ON IDENTITY RELATIONSHIPS FOR 2^{n-r} DESIGNS HAVING WORDS OF EQUAL LENGTH

By Peter W. M. John

University of California, Davis

The effects (main effects and interactions) in a 2^n design may be represented by sets of letters. With the addition of an identity symbol I, these sets form an Abelian group. The elements of an identity relationship for a 2^{n-r} fractional design form a subgroup of order 2^r (the defining contrast subgroup). We shall refer to these elements as words and the number of letters in each word will be called its length. The case in which the defining contrast subgroup contains interactions involving each of the letters is discussed by Burton and Connor [2]. The purpose of this note is to clarify two points that were made in their paper. A necessary and sufficient condition is obtained for the existence of an identity relationship in which the words all have the same length. Burton and Connor [2] give a formula for $V(w)$ in the case $n \leqq 2^r - 1$. It is shown here that the corresponding identity relationship is not necessarily unique.

Let $w_1, w_2, \cdots, w_r$ denote a set of generators of the subgroup and let $t(i)$ denote the number of letters appearing only in the ith generator, $t(ij)$ denote the number appearing in only the ith and jth generators and so on. We suppose that each letter appears in the identity relationship and so $\sum t = n$. We denote word lengths by w. Then $\sum w = 2^{r-1}n$, [1].

Burton and Connor [2] have shown that

$$\sum w^2 = 2^{r-2}(\sum t^2 + n^2),$$

whence the variance of the w's is given by

$$V(w) = 2^{r-2}[(2^r - 1)\sum t^2 - n^2]/(2^r - 1)^2.$$

$V(w)$ takes its minimum value when $\sum t^2$ is a minimum. They argue in a corollary that this occurs when $\sum t^2 = n$, in which case

$$\sum w^2 = 2^{r-2}n(n + 1).$$

If $\sum t^2 = n = \sum t$ we have n of the t's to be unity and the others zero. The corollary cannot hold if $n > 2^r - 1$ for there are only $(2^r - 1)$ t's and at least one of them would have to be greater than unity. Indeed, the 2^{6-2} design defined by

$$I = \underline{ABCD} = \underline{ABEF} = CDEF$$

is an example in which $t(1) = t(2) = t(12) = 2$ and $V(w) = 0$.

$V(w) = 0$ if and only if $n^2 = (2^r - 1)\sum t^2$, i.e., var $(t) = 0$. All the numbers

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involved are integers and $\sum t^2 \geqq n$. Hence, n must be a multiple of $2^r - 1$ and the following result is easily obtained.

A necessary and sufficient condition that there exist an identity relationship with words of equal length is that $n = \lambda(2^r - 1)$ where λ is an integer. Then the identity relationship is obtained by setting each t equal to λ and the length of each word will be $\lambda \cdot 2^{r-1}$.

Burton and Connor also point out that for any given set of t's, the corresponding identity relationship is unique apart from renaming the letters. It is necessary however that the value of each $t(ij)$ be specified.

Let m_j denote the number of t's that are equal to the integer j, $1 \leqq j \leqq n$, and let n_j be the number of words of length j in the identity relationship. Knowledge of the m_j alone does not necessarily give a unique identity relationship. Indeed, the identity relationship is not unique even for the case of minimum variance as the following example shows.

Consider the case $n = 6$, $r = 4$. For minimum variance $\sum_i w_i = 48$, $\sum_i w_i^2 = 168$. There are several different combinations of word lengths satisfying these conditions.

(1) $t(1) = t(2) = t(3) = t(4) = t(12) = t(34) = 1$ gives $n_2 = 6$, $n_4 = 9$.

(2) $t(1) = t(2) = t(3) = t(4) = t(12) = t(13) = 1$ gives $n_1 = 1$, $n_2 = 2$, $n_3 = 6$, $n_4 = 5$, $n_5 = 1$.

(3) $t(1) = t(2) = t(3) = t(4) = t(123) = t(124) = 1$ gives $n_2 = 3$, $n_3 = 8$, $n_4 = 3$, $n_6 = 1$.

(4) $t(1) = t(2) = t(3) = t(4) = t(12) = t(1234) = 1$ gives $n_2 = 4$, $n_3 = 6$, $n_4 = 3$, $n_5 = 2$.

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[1] Brownlee, K. A., Kelly, B. K. and Loraine, P. K (1948). Fractional replication arrangements for factorial experiments with factors at two levels. *Biometrika* **35** 268–276.

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