < \pi$,

$$E[\hat{S}(\omega)^2] = S_K(\omega)^2 + (2\gamma K/N)F'(0)^2 S_y(\omega)^2 + \mathcal{E}, \tag{33}$$

where $\gamma = K^{-1} \sum_{k=1}^{K} c_k^2$ *and $\mathcal{E}$ satisfies the following bounds:*

(i) *for ω such that*

$$S(\omega') = S(\omega) + O(|\omega - \omega'|^{\alpha}), \tag{34}$$

where $0 < \alpha \leqq 1$,

$$\mathcal{E} = O(K^{1-\alpha} N^{-1}), \qquad 0 < \alpha < 1, \tag{35}$$

$$\mathcal{E} = O[(1 + \log K)N^{-1}], \qquad \alpha = 1. \tag{36}$$

(ii) *If ω is such that the derivative $S'(\omega)$ exists, and*

$$S(\omega') = S(\omega) + (\omega' - \omega)S'(\omega) + O(|\omega' - \omega|^{1+\beta}), \qquad \beta > 0, \tag{37}$$

$$\mathcal{E} = O(N^{-1}). \tag{38}$$

Proof. By Lemmas 3 and 11, $S_y(\omega)$ satisfies all the hypotheses for $S(\omega)$.

In Theorem 2, take $a_k = b_k = c_k \cos k\omega$, $G(z) = H(z) = F(z)$. Using the series for $\hat{S}(\omega)$, we find that (32) is true for $K \leqq N$, and

$$\begin{aligned} E[\hat{S}(\omega)^2] = S_K(\omega)^2 &+ 4F'(0)^2 \sum_{k,l=1}^{K} c_k c_l \cos k\omega \cos l\omega \\ &\cdot N^{-2} \sum_{n_1,n_2=1}^{N} [R_y(n_1 - n_2)R_y(n_1 - n_2 + k - l) \\ &+ R_y(n_1 - n_2 + k)R_y(n_1 - n_2 - l)] + O(N^{-1}). \end{aligned}$$

This sum may be expressed in terms of $S_y(\omega)$ by the relation

$$R_y(j) = (2\pi)^{-1} \int_{-\pi}^{\pi} d\omega' S_y(\omega') e^{\pm ij\omega'}.$$

We find that (33) is true, with

$$\begin{aligned} \mathcal{E} = &-(2\gamma K/N)F'(0)^2 S_y(\omega)^2 \\ &+ (F'(0)^2/\pi^2) \int\int d\omega' \, d\omega'' S_y(\omega')S_y(\omega'')\mathcal{K}(\omega, \omega', \omega'') + O(N^{-1}), \end{aligned}$$

where

$$\mathcal{K}(\omega, \omega', \omega'') = \sum_{k,l=1}^{K} c_k c_l \cos k\omega \cos l\omega \cdot N^{-2} \sum_{n_1,n_2=1}^{N} \{\exp [i\omega'(n_1 - n_2) - i\omega''(n_1 - n_2 + k - l)] + \exp [i\omega'(n_1 - n_2 + k) - i\omega''(n_1 - n_2 - l)]\}.$$

Integrating term by term

$$\iint d\omega' \, d\omega'' \mathcal{K}(\omega, \omega', \omega'') = (4\pi^2/N) \sum_{k=1}^{K} c_k^2 \cos^2 k\omega = (2\pi^2/N) \sum_{k=1}^{K} c_k^2 + (2\pi^2/N) \sum_{k=1}^{K} c_k^2 \cos 2k\omega.$$

By the second mean value theorem ([1], p. 256) since c_k^2 is monotonic, for some index $K' \leqq K$,

$$\sum_{k=1}^{K} c_k^2 \cos 2k\omega = c_1^2 \sum_{k=1}^{K'} \cos 2k\omega + c_K^2 \sum_{k=K'+1}^{K} \cos 2k\omega + O(1) = O(1)$$

for $0 < \omega < \pi$. Hence

$$\iint d\omega' \, d\omega'' \mathcal{K}(\omega, \omega', \omega'') = (2\pi^2 K/N)\gamma + O(N^{-1})$$

and

$$\mathcal{E} = [F'(0)^2/\pi^2] \iint d\omega' \, d\omega'' [S_y(\omega')S_y(\omega'') - S_y(\omega)^2]\mathcal{K}(\omega, \omega', \omega'') + O(N^{-1}).$$

Let $d_k = c_k - c_{k+1}$, $k \leqq K - 1$, $d_K = c_K$. Then

$$\mathcal{K}(\omega, \omega', \omega'') = \sum_{j,m=1}^{K} d_j \, d_m \mathcal{K}_{jm}(\omega, \omega', \omega''),$$

where

$$\mathcal{K}_{jm}(\omega, \omega', \omega'') = \sum_{k=1}^{j} \sum_{l=1}^{m} \cos k\omega \cos l\omega \cdot N^{-2} \sum_{n_1,n_2=1}^{N} \{\exp [i\omega'(n_1 - n_2) - i\omega''(n_1 - n_2 + k - l)] + \exp [i\omega'(n_1 - n_2 + k) - i\omega''(n_1 - n_2 - l)]\}. \tag{39}$$

By virtue of the identities

$$\sum_{n_1,n_2=1}^{N} \exp [i(\omega' - \omega'')(n_1 - n_2)] = \sin^2 (N/2)(\omega' - \omega'')/\sin^2 \tfrac{1}{2}(\omega' - \omega''),$$

$$2 \sum_{k=1}^{K} \cos k\omega e^{i\omega_1 k} = \exp \{i[(K + 1)/2](\omega_1 - \omega)\} \cdot \sin (K/2)(\omega_1 - \omega)/\sin \tfrac{1}{2}(\omega_1 - \omega) + \exp \{i[(K + 1)/2](\omega_1 + \omega)\} \cdot \sin (K/2)(\omega_1 + \omega)/\sin \tfrac{1}{2}(\omega_1 + \omega),$$

we have

$$\mathcal{K}_{jm}(\omega, \omega', \omega'') = N^{-2}[\sin^2 (N/2)(\omega' - \omega'')/\sin^2 \tfrac{1}{2}(\omega' - \omega'')] \cdot O\{\sum_{\omega_1=\pm\omega',\pm\omega''} \sum_{p=j,m} [\sin^2 (p/2)(\omega_1 - \omega)/\sin^2 \tfrac{1}{2}(\omega_1 - \omega)]\}. \tag{40}$$

Also, $\mathcal{E} = [F'(0)^2/\pi^2] \sum_{j,m=1}^{K} d_j \, d_m \mathcal{E}_{jm} + O(N^{-1})$, where

$$\mathcal{E}_{jm} = \iint d\omega' \, d\omega'' [S_y(\omega')S_y(\omega'') - S_y(\omega)^2]\mathcal{K}_{jm}(\omega, \omega', \omega'').$$

Since $\sum_{j,m=1}^{K} |d_j d_m| = c_1^2 \leqq 1$, it is sufficient to show that $\mathcal{E}_{jm}$ satisfies (35), (36), or (38).

By (40),

$$\mathcal{E}_{jm} = O\{N^{-2} \textstyle\iint d\omega' \, d\omega'' |S_y(\omega')S_y(\omega'') - S_y(\omega)^2| \cdot [\sin^2 (N/2)(\omega' - \omega'')/\sin^2 \tfrac{1}{2}(\omega' - \omega'')] \cdot \sum_{\omega_1, p} [\sin^2 (p/2)(\omega_1 - \omega)/\sin^2 \tfrac{1}{2}(\omega_1 - \omega)]\}.$$

Using the fact that $S_y(\omega)$ is an even function,

$$\mathcal{E}_{jm} = O\{N^{-2} \textstyle\iint d\omega' \, d\omega'' \, |S_y(\omega')S_y(\omega'') - S_y(\omega)^2| \cdot [\sin^2 (N/2)(\omega' - \omega'')/\sin^2 \tfrac{1}{2}(\omega' - \omega'')] \cdot \sum_{p=j,m} [\sin^2 (p/2)(\omega' - \omega)/\sin^2 \tfrac{1}{2}(\omega' - \omega)]\}. \tag{41}$$

Assume (34). Then

$$S_y(\omega')S_y(\omega'') - S_y(\omega)^2 = O[|\sin \tfrac{1}{2}(\omega' - \omega'')|^\alpha + |\sin \tfrac{1}{2}(\omega' - \omega)|^\alpha].$$

(41) may be estimated by using the relations

$$\textstyle\int_0^\pi [\sin^2 (n/2)\omega'/\sin^2 \tfrac{1}{2}\omega'] \, d\omega' = \pi n,$$

$$\textstyle\int_0^\pi [\sin^2 (n/2)\omega'/|\sin \tfrac{1}{2}\omega'|^{2-\alpha}] \, d\omega' = O[\int_0^{n^{-1}} n^{2-\alpha} \, d\omega' + \int_{n^{-1}}^{\pi} (d\omega'/(\omega')^{2-\alpha})] = \begin{cases} O(n^{1-\alpha}), & 0 < \alpha < 1, \\ O(1 + \log n), & \alpha = 1. \end{cases}$$

We find for $0 < \alpha < 1$,

$$\mathcal{E}_{jm} = O[\textstyle\sum_{p=j,m} (N^{-1}p^{1-\alpha} + pN^{-1-\alpha})] = O(K^{1-\alpha}N^{-1}),$$

and for $\alpha = 1$, $\mathcal{E}_{jm} = O[(1 + \log K)N^{-1}]$, verifying (35) and (36).

Now assume (37). This implies that

$$S_y(\omega') = S_y(\omega) + [(\cos \omega - \cos \omega')/\sin \omega]S_y'(\omega) + O(|\cos \omega - \cos \omega'|^{1+\beta}),$$

$$S_y(\omega')S_y(\omega'') - S_y(\omega)^2 = S_y(\omega)S_y'(\omega)(2 \cos \omega - \cos \omega' - \cos \omega'')/\sin \omega + O(|\cos \omega - \cos \omega'|^{1+\beta} + |\cos \omega - \cos \omega''|^{1+\beta} + |\cos \omega - \cos \omega'| \, |\cos \omega - \cos \omega''|).$$

By a procedure analogous to that above, using the estimate $\int_0^\pi [\sin^2 (n/2)\omega' \, d\omega'/|\sin \tfrac{1}{2}\omega'|^{1-\beta}] = O(1)$, we find that

$$\mathcal{E}_{jm} = [S_y(\omega)S_y'(\omega)/\sin \omega] \cdot \textstyle\iint d\omega' \, d\omega'' (2 \cos \omega - \cos \omega' - \cos \omega'')\mathcal{K}_{jm}(\omega, \omega', \omega'') + O(N^{-1}).$$

This integral may be evaluated by integrating the series for $\mathcal{K}_{jm}$ term by term. Most of the terms give no contribution. We have

$$(4\pi^2)^{-1} \iint d\omega' \, d\omega'' (2 \cos \omega - \cos \omega' - \cos \omega'') \mathcal{K}_{jm}(\omega, \omega', \omega'')$$
$$= (2/N) \cos \omega \sum_{k=1}^{\min(j,m)} \cos^2 k\omega$$
$$- [(2N - 1)/2N^2][\sum_{k=2}^{\min(j,m+1)} \cos k\omega \cos (k - 1)\omega$$
$$+ \sum_{k=1}^{\min(j,m-1)} \cos k\omega \cos (k + 1)\omega]$$
$$= (2/N) \sum_{k=1}^{\min(j,m)} [\cos \omega \cos^2 k\omega - \cos k\omega \cos (k + 1)\omega] + O(N^{-1} + KN^{-2})$$
$$= N^{-1} \sin \omega \sum_{k=1}^{\min(j,m)} \sin 2k\omega + O(N^{-1}) = O(N^{-1}).$$

Hence $\mathcal{E}_{jm} = O(N^{-1})$.

The formula for the variance of $\hat{S}(\omega)$ may be given more explicitly than by Theorem 3 for simple choices of $S(\omega)$. For example if $S(\omega) = 1 + 2R_x(1) \cos \omega$, in the terms of order N^{-1} only the quantities,

$$\rho = E(y_1 y_2),$$
$$\sigma = E(y_1 y_2 y_3 y_4),$$
$$\nu = E(y_1^2 y_2^2),$$
$$\tau = E(y_1 y_2^2 y_3),$$
$$\tau' = E(y_1^2 y_2 y_3)$$

enter. For ordinary partial sums ($c_1, \cdots, c_K = 1$), we find that

$$\pi^{-1} \int_0^\pi \operatorname{Var}[\hat{S}(\omega)] \, d\omega = (2K/N)F'(0)^2 \cdot \pi^{-1} \int_0^\pi S_y(\omega)^2 \, d\omega$$
$$- N^{-1}F'(0)^2(2 + 8\rho + 2\nu + 4\tau + 8\sigma - 18\rho^2)$$
$$- N^{-1}[F'(0)^2 - F'(p)^2](2\nu + 4\tau + 4\sigma - 10\rho^2) + O(KN^{-2}).$$

For small values of $R_x(1)$, the effect of the terms after the first on the right is to decrease the average variance.

4. The value of $F'(0)$. If $\hat{S}(\omega)$ is evaluated by the original method of the introduction [using $y(x) = x$], it is easily verified that the conclusion of Theorem 3 applies: e.g., if $S(\omega)$ satisfies hypothesis (ii) of Theorem 3,

$$E\{\hat{S}(\omega)^2\} = S_K(\omega)^2 + (2\gamma K/N)S(\omega)^2 + O(N^{-1}),$$

for $0 < \omega < \pi$. The term of order K/N given in Theorem 3 differs from this by the replacement of $S(\omega)^2$ by $S_y(\omega)^2$ and the introduction of the factor $F'(0)^2$.

It can be shown that $S_y(\omega)$ always lies between the same bounds as $S(\omega)$, and has the same average value. Thus, the effect of the factor $S_y(\omega)$ is to increase the variance of $\hat{S}(\omega)$ in some places, and decrease it in others.

The factor $F'(0)^2$ gives a uniform increase in variance for all ω. By differentiating Equation (7) in the proof of Lemma 3, we have

$$F'(0) = [f'(0)]^{-1} = E\{xy(x)\}^{-2}.$$

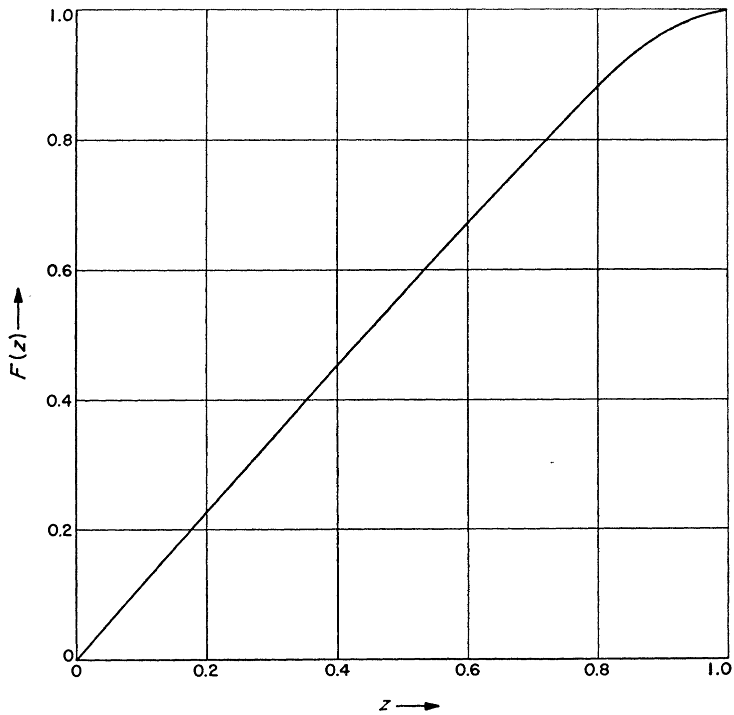

FIG. 1. The function $F(z)$ when y takes two values with the ratio $\frac{1}{4}$.

TABLE 1
Values of $F'(0)$ for $n = 2$

a_1	a_2	a_1/a_2	c_1	$F'(0)$
0.482	1.608	0.300	0.981	1.133
0.402	1.608	$\frac{1}{4}$	0.943	1.137
0.710	1.419	$\frac{1}{2}$	0.976	1.188
0	1.36	0	0.612	1.232

In particular, for $y(x) = x$, $F'(0) = 1$, and for $y(x) = \text{sgn}(x)$, $F'(0) = \pi/2$. It can be shown that for any other choice of $y(x)$ which is an odd nondecreasing function such that $E\{y(x)^2\} = 1$, $F'(0)$ lies between these limits.

It is advantageous to pick a function for $y(x)$ such that the computation of $\hat{R}_y(k)$ can be done rapidly and $F'(0)$ is close to 1. The computation is simple if $y(x)$ is a step function taking only a few values. This leads us to consider a function of the type

$$y(x) = a_j, \quad c_{j-1} < x < c_j, \quad j = 1, \cdots, n,$$
$$y(-x) = -y(x),$$

where $0 = c_0 < c_1 < \cdots < c_n = +\infty, 0 \leqq a_1 < \cdots < a_n$. For a given value of n, the numbers a_j, c_j may be chosen so as to minimize $F'(0)$, given $E[y(x)^2] = 1$. This must be done numerically for $n \geqq 2$.

The best value of $F'(0)$ for $n = 2$ is given in the first line of Table 1, with the corresponding values of the constants. When a_2/a_1 is a power of 2, the circuitry needed to compute $\hat{R}_y(k)$ is simplest, and the computation most rapid. The other lines in the table show best values of $F'(0)$ for several fixed choices of a_1/a_2.

Comparing with the value $F'(0) = 1.57$ for $y = \text{sgn}(x)$, we see that most of the increased error over that for $y = x$ has been removed. The best value of a_1/a_2 which is a power of 2 is $\frac{1}{4}$. The function $F(z)$ is plotted for this case in Figure 1. Note that it is essentially linear until $|z|$ is close to 1.

$F'(0)$ can be decreased further by taking larger values of n. The best case for $n = 3$ is

$$a_1 = 0.327, \quad a_2 = 1.030, \quad a_3 = 1.951,$$
$$c_1 = 0.659, \quad c_2 = 1.447, \quad F'(0) = 1.062.$$

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