A BIVARIATE t DISTRIBUTION

By M. M. Siddiqui

Colorado State University

1. Introduction and summary. In this note we consider the joint distribution of Student variates (t_1, t_2), where t_1 corresponds to the x-observations and t_2 to y-observations from a bivariate normal distribution. No applications are suggested as Hotelling's T^2 is more appropriate whenever estimation of covariance matrix is necessary. Possibly on occasions, when the correlation coefficient, ρ, between x and y may be assumed known, for example from past records, the bivariate (t_1, t_2) may be useful. The main interest in this distribution is theoretical. First, because this type of bivariate (t_1, t_2) has never been worked out before while the joint distribution of $(\bar{x}, \bar{y}, s_1, s_2, r)$ is commonly known. Second, for degrees of freedom $n = 1$ (sample size $N = 2$) the bivariate t distribution is an example of a bivariate Cauchy distribution. Lastly, the asymptotic approximation obtained in Section 3 is an application of the method of steepest descent, which has some methodological interest and can be used in other situations.

There is no loss of generality, as far as the distribution of (t_1, t_2, r) is concerned, in assuming the means of x and y to be zero and variances to be unity. The only parameter which enters into the joint distribution of (t_1, t_2, r) or into that of (t_1, t_2) is ρ. Because of the simplicity of the limiting distribution and the asymptotic approximation we will present them first, while the exact distributions are evaluated only for $n = 1$, and 3 ($N = 2$ and 4). The exact distribution for arbitrary n can be worked out, in double or triple sums, following the method given for $n = 3$.

2. Distribution of (t_1, t_2, r). Let

$$\phi(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2},$$

$$\phi(x, y; \rho) = (2\pi)^{-1}(1 - \rho^2)^{-\frac{1}{2}} \exp[-(2(1 - \rho^2))^{-1}(x^2 + y^2 - 2\rho xy)].$$

Let (x_i, y_i), $i = 1, 2, \cdots, N$, be a random sample from $\phi(x, y; \rho)$, where $|\rho| < 1$, and

$$\text{(2.1)} \qquad \bar{x} = N^{-1} \sum x, \qquad \bar{y} = N^{-1} \sum y, \qquad s_1^2 = N^{-1} \sum (x - \bar{x})^2,$$
$$s_2 = N^{-1} \sum (y - \bar{y})^2, \qquad r = (Ns_1s_2)^{-1} \sum (x - \bar{x})(y - \bar{y}),$$

where the summations extend over the sample values. The joint pdf of $(\bar{x}, \bar{y}, s_1, s_2, r)$ is [1], p. 385,

$$f(\bar{x}, \bar{y}, s_1, s_2, r) = [N^N/2\pi^2\Gamma(N - 2)(1 - \rho^2)^{\frac{1}{2}N}](s_1s_2)^{N-2}(1 - r^2)^{\frac{1}{2}(N-4)}$$
$$\cdot \exp[-[N/2(1 - \rho^2)]\{\bar{x}^2 + s_1^2 + \bar{y}^2 + s_2^2 - 2\rho(\bar{x}\bar{y} + rs_1s_2)\}],$$

with $-\infty < \bar{x} < \infty, -\infty < \bar{y} < \infty, 0 \leqq s_1 < \infty, 0 \leqq s_2 < \infty, -1 \leqq r \leqq 1$.

Received 24 March 1966.

We make the following transformations in succession:

(i) $t_1 = \bar{x}s_1^{-1}(N-1)^{\frac{1}{2}}$, $t_2 = \bar{y}s_2^{-1}(N-1)^{\frac{1}{2}}$, $s_1 = s_1$, $s_2 = s_2$, $r = r$;

(ii) $z_1 = [N/2(1-\rho^2)](1 + t_1^2/(N-1))s_1^2$, $z_2 = [N/2(1-\rho^2)]\cdot(1 + t_2^2/(N-1))s_2^2$, $t_1 = t_1$, $t_2 = t_2$, $r = r$;

(iii) $u_1 = (z_1z_2)^{\frac{1}{2}}$, $u_2 = \frac{1}{2}\log(z_1/z_2)$, $t_1 = t_1$, $t_2 = t_2$, $r = r$. We then integrate out u_1 and u_2 with the help of the gamma integral and

$$\int_0^\infty dx/(\cosh x - a)^N = [B(\tfrac{1}{2}, N)/2^{\frac{1}{2}}(1-a)^{N-\frac{1}{2}}]F(\tfrac{1}{2}, \tfrac{1}{2}; N+\tfrac{1}{2}; \tfrac{1}{2}(1+a)),$$

where

$$F(a, b; c; x) = 1 + (ab/c)(x/1!) + [a(a+1)b(b+1)/c(c+1)]x^2/2! + \cdots,$$

is the Gauss hypergeometric series which converges for $|x| < 1$. For transformations (iii) and the above integral see [1], pp. 386–388. The joint pdf of (t_1, t_2, r) is given by

$$\begin{aligned} f(t_1, t_2, r) = {} & [\Gamma(n+2)(1-\rho^2)^{\frac{1}{2}(n+1)}/(2\pi)^{\frac{1}{2}}\Gamma(n+\tfrac{3}{2})] \\ & \cdot[(1+t_1^2/n)(1+t_2^2/n)]^{-\frac{1}{2}(n+1)}(1-r^2)^{\frac{1}{2}(n-3)} \\ & \cdot(1-b-cr)^{-n-\frac{1}{2}}F(\tfrac{1}{2}, \tfrac{1}{2}; n+\tfrac{3}{2}; \tfrac{1}{2}(1+b+cr)), \end{aligned} \tag{2.2}$$

where $n = N - 1$,

$$b = [\rho t_1t_2/n][(1+t_1^2/n)(1+t_2^2/n)]^{-\frac{1}{2}}, \qquad c = \rho[(1+t_1^2/n)(1+t_2^2/n)]^{-\frac{1}{2}}.$$

It is noted that

$$|1+b+cr| \leqq 1 + |\rho|[1 + |t_1t_2|/n][(1+t_1^2/n)(1+t_2^2/n)]^{-\frac{1}{2}} \leqq 1 + |\rho| < 2,$$

so that the hypergeometric series in (2.2) converges uniformly in (t_1, t_2) and for every $|\rho| < 1$. If we now make the substitution

$$r = \rho + (1-\rho^2)n^{-\frac{1}{2}}v, \qquad t_1 = t_1, \qquad t_2 = t_2, \tag{2.3}$$

we obtain

$$\begin{aligned} f(t_1, t_2, v) = {} & \phi(v)\phi(t_1, t_2; \rho)[1 - (\rho v/2n^{\frac{1}{2}}) \\ & \cdot\{2v^2 - 7 + (t_1^2 + t_2^2 - 2\rho t_1t_2)/(1-\rho^2)\} + O(n^{-1})]. \end{aligned} \tag{2.4}$$

Thus

$$\lim_{n\to\infty} f(t_1, t_2, v) = \phi(v)\phi(t_1, t_2; \rho), \tag{2.5}$$

so that, in the limit, (t_1, t_2, v) are trivariate normal and (t_1, t_2) is independent of r.

3. Asymptotic distribution of (t_1, t_2). We now derive an asymptotic distribution of (t_1, t_2) by integrating out r. This approximation is different from the limit distribution of Section 2, and is of some methodological interest. We need to evaluate

$$\begin{aligned} I_n = {} & \int_{-1}^{1} (1-r^2)^{\frac{1}{2}(n-3)}(1-b-cr)^{-n-\frac{1}{2}} \\ & \cdot F(\tfrac{1}{2}, \tfrac{1}{2}; n+\tfrac{3}{2}; \tfrac{1}{2}(1+b+cr))\, dr. \end{aligned} \tag{3.1}$$

Set

$$u(z) = \tfrac{1}{2}\log(1 - z^2) - \log(1 - b - cz),$$
$$h(z) = (1 - z^2)^{-\frac{1}{2}}(1 - b - cz)F(\tfrac{1}{2}, \tfrac{1}{2}; n + \tfrac{3}{2}; \tfrac{1}{2}(1 + b + cz)),$$

where z is now a complex variable. It is easily verified that $(1 - b)^2 > c^2$ so that the real singularity $(1 - b)/c$ of u and h is outside the interval $(-1, 1)$. The other singularities being 1 and -1, u and h are analytic in the strip $-1 < \operatorname{Re} z < 1$. I_n can be written as $I_n = \int_{-1}^{1} e^{nu}h\, dz$. The saddlepoints are obtained from $u'(z_0) = 0$, which has only one solution $z_0 = c(1 - b)^{-1}$. z_0 is real and an interior point of the interval $(-1, 1)$. Also,

$$u''(z_0) = -(1 - b)^4[(1 - b)^2 - c^2]^{-2} < 0.$$

Thus z_0 is the unique saddlepoint and the real open interval $(-1, 1)$ the line of steepest descent through z_0. From the standard theory we obtain the asymptotic approximation

$$I_n = (2\pi)^{\frac{1}{2}}(-nu_0'')^{-\frac{1}{2}}e^{nu_0}h_0[1 + O(n^{-1})],$$

where u_0, h_0 etc. denote the value of u, h etc. at $z = z_0$. In fact

$$u_0 = -\tfrac{1}{2}\log[(1 - b)^2 - c^2], \qquad h_0 = [(1 - b)^2 - c^2]^{-2}(1 - b)^{\frac{1}{2}}F_0.$$

Finally

$$\begin{aligned} f(t_1, t_2) \sim {} & [\Gamma(n + 2)(1 - \rho^2)^{\frac{1}{2}(n+1)}/2\pi n^{\frac{1}{2}}\Gamma(n + \tfrac{3}{2})] \\ & \cdot[(1 + t_1^2/n)(1 + t_2^2/n)]^{-\frac{1}{2}(n+1)} \\ & \cdot(1 - b)^{\frac{1}{2}}[(1 - b)^2 - c^2]^{-\frac{1}{2}(n+1)}F_0. \end{aligned} \tag{3.2}$$

Improved asymptotic approximations can be obtained by expanding $u(z)$ and $h(z)$ in Taylor series around the point $z = z_0$, and take the form

$$f(t_1, t_2) \sim g_n(t_1, t_2)[1 + A_1/n + A_2/n^2 + \cdots],$$

where $g_n(t_1, t_2)$ is the term on the right hand side of (3.2).

It may be observed that uniformly in every finite two dimensional interval for (t_1, t_2) as $n \to \infty$,

$$\Gamma(n + 2)/n^{\frac{1}{2}}\Gamma(n + \tfrac{3}{2}) \to 1; \qquad (1 - b)^{\frac{1}{2}} \to 1; \qquad F_0 \to 1;$$
$$[(1 + t_1^2/n)(1 + t_2^2/n)]^{-\frac{1}{2}(n+1)} \to e^{-\frac{1}{2}(t_1^2+t_2^2)};$$
$$\tfrac{1}{2}(n + 1)\log(1 - \rho^2) - \tfrac{1}{2}(n + 1)\log[(1 - b)^2 - c^2] \to$$
$$-\tfrac{1}{2}\log(1 - \rho^2) - (2(1 - \rho^2))^{-1}\{\rho^2(t_1^2 + t_2^2) - 2\rho t_1 t_2\},$$

hence

$$\lim_{n\to\infty} f(t_1, t_2) = \lim_{n\to\infty} g_n(t_1, t_2) = \phi(t_1, t_2; \rho).$$

Thus $g_n(t_1, t_2)$ is intermediate between the exact distribution and the normal approximation.

4. Exact distribution for $n = 1$. When $N = 2$, $n = 1$, and $r = \pm 1$ and the derivation (2.2) fails, it is simpler to start with the joint distribution of (x_1, y_1, x_2, y_2), which is $\phi(x_1, y_1; \rho)\phi(x_2, y_2; \rho)$. We then have

$$t_1 = (x_1 + x_2)/|x_1 - x_2|, \qquad t_2 = (y_1 + y_2)/|y_1 - y_2|.$$

Now, the vector $2^{-\frac{1}{2}}(x_1 + x_2, y_1 + y_2, x_1 - x_2, y_1 - y_2)$ has the same distribution as (x_1, y_1, x_2, y_2) and we may as well consider the distribution of $t_1 = x_1/|x_2|$, $t_2 = y_1/|y_2|$. We divide the sample space into four disjoint sets:

(1) $A_1 = \{x_2 \geqq 0, y_2 \geqq 0\}$. Here set $t_1 = x_1/x_2$, $t_2 = y_1/y_2$, $x = x_2$, $y = y_2$.

(2) $A_2 = \{x_2 < 0, y_2 < 0\}$. Here set $t_1 = -x_1/x_2$, $t_2 = -y_1/y_2$, $x = x_2$, $y = y_2$.

(3) $A_3 = \{x_2 < 0, y_2 \geqq 0\}$. Here set $t_1 = -x_1/x_2$, $t_2 = y_1/y_2$, $x = x_2$, $y = y_2$.

(4) $A_4 = \{x_2 \geqq 0, y_2 < 0\}$. Here set $t_1 = x_1/x_2$, $t_2 = -y_1/y_2$, $x = x_2$, $y = y_2$.

The pdf of (t_1, t_2) is then

$$f(t_1, t_2) = \textstyle\sum_{i=1}^{4} h_i(t_1, t_2; \rho)$$

where

$$h_i(t_1, t_2; \rho) = \textstyle\int_{A_i} f(t_1, t_2, x, y)\, dx\, dy.$$

It is easily seen that $h_1(t_1, t_2; \rho) = h_2(t_1, t_2; \rho) = h_3(t_1, t_2; -\rho) = h_4(t_1, t_2; -\rho)$. Thus

$$f(t_1, t_2) = 2[h_1(t_1, t_2; \rho) + h_1(t_1, t_2; -\rho)].$$

Now

$$h_1(t_1, t_2; \rho) = (4\pi^2(1 - \rho^2))^{-1} \textstyle\int_0^\infty \int_0^\infty \exp[-(2(1 - \rho^2))^{-1}\{(1 + t_1^2)x^2 + (1 + t_2^2)y^2 - 2\rho(1 + t_1t_2)xy\}]xy\, dx\, dy.$$

Make the following transformations in succession:

(1) $z_1 = [2(1 - \rho^2)]^{-1}(1 + t_1^2)x^2$, $z_2 = [2(1 - \rho^2)]^{-1}(1 + t_2^2)y^2$,

(2) $u_1 = (z_1z_2)^{\frac{1}{2}}$, $u_2 = \frac{1}{2}\log(z_1/z_2)$,

and set

$$\cos\theta = 2\rho(1 - \rho^2)(1 + t_1t_2)(1 + t_1^2)^{-\frac{1}{2}}(1 + t_2^2)^{-\frac{1}{2}},$$

where $\theta = \theta(t_1, t_2)$ is between 0 and π. On integrating out u_1 and u_2 we obtain

$$(4.1)\quad h_1(t_1, t_2; \rho) = [(1 - \rho^2)\operatorname{cosec}^2\theta/4\pi^2(1 + t_1^2)(1 + t_2^2)] \cdot [1 + (\pi - \theta)\cot\theta].$$

In integrating out u_2 we have used

$$\textstyle\int_0^\infty dx/(\cosh\theta + \cos\theta) = \theta/\sin\theta, \qquad 0 < \theta < \pi,$$

and arrived at (4.1) by differentiation with respect to $\cos\theta$. Finally

$$(4.2)\quad f(t_1, t_2) = [(1 - \rho^2)\operatorname{cosec}^2\theta/\pi^2(1 + t_1^2)(1 + t_2^2)][1 + (\pi/2 - \theta)\cot\theta].$$

When $\rho = 0$, $\theta = \pi/2$, $\operatorname{cosec}\theta = 1$, and $f(t_1, t_2)$ becomes a product of two Cauchy densities.

5. Exact distribution when $n = 3$. If in (3.1) we expand the hypergeometric function and express powers of $(1 + b + cr)$ as powers of $[2 - (1 - b - cr)]$, we get

$$I_n = \sum_{k=0}^{\infty} [\Gamma(n + \tfrac{3}{2})\Gamma^2(k + \tfrac{1}{2})/\Gamma(k + n + \tfrac{3}{2})\Gamma^2(\tfrac{1}{2})k!] \cdot \sum_{j=0}^{k} (-1/2)^j \binom{k}{j} \int_{-1}^{1} (1 - b - cr)^{j-n-\frac{1}{2}}(1 - r^2)^{\frac{1}{2}(n-3)} \, dr.$$

For $N = 4$, $n = 3$, and

$$I_3 = \sum_{k=0}^{\infty} [\Gamma(\tfrac{9}{2})\Gamma^2(k + \tfrac{1}{2})/\Gamma(k + \tfrac{9}{2})\Gamma^2(\tfrac{1}{2})k!] \sum_{j=0}^{k} (-\tfrac{1}{2})^j \binom{k}{j} \cdot [c/c(5 - 2j)][(1 - b - c)^{j-\frac{5}{2}} - (1 - b + c)^{j-\frac{5}{2}}].$$

The terms diminish in magnitude quite rapidly. Thus, if we denote the kth term as u_k, we have

$$u_0 = (2/5c)[(1 - b - c)^{-\frac{5}{2}} - (1 - b + c)^{-\frac{5}{2}}],$$

$$u_1 = (1/18)[u_0 - (1/3c)(1 - b - c)^{-\frac{3}{2}} + (1/3c)(1 - b + c)^{-\frac{3}{2}}],$$

$$u_2 = (1/88)[u_0 - (2/3c)(1 - b - c)^{-\frac{3}{2}} + (2/3c)(1 - b + c)^{-\frac{3}{2}} + (1/2c)(1 - b - c)^{-\frac{1}{2}} - (1/2c)(1 - b + c)^{-\frac{1}{2}}].$$

Finally

$$f(t_1, t_2) = [32 \cdot 2^{\frac{1}{2}}(1 - \rho^2)^2/35\pi^2](1 + t_1^2/3)^{-2}(1 + t_2^2/3)^{-2} I_3 .$$

REFERENCE

[1] KENDALL, M. G. and STUART, ALAN. (1958). *The Advanced Theory of Statistics*. Hafner, New York.