A NOTE ON RISK AND MAXIMAL REGULAR GENERALIZED SUBMARTINGALES IN STOPPING PROBLEMS[1]

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In a recent paper Haggstrom [2] rearranged and amplified on the excellent work of Snell [3] on optimal stopping problems as a preliminary to generalizing the framework to deal with problems in control and experimental design. The purpose of this note is to point out that one of Haggstrom's results can be derived under weaker conditions by using the results of Snell.

Let $\{Z_n, F_n, n \geqq 1\}$ be a stochastic process on a probability space (Ω, F, P) with points ω. A stopping variable (sv) is a random variable (rv) t with values in $\{1, 2, \cdots, \infty\}$ such that $t < \infty$ a.e. and $\{t = n\} \varepsilon F_n$ for each n. For any such sv t, a rv Z_t is defined by $Z_t(\omega) = Z_n(\omega)$ if $t(\omega) = n$ and $Z_t(\omega) = \infty$ if $t(\omega) = \infty$. The optimal stopping problem consists of finding t to minimize the risk $E(Z_t)$.

Random variables are defined in the extended sense, i.e., they can take on the values $+\infty$ and $-\infty$. We regard $E(X)$ as defined as long as either the positive or negative part of X has finite expectation. This extension is usually referred to by the term generalized. The reader is referred to Snell [3] or Haggstrom [2] for detailed discussion of the technical terms used in this note.

Snell [2] obtained results characterizing the solution of the optimal stopping problem in terms of $\{Y_n, F_n, n \geqq 1\}$, the maximal regular generalized submartingale relative to $\{Z_n, F_n, n \geqq 1\}$. In particular, as part of his Theorem 3.6, he proved that

$$E(Y_n) = \inf_{t \varepsilon T_n} E(Z_t) \tag{1}$$

where T_n is the class of sv for which $t \geqq n$ a.e. The proof of this theorem used the hypothesis that

$$E(\inf_n Z_n) > -\infty. \tag{2}$$

Haggstrom [1] introduced, and developed his results in terms of,

$$X_n = \text{ess inf}_{t \varepsilon T_n} E(Z_t \mid F_n) \tag{3}$$

which represents the optimal risk at stage n to the player who, for one reason or another, has not stopped previously. The relation between the results of Snell and Haggstrom, and in fact the motivation of Snell, is clarified by Haggstrom's Theorem 3.5 which states,

$$X_n = Y_n \quad \text{a.e.} \qquad n = 1, 2, \cdots \tag{4}$$

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under (2) and the extra condition that $E(|Z_n|) < \infty$ for each n. Haggstrom pointed out that this extra condition can be relaxed to require only that for each n there exist a sv t in T_n such that $E(Z_t) < \infty$. He referred to the fact that Y. S. Chow independently proved Theorem 3.5 and also proved (4) using $E(|Z_n|) < \infty$ for each n and $E(\sup_n Z_n) < \infty$.

The object of this note is to prove

$$E(Y_n \mid F_r) = \text{ess inf}_{t \varepsilon T_n} E(Z_t \mid F_r) \quad \text{a.e.} \qquad \text{for} \quad n \geqq r \tag{5}$$

as a corollary of Snell's theorem using only condition (2). Equation (4) follows by setting $r = n$.

To this end let B be an arbitrary set of F_r. For $n \geqq r$, let

$$\begin{aligned} Z_n^* &= Z_n, \qquad Y_n^* = Y_n \quad \text{on} \quad B \\ Z_n^* &= 0, \qquad Y_n^* = 0 \quad \text{on} \quad B^c. \end{aligned} \tag{6}$$

Then $(Z_n^*, F_n, n \geqq r)$ is a stochastic process and $(Y_n^*, F_n, n \geqq r)$ is the maximal regular generalized submartingale relative to it. Applying Snell's result

$$E(Y_n^*) = \inf_{t \varepsilon T_n} E(Z_t^*). \tag{7}$$

Hence, with χ_B the characteristic function of B,

$$E[\chi_B E(Y_n \mid F_r)] = \inf_{t \varepsilon T_n} E[\chi_B E(Z_t \mid F_r)]. \tag{8}$$

Suppose now that there is a set of positive measure on which $E(Y_n \mid F_r) >$ ess $\inf_{t \varepsilon T_n} E(Z_t \mid F_r)$. Then there is a subset of positive measure on which $E(Y_n \mid F_r)$ is bounded away from $-\infty$ and ess $\inf_{t \varepsilon T_n} E(Z_t \mid F_r)$ is bounded away from $+\infty$. Hence there exist $t \varepsilon T_n$ and $B \varepsilon F_r$ such that B has positive measure, $E(Y_n \mid F_r) > E(Z_t \mid F_r)$, $E(Y_n \mid F_r)$ is bounded away from $-\infty$, and $E(Z_t \mid F_r)$ is bounded away from $+\infty$ a.e. on B. Then

$$E[\chi_B E(Y_n \mid F_r)] > E[\chi_B E(Z_t \mid F_r)]$$

which contradicts (8). Similarly, assuming $E(Y_n \mid F_r) <$ ess $\inf_{t \varepsilon T_n} E(Z_t \mid F_r)$ on a set of positive measure implies the existence of a set of positive measure $B \varepsilon F_r$ such that for all $t \varepsilon T_n$, $E(Y_n \mid F_r)$ is bounded away from $+\infty$, $E(Z_t \mid F_r)$ is bounded away from $-\infty$, and $E(Y_n \mid F_r) < E(Z_t \mid F_r)$ a.e. on B. This also leads to a contradiction of (8). Hence equation (5) holds and (4) follows as a special case.

A conversation with Robbins based on a draft of this note indicated his feeling that the methods of Chow-Robbins [1] could also be applied to derive this result. A letter arrived the next day from D. Seigmund presenting such a derivation.

REFERENCES

[1] Chow, Y. S. and Robbins, H. (1967). On values to be associated with a stochastic sequence. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.*

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[3] Snell, J. L. (1952). Applications of martingale system theorems. *Trans. Amer. Math. Soc.* **73** 292–312.