ON A THEOREM OF KARLIN REGARDING ADMISSIBILITY OF LINEAR ESTIMATES IN EXPONENTIAL POPULATIONS[1]

By Richard Morton[2] and M. Raghavachari[3]

University of California, Berkeley

1. Summary and introduction. Let the random variable X be distributed according to the probability density $p(x, \omega) = \beta(\omega) \exp(\omega x)$ with respect to a σ-finite measure μ defined on the real line, and ω, the unknown state of nature, belongs to the natural parameter space: $\Omega = \{\omega \mid \int_{-\infty}^{\infty} \exp(\omega x)\, d\mu(x) < \infty\}$ which is an interval of the real line. Let $\bar{\omega}$ and $\underline{\omega}$ be the upper and lower end points of Ω, respectively. $\bar{\omega}$ and $\underline{\omega}$ may or may not belong to Ω, and in some cases $\bar{\omega} = +\infty$ or $\underline{\omega} = -\infty$. The problem under consideration is the estimation of the quantity $\theta(\omega) = E_\omega(x) = -\beta'(\omega)/\beta(\omega)$ from a single observation x on X. The loss-function is the standard squared-error loss function. Karlin [1] considered the admissibility of the linear estimates of the form $a_\gamma(x) = \gamma x = x/(\lambda + 1)$ for $0 \leqq \gamma \leqq 1$ and the following theorem of his gives sufficient conditions for the admissibility of $a_\gamma(x)$.

Theorem 1.1 (Karlin). *Let $p(x, \omega) = \beta(\omega) \exp(\omega x)$ describe the density of the exponential family* wrt *a measure μ. If*

$$\int_c^{\bar{\omega}} \beta^{-\lambda}(\omega)\, d\omega = +\infty \tag{1.1}$$

and

$$\int_{\underline{\omega}}^c \beta^{-\lambda}(\omega)\, d\omega = +\infty, \tag{1.2}$$

where c is an interior point of $\Omega = (\underline{\omega}, \bar{\omega})$, then $\gamma x = x/(\lambda + 1)$ is an admissible estimate of $\theta(\omega) = E_\omega(X)$.

In the sequel we shall refer to the integrals in (1.1) and (1.2) as Karlin's integrals. Karlin [1] conjectured that conditions (1.1) and (1.2) are also necessary for the admissibility of $x/(\lambda + 1)$. We shall, in the following, refer to this as Karlin's conjecture. Let us denote by $I^2(\omega)$ the square of the coefficient of variation given by $[(\beta'(\omega))^2 - \beta(\omega)\beta''(\omega)]/(\beta'(\omega))^2$. Suppose $I^2(\omega)$ ranges between L and L' ($L < L'$) as ω traverses the interval $(\underline{\omega}, \bar{\omega})$. Karlin [1] showed that $x/(\lambda + 1)$ is inadmissible for all $\lambda < L$ and $\lambda > L'$. While criteria for inadmissibility of $x/(\lambda + 1)$ are given in terms of L and L', the conditions for admissibility are in terms of integrability of $\beta^{-\lambda}(\omega)$ near the end points of Ω. The purpose of this paper is to link up these two criteria and characterize Karlin's integrability conditions in terms of the behavior of $I^2(\omega)$ near $\underline{\omega}$ and $\bar{\omega}$ (Theorem

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[2] Now at University of Manchester.

[3] Now at Indian Institute of Management, Vastrapur, Ahmedahad-6.

2.2). It is also shown (Theorem 2.1) that the range of λ for which both of Karlin's integrals are infinite form a sub-interval of $(0, \infty)$. While Karlin's conjecture still remains open, the characterizations we obtained serve to settle the conjecture for a class of cases, an example of which is given in Example 2.1. It is also interesting to regard $I^2(\omega)$ as the ratio of the risks of two linear estimates $x/(\lambda + 1)$ corresponding to $\lambda = 0$ and $\lambda = \infty$, the boundary points of the range of λ.

2. Main theorems.

LEMMA 2.1. *$\beta(\omega)$ has at most one local extremum.*

PROOF. Since $\theta'(\omega) = \text{Var}(X \mid \omega) > 0$, $\theta(\omega)$ is a strictly increasing function. So $\theta(\omega) = -\beta'(\omega)/\beta(\omega)$ is zero at most once, and consequently $\beta'(\omega)$ is zero at most once.□

THEOREM 2.1. *The range of λ for which both of Karlin's integrals are infinite forms a single (possibly infinite) interval.*

PROOF. By Lemma 2.1 there is a number $\bar{c}$ such that for all ω between $\bar{c}$ and $\bar{\omega}$, $\beta(\omega)$ is $\leqq 1$ (or $\geqq 1$) in which case the integrand $\beta^{-\lambda}(\omega)$ is an increasing (decreasing) function of λ.

It follows that the range of λ for which the first integral is infinite is a semi-infinite or infinite interval which may be open or closed. For any $c > \bar{c}$ the integral is still infinite.

A similar argument holds for the second integral, which is therefore infinite for values of λ is a second interval. The intersection of these two intervals is the range of λ for which both intervals are infinite, and forms a single interval.

We remark that if Karlin's conjecture is true, the admissible values of λ also form an interval.

The proof indicates how we may verify, upon inspection of the behaviour of $\beta(\omega)$ near the extremes, whether the interval is finite, semi-infinite, or infinite.

Lemma 2.1 implies that lim $\beta(\omega)$ exists as $\omega \to \underline{\omega}$ or $\bar{\omega}$. If this limit is neither 0 nor ∞, Karlin's integrals are trivially infinite or finite according as the range Ω of ω is infinite or finite.

In most natural examples $\beta(\omega) \to 0$ or ∞, but it is possible to have other limits—for example $d\mu(x)/dx = [\pi(1 + x^2)]^{-1}e^{-|x|}$, in which both limits (as $|\omega| \to 1$) of $\beta(\omega)$ are 1.

We now consider the case where $\beta(\omega) \to 0$ or ∞. In all that follows sgn $[y] = +1$ if $y > 0$ and $= -1$ if $y < 0$.

THEOREM 2.2. *If $\beta(\omega) \to 0$ or $+\infty$ as $\omega \to \omega_0$ (where ω_0 may be $\underline{\omega}$ or $\bar{\omega}$) and if $I^2(\omega) \to \lambda_0$ as $\omega \to \omega_0$, then for any $\lambda \neq \lambda_0$,*

$$\int_a^b \beta^{-\lambda}(\omega)\, d\omega = +\infty \qquad \text{if } \operatorname{sgn}[(\lambda_0 - \lambda)\beta'(\omega)(\omega_0 - \omega)] \text{ is } (+1)$$

$$< +\infty \qquad \text{if } \operatorname{sgn}[(\lambda_0 - \lambda)\beta'(\omega)(\omega_0 - \omega)] \text{ is } (-1)$$

eventually (for ω close to ω_0) where (a, b) represents (c, ω_0) or (ω_0, c) whichever is in numerical order. $I^2(\omega)$ is defined in Section 1.

PROOF. Let us first prove the theorem for the case $\omega_0 = \bar{\omega}$. We use the relations

$$(2.1) \qquad \int_c^{\bar{\omega}} \beta^{-\lambda}(\omega)\, d\omega = \int_c^{\bar{\omega}} (\beta^{-\alpha}(\omega)/|\theta(\omega)|)\cdot(|\beta'(\omega)|/\beta^{\lambda-\alpha+1}(\omega))\, d\omega$$

and

$$(2.2) \qquad (d/d\omega)[\beta^{-\alpha}(\omega)/|\theta(\omega)|] = (I^2(\omega) - \alpha)\beta^{-\alpha}(\omega) \text{ sgn } [\beta'(\omega)].$$

For any values of $\alpha \neq \lambda_0$, sgn $[I^2(\omega) - \alpha]$ = sgn $[\lambda_0 - \alpha]$ eventually, and since by Lemma 2.1 $\beta'(\omega)$ can change sign at most once, $(\lambda_0 - \alpha)\beta'(\omega)$ is eventually of constant sign. It follows that there exists a constant $A > 0$ such that

$$(2.3) \qquad \begin{aligned} \beta^{-\alpha}(\omega)/|\theta(\omega)| &> A \quad \text{eventually according as } (+) \\ &< A \quad \text{eventually according as } (-), \end{aligned}$$

where $(+)$ and $(-)$ indicate the eventual sign of $(\lambda_0 - \alpha)\beta'(\omega)$.

Consider now

$$(2.4) \qquad \int_c^{\bar{\omega}} |\beta'(\omega)|/\beta^{\lambda-\alpha+1}(\omega)\, d\omega = \text{sgn } [(\alpha - \lambda)\beta'(\omega)][\beta^{-\lambda+\alpha}(\omega)]_{\omega=c}^{\omega=\bar{\omega}}.$$

It can easily be checked that the above integral is infinite or finite according as sgn $[(\alpha - \lambda)\beta'(\omega)]$ equals 1 or -1. Since $\lambda \neq \lambda_0$, we may choose α to lie between λ and λ_0 and then sgn $(\alpha - \lambda)$ = sgn $(\lambda_0 - \lambda)$ = sgn $(\lambda_0 - \alpha)$. Hence

$$(2.5) \qquad \begin{aligned} \int_c^{\bar{\omega}} |\beta'(\omega)|/\beta^{\lambda-\alpha+1}(\omega)\, d\omega &= +\infty \\ &= <\infty \end{aligned}$$

according as

$$(2.6) \qquad \begin{aligned} \text{sgn } [(\lambda_0 - \lambda)\beta'(\omega)] \text{ is eventually } &= +1. \\ &= -1. \end{aligned}$$

Combining Equations (2.1), (2.3) and (2.5), we have

$$\begin{aligned} \int_c^{\bar{\omega}} \beta^{-\lambda}(\omega)\, d\omega &= +\infty \\ &= <\infty \end{aligned}$$

according as (2.6) is true. This proves the theorem for $\omega_0 = \bar{\omega}$. The case $\omega_0 = \underline{\omega}$ is similarly dealt with.

COROLLARY 2.1. *Under the conditions of Theorem* 2.2 *and for the case* $\lambda = \lambda_0$,

$$\int_a^b \beta^{-\lambda_0}(\omega)\, d\omega = +\infty \quad \textit{if} \quad [I^2(\omega) - \lambda_0]\beta'(\omega)(\omega_0 - \omega) \geqq 0$$

eventually as $\omega \to \omega_0$.

PROOF. From (2.2) with $\alpha = \lambda_0$, we see that there exists $A > 0$ such that

$$(2.7) \qquad \beta^{-\lambda_0}(\omega)/|\theta(\omega)| > A \quad \text{eventually as} \quad \omega \to \omega_0$$

$$\text{if} \quad [(I^2(\omega) - \lambda_0)\beta'(\omega)(\omega_0 - \omega)] \geqq 0 \quad \text{eventually as} \quad \omega \to \omega_0.$$

Also

$$\int_a^b |\theta(\omega)|\, d\omega = [\operatorname{sgn}(\beta'(\omega)) \log \beta(\omega)]_{\omega=a}^{\omega=b} = \infty, \tag{2.8}$$

since $\beta(\omega) \to 0$ or $+\infty$ as $\omega \to \omega_0$.

The proof of the corollary is completed by putting together (2.7) and (2.8).

Before we proceed we make two remarks. The first is that although the cases where sgn $[(I^2(\omega) - \lambda_0)\beta'(\omega)(\omega_0 - \omega)]$ is not $+1$ eventually are not covered by Corollary 2.1, it is possible to prove further results to cover some of these cases by considering the manner in which $I^2(\omega)$ converges to its limit. The second remark is that there are examples where $I^2(\omega)$ does not converge to any limit, but may oscillate in any continuous manner. As our construction for such examples is rather lengthy we shall omit them here.

The following lemma gives a condition under which $I^2(\omega)$ does converge.

LEMMA 2.2. *If $\beta(\omega) \to 0$ or $+\infty$ and if for some λ, $\beta^\lambda(\omega)\theta(\omega)$ converges to a limit as $\omega \to \omega_0$ which is neither zero nor infinity, then $I^2(\omega) \to \lambda$. Here ω_0 is the same as in Theorem* 2.2.

PROOF. First the case $\lambda \neq 0$. Since the derivatives of $1/\theta(\omega)$ and $\beta^\lambda(\omega)$ are respectively $-I^2(\omega)$ and $-\lambda\beta^\lambda(\omega)\theta(\omega)$, we see by applying L'Hopital's rule to $\beta^\lambda(\omega)/[\theta(\omega)]^{-1}$ that

$$\lim_{\omega\to\omega_0} \beta^\lambda(\omega)\theta(\omega) = \lim_{\omega\to\omega_0} [\beta^\lambda(\omega)\theta(\omega)\lambda/I^2(\omega)].$$

It follows that $\lim_{\omega\to\omega_0} I^2(\omega) = \lambda$.

Since $\theta(\omega) = -\beta'(\omega)/\beta(\omega)$, and the case $\lambda = 0$ implies that $\theta(\omega)$ converges to a limit which is neither zero nor infinity, we may again apply L'Hopital's rule to the expression to obtain $\lim \theta(\omega) = \lim [(1 - I^2(\omega))\theta(\omega)]$. Consequently $\lim I^2(\omega) = 0$ and the lemma is proved.

The following corollary gives a sufficient condition for Karlin's integrals to be infinite.

COROLLARY 2.2. *If* $\liminf [\beta^{-\lambda}(\omega)/|\theta(\omega)|] > 0$ *as $\omega \to \omega_0$ then* $\int_a^b \beta^{-\lambda}(\omega)\, d\omega = +\infty$, *where ω_0, a, and b are the same as in Theorem* 2.2.

PROOF. We have $\beta^{-\lambda}(\omega)/|\theta(\omega)| \geqq A > 0$ eventually as $\omega \to \omega_0$ and $\int_c^b |\theta(\omega)| = +\infty$ in virtue of (2.8). The proof of the corollary follows.

REMARK. In many cases where $I^2(\omega) \to L$ or L' as ω tends to its extremes, by Theorem 2.2, Karlin's integrals are infinite for all λ in the open interval (L, L'). When Corollary 2.2 applies with $\lambda = L$ or L', Karlin's integrals are infinite for all λ taking the end points L and L'. Since Karlin has shown that the estimate $x/(\lambda + 1)$ is inadmissible for $\lambda < L$ and $\lambda > L'$, Karlin's conjecture is verified. An example to illustrate the remark is given below.

EXAMPLE 2.1. Let (i) $\Omega = (-\infty, \infty)$, (ii) $\mu\{(-\infty, 0]\} = 0$, (iii) $I^2(\omega) \to L'$ as $\omega \to -\infty$ and (iv) $\lim_{\omega\to-\infty} \beta^{-L'}(\omega)/\theta(\omega) \neq 0, \infty$. Here $\int_c^{+\infty} \beta^{-\lambda}(\omega)\, d\omega = +\infty$ for all λ since $\beta(\omega) \to 0$ as $\omega \to +\infty$. By Theorem 2.2 and Corollary 2.2, $\int_{-\infty}^c \beta^{-\lambda}(\omega)\, d\omega = \infty$ for $\lambda \leqq L'$ and $< \infty$ for $\lambda > L'$. Karlin's conjecture is therefore verified. It may be noted that, in this example, $I^2(\omega) \to 0$ as $\omega \to +\infty$.

REMARK. It is to be noted that in order to settle Karlin's conjecture for the case Ω: $(-\infty, \infty)$, one has to consider only the case where $\mu\{(-\infty, 0]\} = 0$ (or $\mu\{[0, \infty)\} = 0$).

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REFERENCE

[1] KARLIN, SAMUEL (1958). Admissibility for estimation with quadratic loss. *Ann. Math. Statist.* **29** 406–436.