ON A DISTRIBUTION-FREE METHOD OF ESTIMATING ASYMPTOTIC EFFICIENCY OF A CLASS OF NON-PARAMETRIC TESTS[1]

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1. Summary. This paper deals with some aspects of nonparametric confidence intervals for shift and scale parameters which may be obtained from a celebrated class of rank order tests for the location and scale problems. This also provides a distribution-free method of estimating asymptotic efficiency of a class of tests and estimates (point as well as intervals) that may be derived from the same class of rank order statistics. Further, the proposed method is also applicable for estimating certain functionals of the distribution function which may not be otherwise estimated in a simple manner.

2. Introduction. Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two independent random samples of sizes m and n, drawn from populations with absolutely continuous cumulative distribution functions (cdf) $F(x)$ and $G(x)$, respectively. For testing the null hypothesis of the identity of F and G, a class of non-parametric tests can be represented in terms of Chernoff-Savage [1] type of test-statistics of the form

$$mT_N = \sum_{i=1}^{N} E_{Ni}Z_{Ni}, \qquad N = m + n, \tag{2.1}$$

where $Z_{Ni} = 1$, if the ith smallest observation from the combined sample is from the first sample, and $Z_{Ni} = 0$, otherwise; $\{E_{Ni}\}$'s are N constants, which we may represent in the Chernoff-Savage form as

$$E_{Ni} = J_N(i/N), \qquad 1 \leqq i \leqq N, \tag{2.2}$$

where the function $J_N(H)$ has the limit (as $N \to \infty$)$J(H)$ for all $0 < H < 1$, and where $J_N(H)$ satisfies all the four regularity conditions of Theorem 1 of Chernoff and Savage ([1], p. 974).

Statistics of the type (2.1) are known as rank order statistics and they play a very important role in the theory of nonparametric inference. First, for testing the null hypothesis of identity of F and G against translation or scale type of alternatives that may be put in the form

$$G(x) = F(x - \theta), \qquad \theta \neq 0 \tag{2.3}$$

and

$$G(x - \mu) = F([x - \mu]/\sigma), \qquad \sigma \neq 1, \tag{2.4}$$

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the tests based on such rank order statistics are known to possess various desirable properties (cf. [1], [2], [3]). Second, Hodges and Lehmann [4] have shown that these rank order statistics can also be used to attach a translation invariant point estimate to the shift parameter θ in (2.3). Subsequently, Lehmann [7] has also obtained a confidence interval for θ, based on Wilcoxon's statistics. Both the point and interval estimates are also obtained by Sen [13] in connection with a problem in bioassay. It follows from these results that the relative asymptotic efficiencies of these estimates are the same as the corresponding Pitman-efficiencies of the tests which derive them.

The present investigation attempts to generalize the above findings in two directions. Firstly, it generalizes Lehmann's [7] and Sen's [13] method of deriving confidence intervals for a shift parameter to a wider class of rank order statistics. It also deals with the problem of point as well as interval estimation of the scale parameter σ, in (2.4), which may be obtained from a similar class of rank order tests. Secondly, it provides a method of estimating relative asymptotic efficiency of the class of tests and estimates referred to earlier. This we sketch briefly as below.

If we consider the sequence of translation type of alternatives, viz.,

$$H_N : G(x) = F(x - N^{-\frac{1}{2}}\theta), \tag{2.5}$$

where θ is real, and if the cdf $F(x)$ satisfies the regularity conditions of Lemma 7.2 of Puri [8], then for two sequences of tests $\{T_N\}$ and $\{T_N^*\}$ both of the type (2.1) with limiting functions J and J^* respectively), the asymptotic Pitman-efficiency of $\{T_N\}$ with respect to $\{T_N^*\}$ for testing $H_0 : F \equiv G$ against $\{H_N\}$ is

$$e^{(H)}_{\{T_N\},\{T_N^*\}} = [A^*(F)B(F)/A(F)B^*(F)]^2, \tag{2.6}$$

where

$$A^2(F) = \textstyle\int_0^1 J^2(x)\,dx - [\int_0^1 J(x)\,dx]^2, \tag{2.7}$$

$$B(F) = \textstyle\int_{-\infty}^{\infty} (d/dx)J[F(x)]\,dF(x), \tag{2.8}$$

and $A^*(F)$ and $B^*(F)$ are obtained by replacing J by J^* in (2.7) and (2.8), respectively. It may be noted that $A(F)$ is a distribution-free constant, while the functional $B(F)$ depends explicitly on the cdf F. Again, if $F(x)$ possesses a finite variance $\sigma^2(F)$, then the Pitman-efficiency of $\{T_N\}$ with respect to the Student's t-test is

$$e^{(H)}_{\{T_N\},\{t_N\}} = [\sigma(F)B(F)/A(F)]^2. \tag{2.9}$$

Puri [8] while extending the Chernoff-Savage theorem to the $c(\geqq 2)$ sample case, has observed that the same expressions for the asymptotic efficiency (i.e., (2.6) and (2.9)) also hold for the c-sample case, (where, Student's t-test has to be replaced by the classical analysis of variance test).

Hodges and Lehmann [4] have observed that the asymptotic relative efficiency of any two point estimators of θ, in (2.3), would be the same as the corresponding

Pitman-efficiency of the two sequences of tests based on the same rank order statistics, provided the latter exists. Lehmann [7] has shown that the asymptotic efficiency of the confidence interval for θ obtained by using Wilcoxon's test is the same as that of the test itself, and we shall see later on that this statement applies to the general class of rank order tests and estimates.

Further, if we consider the sequence of scale type of alternatives, viz.,

$$K_N : G(x - \mu) = F([x - \mu][1 - N^{-\frac{1}{2}}\theta]), \tag{2.10}$$

where θ is real; then under similar conditions

$$e^{(K)}_{\{T_N\},\{T_N^*\}} = [A^*(F)\cdot C(F)/A(F)C^*(F)]^2, \tag{2.11}$$

where

$$C(F) = \int_{-\infty}^{\infty} x(d/dx)J[F(x))]\, dF(x), \tag{2.12}$$

and $C^*(F)$ is obtained by replacing J by J^* in (2.10). Finally, if $F(x)$ has finite fourth order moment, so that $\beta_2(F)$, the usual Pearsonian measure of kurtosis of $F(x)$, exists, we have for the classical variance ratio (F-) test,

$$e^{(K)}_{\{T_N\},\{F_N\}} = \beta_2(F)[C(F)/A(F)]^2/4. \tag{2.13}$$

It now follows from (2.6) (and (2.11)) that if we want to study the comparative performance of two Chernoff-Savage type of tests for location (or scale), we require the knowledge of the functional $B(F)$ (or $C(F)$), and the comparison with the parametrically optimum tests, deems further the knowledge of $\sigma^2(F)$ (or $\beta_2(F)$).

Now in non-parametric situations, the parent cdf $F(x)$ is not explicitly stated, and as a result, the functionals $B(F)$, $C(F)$ and $\beta_2(F)$ are all unknown. This makes the study of the exact expression for the power-efficiency impracticable, and the current literature on this topic contains various studies relating to (2.6), (2.9), (2.11) and (2.13) for various common types of cdf's, and in some situations with the universal bounds for the same, valid for all F (cf. Hodges and Lehmann ([2], [3]), Klotz [5], Sen [12], among many others). This study undoubtedly supplies some information about the relative performance of two rival tests under suitable sequences of alternative hypotheses and for typical parent cdf's, but it fails to provide us with any idea as to how to compare two tests when nothing is known about $F(x)$.

The object of the present investigation is to propose and study a general method of estimating the functionals $B(F)$ and $C(F)$, without assuming the nature of the cdf $F(x)$, and with the aid of these estimates to provide means of estimating the efficiency-factors when really $F(x)$ is neither known nor is assumed to be specified even under the alternative hypotheses.

Incidentally, the proposed method can also be readily applied to estimate certain functionals of absolutely continuous cdf's, and the same has been briefly illustrated in Section 6.

3. Nonparametric confidence intervals for a shift parameter and estimation of $B(F)$. Here we propose to estimate $B(F)$, defined in (2.8), when J_N, defined in (2.2), satisfies all the regularity conditions of Theorem 1 of Chernoff and Savage [1], and when $F(x)$ is absolutely continuous and it satisfies the regularity conditions of Lemma 7.2 of Puri [8]. Looking at (2.8), we observe that $(d/dx)J[F(x)]$ depends explicitly on the unknown $F(x)$, and hence, the usual methods of estimation may not be applicable.

We will use a technique by Lehmann [7] and Sen [13] of attaching a distribution-free confidence limit to the shift parameter, θ in (2.3), to derive our estimate of $B(F)$.

Let us first consider the problem of interval estimation and write

$$(3.1)\quad \mathbf{X}_m = (X_1, \cdots, X_m), \qquad \mathbf{Y}_n = (Y_1, \cdots, Y_n), \qquad \mathbf{I}_l = (1, \cdots, 1),$$

and rewrite (2.1)

$$(3.2)\qquad T_N = h_N(\mathbf{X}_m, \mathbf{Y}_n),$$

where h_N depends on the ranks of $\mathbf{X}_m$ and $\mathbf{Y}_n$ in the combined sample of size N We shall assume further that J_N is such that

$$(3.3)\qquad h_N(\mathbf{X}_m + a\mathbf{I}_m, \mathbf{Y}_n) \text{ is } \uparrow \text{ in } a, \quad \text{for all } \mathbf{X}_m \text{ and } \mathbf{Y}_n.$$

It may be noted that almost all the location tests based on Chernoff-Savage type of test statistic T_N, satisfy (3.3); and hereafter, in this section, it will be assumed that for the estimation of $B(F)$, (3.3) holds i.e., $J(F)$ is non-decreasing in F: $0 < F < 1$. Let now

$$(3.4)\qquad \lambda_N = m/N \to \lambda : 0 < \lambda < 1 \quad \text{as} \quad N \to \infty,$$

$$(3.5)\qquad \mu = \textstyle\int_0^1 J(x)\, dx \quad \text{and} \quad A^2 = \int_0^1 J^2(x)\, dx - \mu^2.$$

Since $J(F)$ is known, both μ and A^2 are known. Then, under the hypothesis $H: G(x) = F(x - \theta)$, it follows from elementary considerations that

$$(3.6)\qquad Z_N = \{N\lambda_N/A^2(1 - \lambda_N)\}^{\frac{1}{2}} \cdot \{h_N(\mathbf{X}_n + \theta\mathbf{I}_m, \mathbf{Y}_n) - \mu\}$$

is a strictly distribution-free statistic, and it follows from Chernoff-Savage theorem [1] that asymptotically Z_N has a normal distribution with zero mean and unit variance. We shall now apply Lehmann [7] and Sen's [13] method of determining the confidence limits for θ based on Z_N. Let $Z_N^{(1)}$ and $Z_N^{(2)}$ be such that

$$(3.7)\qquad P_\theta\{Z_N \leqq Z_N^{(1)}\} = \alpha_1, \qquad P_\theta\{Z_N \geqq Z_N^{(2)}\} = \alpha_2, \quad \alpha_1 + \alpha_2 = \alpha;$$

(ideally α_1 and α_2 are equal). Then the values of $Z_N^{(1)}$ and $Z_N^{(2)}$ can be computed from the known distribution of Z_N (independently of F). For large N, we take $\alpha_1 = \alpha_2 = \alpha/2$, and get the following

$$(3.8)\qquad Z_N^{(1)} + Z_N^{(2)} \to 0, \qquad Z_N^{(1)} \to \tau_{\alpha/2},$$

where $\tau_{\alpha/2}$ is the lower 100 $\alpha/2$ % point of a standardized normal distribution.

Let now

$$\hat{\theta}_{U\cdot N} = \text{Sup}\,\{\theta: Z_N < Z_N^{(2)}\}, \tag{3.9}$$
$$\hat{\theta}_{L\cdot N} = \text{Inf}\,\{\theta: Z_N > Z_N^{(1)}\},$$

and let $I_N = \{\theta: \hat{\theta}_{L\cdot N} \leqq \theta \leqq \hat{\theta}_{U\cdot N}\}$ be an interval in θ whose width is

$$\hat{\delta}_N = \hat{\theta}_{U\cdot N} - \hat{\theta}_{L\cdot N}\,. \tag{3.10}$$

Then $\{\hat{\theta}_{L\cdot N} \leqq \hat{\theta} \leqq \hat{\theta}_{U\cdot N}\}$ supplies a $100(1 - \alpha)$ % confidence interval for θ, and the width of the confidence interval is equal to $\hat{\delta}_N$.

It may be noted that the estimates $\hat{\theta}_{U\cdot N}$ and $\hat{\theta}_{L\cdot N}$ are both translation invariant (which may be proved on the same line as in [4], p. 605); and as a result, $\hat{\delta}_N$ is also translation invariant. Further, proceeding precisely on the same line as in Lemma 1 of Lehmann [7], it readily follows that the asymptotic relative efficiency of any two such sequences of confidence intervals derived from two sequences of rank order statistics $\{T_N\}$ and $\{T_N^*\}$ (having the limiting functions $J(F)$ and $J^*(F)$, respectively) will be the same as in (2.6).

Let us now consider the following theorems which generalize Theorem 1 and Lemma 4 of Lehmann [7] to a more general class of rank order statistics.

THEOREM 1. *If $F(x)$ in (2.3) satisfies the regularity conditions of Lemma 7.2 of* [8], *and T_N in (3.2) satisfies (3.3) in addition to the regularity conditions of Theorem 1 of* [1], *then under (3.4), the joint distribution of $N_0^{\frac{1}{2}}((\hat{\theta}_{U\cdot N} - \theta), (\hat{\theta}_{L\cdot N} - \theta))$ converges in law to a normal distribution on the plane concentrating on a line*

$$N_0^{\frac{1}{2}}(\hat{\theta}_{U\cdot N} - \theta) = N_0^{\frac{1}{2}}(\hat{\theta}_{L\cdot N} - \theta) + 2A|\tau_{\alpha/2}|/B(F), \tag{3.12}$$

where $N_0 = N\lambda_N(1 - \lambda_N)$, and $B(F)$ and A are defined in (2.7) and (2.8), respectively.

OUTLINE OF PROOF. Let a be any real and finite quantity, and we define

$$H_{(a)}(x) = \lambda_N F(x) + (1 - \lambda_N)F(x + N_0^{-\frac{1}{2}}a). \tag{3.13}$$

Then, under the stated regularity conditions, it can be shown precisely on the same line as in the proof of Theorem 1 (as well as Corollary 2) of [1] that for any real and finite a

$$[N_0^{\frac{1}{2}}/A(1 - \lambda_N)]\{h_N(\mathbf{X}_m + (\theta + N_0^{-\frac{1}{2}}a)\mathbf{I}_m\,, \mathbf{Y}_n) - \mu\} \tag{3.14}$$
$$= aB(F)/A + (N_0^{\frac{1}{2}}/A)[B_{1,N}^{(a)} - B_{2,N}^{(a)}] + o_p(1),$$

where μ is defined in (3.5), and

$$B_{1,N}^{(a)} = \int_{-\infty}^{\infty} [G_n(x + N_0^{-\frac{1}{2}}a) - F(x + N_0^{-\frac{1}{2}}a)]J'(H_{(a)}(x))\,dF(x), \tag{3.15}$$

$$B_{2,N}^{(a)} = \int_{-\infty}^{\infty} [F_m(x) - F(x)]J'(H_{(a)}(x))\,dF(x + N_0^{-\frac{1}{2}}a), \tag{3.16}$$

$$F_m(x) = m^{-1}[\text{Number of } X_i \leqq x], \tag{3.17}$$

$$G_n(x) = n^{-1}[\text{Number of } (Y_i - \theta) \leqq x]. \tag{3.18}$$

Further, the proofs of Theorem 1 and Corollary 2 of [1] imply that for any real and finite a, $(N_0^{\frac{1}{2}}/A)[B_{1,N}^{(a)} + B_{2,N}^{(a)}]$ converges in law to a normal distribution with zero mean and unit variance. Hence, using the same technique as in Theorem 4 of [4], we get from (3.9) that

$$(3.19)\quad \lim_{N=\infty} P_\theta\{N_0^{\frac{1}{2}}(\hat{\theta}_{U\cdot N} - \theta) \leqq a\} = \Phi(\tau_{\alpha/2} + aB(F)/A); \quad \Phi(\tau_{\alpha/2}) = \alpha/2,$$

where Φ is the cdf of a standardized normal variate. (3.19) implies that

$$(3.20)\qquad |N_0^{\frac{1}{2}}(\hat{\theta}_{U\cdot N} - \theta) + \tau_{\alpha/2}A/B(F)| \text{ is bounded in probability.}$$

Similarly, it can be shown that

$$(3.21)\qquad |N_0^{\frac{1}{2}}(\hat{\theta}_{L\cdot N} - \theta) - \tau_{\alpha/2}A/B(F)| \text{ is bounded in probability.}$$

Now, if (a, b) be any two real and finite quantities, then it can be shown using (3.15) and some simple but somewhat lengthy algebraic manipulations that

$$(3.22)\qquad E(B_{1,N}^{(a)} - B_{1,N}^{(b)}) = 0,$$

$$(3.23)\qquad nE\{[B_{1,N}^{(a)}]^2\} = A^2 + o(1), \qquad nE\{[B_{1,N}^{(b)}]^2\} = A^2 + o(1);$$

$$(3.24)\qquad nE\{B_{1,N}^{(a)} \cdot B_{1,N}^{(b)}\} = A^2 + o(1).$$

Thus, from (3.22), (3.23) and (3.24), we get that

$$(3.25)\qquad nE\{[B_{1,N}^{(a)} - B_{1,N}^{(b)}]^2\} = o(1),$$

and hence by Chebyshev's lemma, we get that

$$(3.26)\qquad |n^{\frac{1}{2}}[B_{1,N}^{(a)} - B_{1,N}^{(b)}]| = o_p(1).$$

Similarly, we have

$$(3.27)\qquad |m^{\frac{1}{2}}[B_{2,N}^{(a)} - B_{2,N}^{(b)}]| = o_p(1).$$

Consequently, from (3.14), (3.15), (3.16), (3.26) and (3.27), we get that for any two real and finite (a, b)

$$(3.28)\quad [N_0^{\frac{1}{2}}/A(1 - \lambda_N)]\{h_N(\mathbf{X}_m + (\theta + N_0^{-\frac{1}{2}}a)\mathbf{I}_m, \mathbf{Y}_n) - h_N(\mathbf{X}_m + (\theta + N_0^{-\frac{1}{2}}b)\mathbf{I}_m, \mathbf{Y}_n)\} = (a - b)B(F)/A + o_p(1).$$

Thus, from (3.3), (3.9), (3.20), (3.21) and (3.28), we may conclude that

$$(3.29)\qquad N_0^{\frac{1}{2}}(\hat{\theta}_{U\cdot M} - \hat{\theta}_{L\cdot N})B(F)/A = 2|\tau_{\alpha/2}| + o_p(1),$$

which asserts the truth of (3.12). The rest of the theorem follows directly from (3.14), (3.28), (3.29) and the asymptotic normality of $(N_0^{\frac{1}{2}}/A)[(B_{1,N}^{(a)} - B_{2,N}^{(a)}), (B_{1,N}^{(b)} - B_{2,N}^{(b)})]$.

Hence, the theorem.

THEOREM 2. *Under the conditions of Theorem* 1,

$$(3.30)\qquad [N\lambda_N(1 - \lambda_N)]^{\frac{1}{2}}\hat{\delta}_N \to_P 2A|\tau_{\alpha/2}|[B(F)]^{-1}.$$

The proof follows readily from Theorem 1 and hence is omitted.

Consequent on Theorem 2 we arrive at the following estimate

$$(3.31)\quad \hat{B}(F) = \{(Z_N^{(2)} - Z_N^{(1)})/\hat{\delta}_N[N\lambda_N(1 - \lambda_N)]^{\frac{1}{2}}\} \sim_P \{2A|\tau_{\alpha/2}|/\hat{\delta}_N[N\lambda_N(1 - \lambda_N)]^{\frac{1}{2}}\}.$$

Since, $\hat{\delta}_N$ has been shown to be translation invariant, we get from (3.16) that $\hat{B}(F)$ is also so. This property of $\hat{B}(F)$ ensures its consistency even when the null hypothesis $H_0 : F \equiv G$ does not hold. Again, substituting (3.16) in (2.6), we observe that the estimated value of (2.6) reduces to

$$(3.32)\qquad \hat{e}_{\{T_N\},\{T_N^*\}} = (\hat{\delta}_N^*/\hat{\delta}_N)^2.$$

From (3.30) and (3.22), we readily arrive at the following:

THEOREM 3. *If corresponding to a given confidence coefficient* $1 - \alpha$, $\{I_N\}$ *and* $\{I_N^*\}$ *be two sequences of confidence intervals for the shift parameter* θ *in* (2.3), *based on two sequences of rank order statistics* $\{T_N\}$ *and* $\{T_N^*\}$ *both satisfying the conditions of Theorem* 1, *then the asymptotic ratio of squared lengths of the confidence intervals is equal to the relative efficiency of the test which generates them.*

The above results can be easily extended to the one sample location problem. As these will follow as the direct generalizations of the results of Lehmann ([7], p. 511) and along the lines of the previous three theorems, the details are omitted.

4. Nonparametric confidence intervals for a scale parameter and estimation of $C(F)$**.** In the scale problem there are two main difficulties which make the use of Hodges and Lehmann [4] type of point estimates and Lehmann [7] and Sen's [13] confidence limits somewhat dubious in small samples. The first difficulty is caused by the value of μ in (2.8) (no matter, known or unknown), as a result of which the distribution-freeness and monotonicity of T_N (in (2.1) or (3.2)) under scale change usually do not hold simultaneously; (usually we require $h_N(\theta\mathbf{X}_m, \mathbf{Y}_n)$ (referred to (3.2)) to be distribution-free when $G(x - \mu) = F((x - \mu)\theta)$ and also we require $h_N(\theta\mathbf{X}_m, \mathbf{Y}_n)$ to be $\uparrow$ in θ). For a class of tests based on generalized U-statistics, however, this difficulty can be removed by an approach by Sen ([13], p. 542), but usually the rank-order tests can not be tackled in that manner. The second difficulty is that the estimate of $C(F)$ should be valid even when the null hypothesis $F \equiv G$ is not true, (as is the case with $B(F)$). This imposes some further restrictions on F.

The confidence interval for σ in (2.4) and the estimator of $C(F)$, considered here, are also based on all the assumptions regarding J_N and F, as stated at the beginning of Section 3. Besides, it requires an additional assumption on F, namely that $F(x)$ is symmetric. We also assume that $J(u)$ is symmetric about $u = \frac{1}{2}$, and

$$(4.1)\qquad J'(F(x - t))f(x - t) \leqq T(x) \quad \text{for all} \quad 0 \leqq |t| < c < \infty$$

where $T(x)$ is quadratically integrable with respect to $F(x)$. If (4.1) holds, obviously the conditions of Lemma 7.2 of Puri [8] also hold.

Let now $\hat{X}$ and $\hat{Y}$ be some estimate of the location parameters (μ_1 and μ_2) of X and Y respectively, and are such that $\hat{X}$ and $\hat{Y}$ are scale invariant and

$$(4.2) \qquad |N^{\frac{1}{2}}(\hat{X} - \mu_1)| = O_p(K), \qquad |N^{\frac{1}{2}}(\hat{Y} - \mu_2)| = O_p(K),$$

where K is a finite constant and where (3.4) is assumed to hold. We then consider a rank-order statistic of the form (2.1) but based on observations centered at the respective estimated locations, and write it as

$$(4.3) \qquad \hat{T}_N = h_N(\mathbf{X}_m - \hat{X}\mathbf{I}_m, \mathbf{Y}_n - \hat{Y}\mathbf{I}_n),$$

where J_N is such that

$$(4.4) \qquad h_N(a[\mathbf{X}_m - \hat{X}\mathbf{I}_m], \mathbf{Y}_n - \hat{Y}\mathbf{I}_n) \text{ is } \uparrow \text{ in } a.$$

It may be noted that (4.4) is the usual characteristic of the rank-order scale tests. Defining then λ_N, μ and A as in (3.4) and (3.5), it can be shown precisely on the same line as in Raghavachari [10] that under $H: G(x - \mu_2) = F(\theta^{-1}[x - \mu_1])$

$$(4.5) \qquad Z_N = \{N\lambda_N/A^2(1 - \lambda_N)\}^{\frac{1}{2}} \cdot \{h_N(\theta[\mathbf{X}_m - \hat{X}\mathbf{I}_m], \mathbf{Y}_n - \hat{Y}\mathbf{I}_n) - \mu\}$$

is asymptotically distribution-free and has a normal distribution with zero mean and unit variance. We then define the estimates $\hat{\theta}_{U \cdot N}$, $\hat{\theta}_{L \cdot N}$ and $\hat{\delta}_N$ as in (3.9) and (3.10), and for the point estimation of σ, we essentially use Hodges and Lehmann's ([4], p. 599) technique. The details are omitted for the intended brevity. Further, Theorem 1 will also hold for the scale problem and for Theorem 2, the only change that has to be made, is to replace $B(F)$ by $C(F)$ defined by (2.12). A similar change is necessary with (3.31), and again Theorem 3 will hold in this case.

It is also easily seen that the interval estimates as well as the estimator of $C(F)$ derived by this technique are all scale invariant.

5. Estimation of $B(F)$ and $C(F)$ in the c sample case. In this case, there are c independent samples of sizes $n_1, \cdots, n_c$. We define $N = \sum_{k=1}^{c} n_k$, and for the kth sample, we define $T_{N,k}$ as in Puri ([8], p. 103), with $\{E_{Ni}\}$ defined in (2.2). Also, we define (following Puri [8])

$$(5.1) \qquad \mathfrak{L}_N = \sum_{k=1}^{c} n_k[T_{N,k} - \mu]^2/A^2,$$

where μ and A^2 are defined in (3.5). It may be noted that $T_{N,1}, \cdots, T_{N,c}$ are subject to a single constraint, and hence, only $(c - 1)$ of them are linearly independent. Let us now denote the c cdf's by $F_1(x), \cdots, F_c(x)$, respectively, and take $F_1 \equiv F$, while

$$(5.2) \qquad F_k(x) = F(x - N^{-\frac{1}{2}}\theta_k) \quad \text{or} \quad F(x[1 - N^{-\frac{1}{2}}\theta_k]),$$

according as translation or scale alternatives, where $\theta_2, \cdots, \theta_c$ are all real. In this case, the asymptotic power-efficiency of the $\mathfrak{L}_N$-test with respect to a similar $\mathfrak{L}_N{}^*$-test or with respect to the classical analysis of variance test (for location) or the classical homogeneity of variance test (for scale) is independent of $c(\geqq 2)$ and becomes identical with the expressions in Section 2, in the various cases (cf. Puri [8], [9]). Thus, here also we require to estimate $B(F)$ and $c(F)$ only.

One possible method of estimation is to consider all possible $c(c-1)/2$ different pairs of samples, and for each such pair of samples, to employ the methods used in Sections 3 and 4 to estimate $B(F)$ or $C(F)$; finally to pool all these $C(c-1)/2$ estimates into a single measure, having some optimum properties. Alternatively, we may use $\mathcal{L}_N$ in (5.1) to derive suitable estimates of $B(F)$ and $C(F)$. We consider here the second approach and only the case of $B(F)$; the estimation of $c(F)$ will follow precisely on the same line. In this case, we write

$$\mathbf{X}_k = (X_{k1}, \cdots, X_{kn_k}), \qquad k = 1, \cdots, c \tag{5.3}$$

as the c sample observation vectors, and rewrite $\mathcal{L}_N$ as

$$\mathcal{L}_N = \textstyle\sum_{k=1}^{c} n_k[h_N^{(k)}(\mathbf{X}_1, \cdots, \mathbf{X}_c) - \mu]^2/A^2 \tag{5.4}$$

where $h_N^{(k)}$ depends only on the ranks of $\mathbf{X}_k$ $(k = 1, \cdots, c)$ in the combined sample, and

$$h_N^{(k)}(\mathbf{X}_1, \cdots, \mathbf{X}_{k-1}, \mathbf{X}_k + a_k\mathbf{I}_{n_k}, \mathbf{X}_{k+1}, \cdots, \mathbf{X}_c) \quad \text{is} \quad \uparrow \quad \text{in} \quad a_k \tag{5.5}$$

for all $\mathbf{X}_1, \cdots, \mathbf{X}_c$ and all $k = 1, \cdots, c$. As, we can rewrite $\mathcal{L}_N$ in (5.4) as a positive definite quadratic form in $h_N^{(2)}, \cdots, h_N^{(c)}$, we consider the following $(c-1)$ equations

$$h_N^{(k)}(\mathbf{X}_1, \mathbf{X}_2 + a_2\mathbf{I}_{n_2}, \cdots, \mathbf{X}_c + a_c\mathbf{I}_{n_c}) = \mu, \qquad k = 2, \cdots, c \tag{5.6}$$

solving which along with the convention of Hodges and Lehmann ([4], p. 599; (2.3)), we get the extended Hodges-Lehmann estimates $\hat{\theta}_2, \cdots, \hat{\theta}_c$ of $\theta_1, \cdots, \theta_c$ respectively, in the model

$$F_k(x) = F_1(x - \theta_k), \qquad k = 2, \cdots, c. \tag{5.7}$$

All these estimates are translation invariant. Further, under (5.7), it follows from Puri's [8] extension of Chernoff-Savage theorem that

$$\textstyle\sum_{k=1}^{c} n_k[h_N^{(k)}(\mathbf{X}_1, \mathbf{X}_2 + \theta_2\mathbf{I}_{n_2}, \cdots, \mathbf{X}_c + \theta_c\mathbf{I}_{n_c}) - \mu]^2/A^2 \sim \chi_{c-1}^2, \tag{5.8}$$

where χ_{c-1}^2 has the chi square distribution with $(c-1)$ degrees of freedom (d.f.). Now extending Theorem 4 of Hodges and Lehmann ([4], p. 608) in a more or less straight forward way to the c-sample case, it follows after some essentially simple steps that under

$$n_{k/N} \to \lambda_k : 0 < \lambda_k < 1, \; (\textstyle\sum_{k=1}^{c} \lambda_k = 1), \quad \text{as} \quad N \to \infty; \tag{5.9}$$

(5.8) reduces to

$$[B(F)/A]^2\{\textstyle\sum_{k=2}^{c}\sum_{q=2}^{c} \lambda_k(\delta_{kq} - \lambda_q)N(\hat{\theta}_k - \theta)(\hat{\theta}_q - \theta)\} \sim \chi_{c-1}^2 \tag{5.10}$$

where δ_{kq} is the Kronecker-delta. Let now $\chi_{\alpha \cdot c-1}^2$ be the $100(1-\alpha)$ % point of a χ^2 distribution with $(c-1)$ d.f., and let $\boldsymbol{\theta} = (\theta_2, \cdots, \theta_c)$ be a point in the

$(c - 1)$ dimensional Euclidean space. We equate the left hand side of (5.8) to $\chi^{*}_{\alpha \cdot c-1}$, and solve for the values of $h_N^{(k)}$, $k = 1, \cdots, c$. Let $\mathbf{h}_N^* = (h_N^{*(1)}, \cdots, h_N^{*(c)})$ be any solution of this equation. Then the set of points $\{\mathbf{h}_N^*\}$ describe an ellipsoid in a c dimensional space, whose origin is $\boldsymbol{\mu} = (\mu, \cdots, \mu)$. Given any $\mathbf{h}_N^*$ we consider again the $(c - 1)$ equations

$$h_N^{(k)}(\mathbf{X}_1, \mathbf{X}_2 + a_2\mathbf{I}_{n_2}, \cdots, \mathbf{X}_c + a_c\mathbf{I}_{n_c}) = h_N^{*(k)}, \quad k = 2, \cdots, c. \tag{5.11}$$

Solving these, we get the estimate $\boldsymbol{\theta}^* = (\theta_2^*, \cdots, \theta_c^*)$ of the model (5.7), where we adopt a convention similar to (3.10) i.e., we take $\boldsymbol{\theta}^*$ as farthest away from $\hat{\boldsymbol{\theta}}$, in the given direction, for which (5.11) holds. Then from (5.11), we get that

$$[B(F)/A]^2\{\textstyle\sum_{k=2}^{c}\sum_{q=2}^{c}\lambda_k(\delta_{kq} - \lambda_q)N(\theta_k^* - \hat{\theta}_k)(\theta_q^* - \hat{\theta}_q)\} \sim \chi^2_{\alpha \cdot c-1}. \tag{5.12}$$

Thus, if we compute $\hat{\theta}$ and θ^* (in any direction), we can estimate $B(F)$ from (5.12). To eliminate the arbitrariness of the direction of θ^*, we may estimate $B(F)$ along each of the $(c - 1)$ principal axes of the ellipsoid in (5.12), and combine these $(c - 1)$ estimates into a single measure (preferably with weights $\lambda_2, \cdots, \lambda_c$).

The estimation of $c(F)$ follows precisely on the same line, and hence, is not considered.

6. Estimation of certain functionals of distribution functions. Often, in many situations, we come across certain functionals of the form

$$\theta(F) = \int_{-\infty}^{\infty} g[F(x)]\, dF(x) \tag{6.1}$$

for which it may be considerably difficult to find an unbiased estimator; the density function of any absolutely continuous cdf at any specified quantile being one of the examples of such functionals. Incidentally, if in such a case, $g(F)$ satisfies the following conditions:

(i) $g(F) \geqq 0$ for all $0 \leqq F \leqq 1$,

(ii) $g(F) = (d/dx)G[F(x)]$, where $G(F)$ satisfies the conditions of Lemma 7.2 of Puri [8], then it readily follows from our results in Section 3 that if we work with a rank order statistic T_N of the type (3.2) whose weight-function $J(F)$ is $G(F)$, then the estimate (3.16) will be a translation invariant consistent estimator of $\theta(F)$ in (6.1).

Let us now consider, in brief, how our technique provides a simple method of estimating the density function of any absolutely continuous distribution at any quantile $\xi_p : 0 < p < 1$. We shall consider here both the one sample and two sample procedures. For detailed study of nonparametric estimation of density functions, the reader is referred to Rosenblatt [11] and Leadbetter and Watson [6], and the relation of our method with these will also be indicated here.

We consider first the one sample situation. In a sample of size n drawn from a population having a continuous density function $f(x)$ in the neighbourhood of the pth quantile $\xi_p : 0 < p < 1$, we require to provide an estimate of $f(\xi_p)$. It is well-known that we can always find two values of r_n and s_n, satisfying $r_n \geqq [np] + 1 \geqq s_n$, ($[s]$ being the largest integer contained in s), such that

(6.2) $$P\{X_{(s_n)} \leqq \xi_p \leqq X_{(r_n)}\} = 1 - \alpha,$$

α being any preassigned quantity and $X_{(r)}$ being the rth smallest observation in a sample of size n. The confidence interval (6.2) is obtained by using the sign-test statistic

(6.3) $$T_n = \sum_{i=1}^{n} c(X_i - \xi_p), \qquad c(u) = 1, \quad \text{if} \quad u \leqq 0,$$
$$= 0, \quad \text{otherwise};$$

and the confidence interval for T_n is given by

(6.4) $$P\{s_n \leqq T_n < r_n \mid F(\xi_p) = p\} = 1 - \alpha.$$

Further, it is well-known that for the sequence of alternatives $H_n : F_n(x) \equiv F(x + n^{-\frac{1}{2}}\theta)$, $n^{\frac{1}{2}}(T_n - p)/[p(1 - p)]^{\frac{1}{2}}$ has asymptotically a normal distribution with mean $\theta f(\xi_p)/[p(1 - p)]^{\frac{1}{2}}$ and unit variance. Thus, if we use the result in Theorem 2 (as adapted in the one sample case), we may conclude that the following

(6.5) $$f_n(\xi_p) = (r_n - s_n - 1)/n[X_{(r_n)} - X_{(s_n)}]$$

is a translation invariant consistent estimate of $f(\xi_p)$. This estimate has a close analogy with Rosenblatt's estimate. He considered a nonnegative weight function $w_n(u)$ such that

(6.6) $$\int_{-\infty}^{\infty} w_n(u)\, du = 1, \qquad \int_{|u|<\epsilon} w_n(u)\, du \to 1,$$

for any $\epsilon > 0$, and his estimate is

(6.7) $$f_n(y) = \int_{-\infty}^{\infty} w_n(y - x)\, dF_n(x) = n^{-1} \sum_{j=1}^{n} w_n(y - X_j).$$

In order that $f_n(y)$ has a negligible bias, he further showed that $w_n(u)$ should satisfy

(6.8) $$\int_{-\infty}^{\infty} u w_n(u)\, du = 0.$$

If we rewrite (6.5) in the form of (6.7) then it can be easily shown that the weight function corresponding to $f_n(\xi_p)$ in (6.5) has stochastically the same properties as a Rosenblatt weight function.

Finally let us consider the two sample problem. We consider a statistic of the form (2.1), where we take

(6.9) $$E_{Ni} = 1 \quad \text{if} \quad i \leqq r = [Np] + 1,$$
$$= 0 \quad \text{otherwise},$$

so that μ and A^2 defined in (3.5) reduce to p and $p(1 - p)$, respectively. In this case, it is easily shown on using Theorem 2 that the estimator (3.31) is nothing but a translation invariant consistent estimator of $f(\xi_p)$. By varying $r(=[Np] + 1)$ over the range of $p: 0 < p < 1$, we can estimate $f(\xi_p)$ for various values of p.

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