A NOTE ON GENERALIZED INVERSES IN THE LINEAR HYPOTHESIS NOT OF FULL RANK

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1. Introduction. The object of this note is to co-ordinate some results presented by Chipman [1], Goldman and Zelen [2], and John [3].

We shall consider the model in the form

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{1}$$

where $\mathbf{X}$ is of order $n \times k$ of rank $k - s$, and $\boldsymbol{\varepsilon}$ has zero mean and variance matrix $\sigma^2\mathbf{I}$. It is desired to estimate the parameters $\boldsymbol{\beta}$ subject to the set of s linearly independent constraints $\mathbf{H}\boldsymbol{\beta} = \mathbf{c}$, where $\mathbf{H}$ is complementary to $\mathbf{X}$.

2. Generalized inverse matrices. A unique generalized inverse of a matrix $\mathbf{X}$ has been defined (e.g. [2]) as a matrix $\mathbf{A}$ satisfying

$$\mathbf{XAX} = \mathbf{X}, \qquad \mathbf{AXA} = \mathbf{A}, \qquad (\mathbf{XA})' = \mathbf{XA}, \qquad (\mathbf{AX})' = \mathbf{AX}. \tag{2}$$

In this note we introduce a generalized inverse which satisfies the first three conditions only. Such a matrix will be denoted by $\mathbf{X}^-$, and is essentially the same as the "weak generalized inverse" of Goldman and Zelen [2], the slight change being necessitated by their use of $\mathbf{X}'$ in place of $\mathbf{X}$ in (1). Rao [6] also considered a class of generalized inverses satisfying only the first condition of (2).

Chipman [1, Lemma 1.1] shows that there is a matrix $\mathbf{B}$ such that $\mathbf{XB} = \mathbf{0}$ and $\mathbf{HB} = \mathbf{I}_s$. If

$$\mathbf{W} = \begin{bmatrix} \mathbf{X} \\ \mathbf{H} \end{bmatrix},$$

$\mathbf{W}'\mathbf{W} = \mathbf{X}'\mathbf{X} + \mathbf{H}'\mathbf{H}$, which is of order $k \times k$ of rank k, and $(\mathbf{W}'\mathbf{W})^{-1}\mathbf{X}' = \mathbf{X}^-$ i.e. it satisfies the first three conditions of (2). Similarly, $\mathbf{B} = (\mathbf{W}'\mathbf{W})^{-1}\mathbf{H}' = \mathbf{H}^-$ and we have the relations

$$\mathbf{XH}^- = \mathbf{0}, \qquad \mathbf{HX}^- = \mathbf{0}, \qquad \mathbf{HH}^- = \mathbf{I}_s. \tag{3}$$

For the special case here considered, $\mathbf{X}^-$ and $\mathbf{H}^-$ are the matrices denoted by Chipman (Theorem 1.1) as $\mathbf{X}^\ddagger$ and $\mathbf{Y}^\ddagger$. We note that $\mathbf{X}^-$ and $\mathbf{H}^-$ are not unique, being dependent on choice of $\mathbf{H}$ and $\mathbf{X}$ respectively, but are unique (as defined by Chipman) for a particular $\mathbf{H}$ or $\mathbf{X}$ respectively.

3. Solution of normal equations. Plackett [5], in deriving the solution (in our notation)

$$\hat{\boldsymbol{\beta}} = \mathbf{X}^-\mathbf{y} + \mathbf{H}^-\mathbf{c}, \tag{4}$$

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makes use of a matrix $\mathbf{D}$ such that $\mathbf{XD} = \mathbf{0}$ and $\mathbf{HD}$ is nonsingular of rank s. John [3], uses the same matrix in showing that the Rao generalized inverse of $\mathbf{S}$ $(= \mathbf{X}'\mathbf{X})$ implicitly used by Plackett, viz. $(\mathbf{W}'\mathbf{W})^{-1}$, is not the same as the matrix $\mathbf{P}$ (also a Rao generalized inverse) in

$$\begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{R} \end{bmatrix} = \begin{bmatrix} \mathbf{S} & \mathbf{H}' \\ \mathbf{H} & \mathbf{0} \end{bmatrix}^{-1}, \tag{5}$$

used by other authors (e.g. Kempthorne [4], p. 72) in solving the normal equations. However, it is evident from the above that $\mathbf{D}$ may be taken as $\mathbf{H}^-$ so that $\mathbf{HH}^- = \mathbf{I}_s$, with considerable simplification of the algebra.

4. Relationship between the two Rao generalized inverses. This may be established in a rather more direct manner than that used by John. From (5) we have

$$\mathbf{SP} + \mathbf{H}'\mathbf{Q}' = \mathbf{I}_k, \qquad \mathbf{SQ} + \mathbf{H}'\mathbf{R} = \mathbf{0}.$$

$$\mathbf{HP} = \mathbf{0}, \qquad \mathbf{HQ} = \mathbf{I}_s.$$

Pre-multiplication of the first two equations by $(\mathbf{H}^-)'$ gives, by virtue of (3), $\mathbf{Q} = \mathbf{H}^-$ and $\mathbf{R} = \mathbf{0}$. Pre-multiplication of the first equation by $\mathbf{P}$ and by $(\mathbf{W}'\mathbf{W})^{-1}$ gives, respectively,

$$\mathbf{PSP} = \mathbf{P}, \tag{6}$$

(showing that $\mathbf{S}$ is a Rao generalized inverse of $\mathbf{P}$, as well as *vice versa*) and

$$\mathbf{X}^-\mathbf{XP} = (\mathbf{W}'\mathbf{W})^{-1} - \mathbf{H}^-(\mathbf{H}^-)'. \tag{7}$$

The right hand side of (7) is actually John's expression for $\mathbf{P}$. To show this we use (3), (6), and (7) as follows:

$$\mathbf{XX}^-\mathbf{XP} = \mathbf{XP} = (\mathbf{X}^-)';$$

and

$$\mathbf{P} = \mathbf{PSP} = (\mathbf{XP})'\mathbf{XP} = \mathbf{X}^-(\mathbf{X}^-)'.$$

This gives the variance matrix of $\hat{\beta}$, as is also evident from (4).

The fact that apparently non-unique sub-matrices, e.g. $\mathbf{Q} = \mathbf{H}^-$, appear in the inverse matrix (5), necessarily unique, is explainable by the remark at the end of Section 2.

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