MATRICVARIATE GENERALIZATIONS OF THE MULTIVARIATE t DISTRIBUTION AND THE INVERTED MULTIVARIATE t DISTRIBUTION[1]

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1. Introduction. Consider the random p-vector,

$$t = v^{-\frac{1}{2}}y,$$

where $v \sim \chi_\nu^2/Q$, independently of the vector $y \sim N(0, P^{-1})$. As is well known (Cornish, 1954), t has the multivariate t distribution with density,

$$\rho_t = \Gamma[\tfrac{1}{2}(\nu + p)][\Gamma(\tfrac{1}{2}\nu)\pi^{p/2}]^{-1}Q^{\nu/2}\cdot|P|^{\frac{1}{2}}[Q + t'Pt]^{-(\nu+p)/2}. \tag{1.1}$$

(Traditionally, $Q = \nu$.) It is also known (Ando and Kaufman, 1965, eq. (6c)) that t has the representation,

$$t \sim (U^{\frac{1}{2}\prime})^{-1}x,$$

where $U^{\frac{1}{2}}U^{\frac{1}{2}\prime} = U \sim W(P, \nu + p - 1)$, the Wishart distribution with covariance-matrix parameter P and $\nu + p - 1$ degrees of freedom, independently of the vector $x \sim N(0, QI_p)$, with I_p denoting the $p \times p$ indentity matrix. In this note we study the random $p \times q$ matrix T with density proportional to

$$|Q + T'PT|^{-m/2},$$

where Q is now a $q \times q$ matrix. Such a density appears in Kshirsagar (1960), Olkin and Rubin (1964, eq. (4.2)), Tiao and Zellner (1964), Geisser (1965), Kiefer and Schwartz (1965, eq. (4.1)).

We give several representations for T (Section 3), a Bayesian application involving a new conjugate prior distribution (Section 4), and a matric generalization (Section 5) of the random vector $r = [1 + (1/Q)t'Pt]^{-\frac{1}{2}}t$ (Raiffa and Schlaifer, 1961, p. 259), which is, in turn, a generalization of the scalar sample correlation coefficient r from two independent univariate normal populations.

2. Notation and preliminaries. From Wishart's density for a symmetric $p \times p$ matrix U, if $\lambda > \frac{1}{2}(p - 1)$, $M > 0$, then

$$\int_{U>0} |U|^{\lambda-(p+1)/2} \operatorname{etr}(-MU)\, dU = |M|^{-\lambda}\Gamma_p(\lambda), \tag{2.1}$$

where

$$\Gamma_p(\lambda) = \pi^{p(p-1)/4}\Gamma(\lambda)\Gamma(\lambda - \tfrac{1}{2}) \cdots \Gamma(\lambda - \tfrac{1}{2}p + \tfrac{1}{2}), \tag{2.2}$$

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and, as thoughout the paper, $A > 0$ indicates that a matrix A is symmetric and positive definite.

Recall (Anderson, 1958) that for a partitioned matrix $A = (A_{ij})$, $i, j = 1, 2$, if A_{11} and A_{22} are nonsingular,

$$|A| = |A_{11\cdot 2}|\cdot|A_{22}| = |A_{11}|\cdot|A_{22\cdot 1}|, \tag{2.3}$$

where as usual, $A_{ii\cdot j} = A_{ii} - A_{ij}A_{jj}^{-1}A_{ji}$.

By equation (2.3) and the density (1.1) of the multivariate t distribution, if $p \times p$ $M > 0$ and $m > p$,

$$\int_{R^p} |M + tt'|^{-m/2}\, dt = \pi^{p/2}\Gamma[\tfrac{1}{2}(m - p)]/\Gamma(\tfrac{1}{2}m)\cdot|M|^{-(m-1)/2}.$$

Hence, for the $p \times q$ matrix T, if $m > p + q - 1$,

$$\begin{aligned} k(m, p, q) &\equiv \int_{R^{pq}} |I_q + T'T|^{-m/2}\, dT \\ &= \int_{R^{pq}} |I_p + TT'|^{-m/2}\, dT = \pi^{pq/2}\Gamma_q[\tfrac{1}{2}(m - p)]/\Gamma_q(\tfrac{1}{2}m). \end{aligned} \tag{2.4}$$

Since $|I_q + T'T| = |I_p + TT'|$ and $dT = dT' = \Pi_{i,j}\, dt_{ij}$, then

$$k(m, p, q) = k(m, q, p). \tag{2.5}$$

Given a symmetric $p \times p$ matrix A, let $A^{\frac{1}{2}}$ denote any measurable $p \times p$ square root of A, $A^{\frac{1}{2}}A^{\frac{1}{2}'} = A$. If A is nonnegative definite, $A = S^2$, where $S = B\Lambda^{\frac{1}{2}}B'$ with $A = B\Lambda B'$, $B'B = I_p$ and Λ diagonal. If A is positive definite, S is the only symmetric square root of A, and then, since $S^{-1}A^{\frac{1}{2}} = S\,(A^{\frac{1}{2}'})^{-1}$, every $p \times p$ square root is of the form,

$$A^{\frac{1}{2}} = SO, \qquad OO' = I_p \tag{2.6}$$

3. The Matrix T.

THEOREM 3.1. *Let T be the random $p \times q$ matrix,*

$$T = (U^{\frac{1}{2}'})^{-1}X, \tag{3.1}$$

where $U \sim W(P, m - q)$, $P > 0$, $m > p + q - 1$, independently of X, the **row vectors** *of which are independently $N(0, Q)$ distributed, $Q > 0$. Then T has the density,*

$$\begin{aligned} \rho_T &= [k(m, q, p)]^{-1}|P|^{-(m-q)/2}|Q|^{-p/2}|P^{-1} + TQ^{-1}T'|^{-m/2} \\ &= [k(m, p, q)]^{-1}|Q|^{(m-p)/2}|P|^{q/2}|Q + T'PT|^{-m/2}. \end{aligned} \tag{3.2}$$

PROOF. The joint densities of U and X and of U and T satisfy the proportionalities,

$$\rho_{U,X} \propto |U|^{(m-q-p-1)/2}\,\mathrm{etr}\,[-\tfrac{1}{2}(P^{-1}U + XQ^{-1}X')],$$

$$\rho_{U,T} \propto |U|^{(m-p-1)/2}\,\mathrm{etr}\,[-\tfrac{1}{2}(P^{-1} + TQ^{-1}T')U].$$

By equation (2.1),

$$\rho_T \propto |P^{-1} + TQ^{-1}T'|^{-m/2}.$$

The normalizing constant follows with linear transformations from (2.4). Equation (2.3) implies the second form of ρ_T.

The particular random matrix denoted by T in the theorem will be continually referred to below. Merely for temporary notational convenience, we shall say that $T + C$, with C a constant matrix, is $T(P, Q, C, m)$ distributed (hence $T \sim T(P, Q, 0, m) \sim T(lP, l^{-1}Q, 0, m)$, scalar $l > 0$).

The limit in distribution, $T \to (P^{\frac{1}{2}\prime})^{-1}X_0M^{\frac{1}{2}\prime}$, as $m \to \infty$, where $Q = mM$ and X_0 has independent standard normal entries, follows immediately from the definition (3.1).

The two forms of the density in the theorem imply

$$T' \sim T(Q^{-1}, P^{-1}, 0, m). \tag{3.3}$$

COROLLARY 3.1.

$$T \sim Y(V^{\frac{1}{2}})^{-1}, \tag{3.4}$$

where $V \sim W(Q^{-1}, m - p)$, *independently of* Y, *the* **column vectors** *of which are independently* $N(0, P^{-1})$ *distributed.*

Consider the special case of $T(P = I_p, Q = I_q)$,

$$T_0 = (U_0^{\frac{1}{2}\prime})^{-1}X_0, \tag{3.5}$$

where the *row vectors,* and thus the *column vectors,* of X_0 are independent and standard normal, and $U_0 \sim W(I_p, m - q)$. Then

$$T \sim (P^{\frac{1}{2}\prime})^{-1}T_0Q^{\frac{1}{2}\prime}.$$

Since the density of T_0 is invariant under constant orthogonal transformations, we can write

$$T_0 \sim (U_0^{(\frac{1}{2})})^{-1}X_0, \tag{3.6}$$

where $U_0^{(\frac{1}{2})}$ is a square root of the form (2.6) for *constant* orthogonal O. Olkin and Rubin (1964, Theorem 4.2) obtain the density of $(U_0^{\frac{1}{2}})^{-1}X_0$ for particular choices of *random* orthogonal O.

The matrix T_0T_0' is of interest for its importance in multivariate testing. Equations (3.5) and (3.6) (with (3.3)) imply

$$T_0T_0' \sim (U_0^{\frac{1}{2}\prime})^{-1}W_0(U_0^{\frac{1}{2}})^{-1} \sim Y_1V_1^{-1}Y_1' \tag{3.7a}$$

where $W_0 \sim W(I_p, q)$, and where given arbitrary $q \times q$ $\Sigma_1 > 0$, $V_1 \sim W(\Sigma_1, m - p)$, independently of Y_1, the *row vectors* of which are independently $N(0, \Sigma_1)$ distributed. If $q \geqq p$, T_0T_0' has the density,

$$\rho_{T_0T_0'} = [B_p((m - q)/2, q/2)]^{-1}|T_0T_0'|^{(q-p-1)/2}|I_q + T_0T_0'|^{-m/2}, \tag{3.7b}$$

where

$$B_p(a, b) = \Gamma_p(a)\Gamma_p(b)/\Gamma_p(a + b).$$

The density (3.7b) is that of a particular multivariate beta distribution, generalizing the F distribution. It was derived as the density of T_0T_0' by Olkin and Rubin: by Hsu's lemma (Anderson, 1958, p. 319),

$$\rho_{T_0T_0'} = [\pi^{pq/2}/\Gamma_p(\tfrac{1}{2}q)]|T_0T_0'|^{(q-p-1)/2}\rho_{T_0}\,.$$

Geisser (1965) observed that if $U = |Q|/|Q + T'PT|$, then $U \sim U_{q,p,m-p}$ (Anderson, 1958, p. 194), a product of beta variables.

Consider the usual estimate $p \times q$ $B = S_{11}^{-1}S_{12}$ of a matrix $\beta = \Sigma_{11}^{-1}\Sigma_{12}$ of regression coefficients, where $(p + q) \times (p + q) S \sim W(\Sigma, \nu)$, $\Sigma > 0$, $\nu > p + q - 1$. Kshirsagar (1960) proved that B is T distributed,

$$B \sim T(\Sigma_{11}\,, \Sigma_{22\cdot 1}\,, \beta, \nu + q).$$

Noting that B' is also T distributed, we obtain two more representations for T. If $m > p + 2q - 1$,

$$T \sim S_{11}^{-1}S_{12} - P^{-1}\Sigma_{12}\,, \tag{3.8}$$

where $S \sim W(\Sigma, m - q)$,

$$\Sigma = \begin{pmatrix} P & \Sigma_{12} \\ \Sigma_{21} & Q + \Sigma_{21}P^{-1}\Sigma_{12} \end{pmatrix},$$

and Σ_{12} is arbitrary ($\Sigma > 0$, by Lemma 3.1 below). Also, by (3.3), under a different partitioning,

$$T \sim S_{21}S_{11}^{-1} - \Sigma_{21}Q, \tag{3.9}$$

but now,

$$\Sigma = \begin{pmatrix} Q^{-1} & \Sigma_{12} \\ \Sigma_{21} & P^{-1} + \Sigma_{21}Q\Sigma_{12} \end{pmatrix}.$$

The remainder of Section 3 concerns marginal and conditional distributions of submatrices of T, which are also T distributions. In particular, the marginal distribution of a row or column vector of T is multivariate t.

Lemma 3.1. *If*

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

and A_{11} is nonsingular,

$$X'AX = (X_1 + A_{11}^{-1}A_{12}X_2)'A_{11}(X_1 + A_{11}^{-1}A_{12}X_2) + X_2'A_{22\cdot 1}X_2\,.$$

Note. If $A^{-1} = M$, $A_{11}^{-1}A_{12} = -M_{12}M_{22}^{-1}$ and $A_{22\cdot 1} = M_{22}^{-1}$.

Proof. Writing $X = (x_1\,, \cdots, x_q)$, $x_j' = (x_{j1}', x_{j2}')$, complete the bilinear form in $x_{i1}\,, x_{j1}$ in the i, jth entry of $X'AX = (x_i'Ax_j)$.

Theorem 3.2. *If $T' = (X_1\,, X_2)$, the conditional distribution of X_1, given X_2, is T with parameters $P_{11}\,, Q + X_2'P_{22\cdot 1}X_2\,, -P_{11}^{-1}P_{12}X_2\,, m$. If $T = (T_1\,, T_2)$, the conditional distribution of T_1 given T_2 is T with parameters $(P^{-1} + T_2Q_{22}^{-1}T_2')^{-1}$, $Q_{11.2}\,, T_2Q_{22}^{-1}Q_{21}\,, m$.*

PROOF. Apply the lemma to the second form of the density (3.2) of T to recognize the functional form of the conditional density of X_1. Apply the first statement in Theorem 3.2 to T' to obtain the second statement.

THEOREM 3.3. *If* $T' = (X_1, X_2)$, $X_i\, p_i \times q$, *the marginal distribution of* X_2 *is* T *with parameters* $P_{22\cdot 1}, Q, 0, m - p_1$. *If* $T = (T_1, T_2)$, $T_j\, p \times q_j$, *the marginal distribution of* T_2 *is* T *with parameters* $P, Q_{22}, 0, m - q_1$.

PROOF. Either, using the Lemma 3.1 and Theorem 3.2, perform the integrations with respect to X_1, T_1 in the joint density of T, or merely refer to the synthetic representations in Theorem 3.1 and Corollary 3.1.

4. Bayesian applications. Let the q sets of independent random p-vectors $x_{j1}, \cdots, x_{jN_j}, j = 1, \cdots, q$, have normal experimental distributions with proportional unknown covariance matrices $(l_jH)^{-1}$ and unknown mean vectors μ_j. The resulting likelihood function for the l_jH and μ_j is proportional to

$$(4.1)\quad \Pi_j\, l_j^{pN_j/2}|H|^{N_j/2} \operatorname{etr}\{-\tfrac{1}{2}l_j[A_j + N_j(\bar{x}_j - \mu_j)(\bar{x}_j - \mu_j)']H\}$$
$$= (\Pi_j l_j^{N_j})^{p/2}|H|^{N/2} \operatorname{etr}\{-\tfrac{1}{2}[\Sigma_j l_j A_j + (M - \bar{X})L(M - \bar{X})']H\},$$

where $\bar{x}_j = \sum_n x_{jn}/N_j$, $A_j = \sum_n (x_{jn} - \bar{x}_j)(x_{jn} - \bar{x})'$, $\bar{X} = (\bar{x}_1, \cdots, \bar{x}_q)$, $M = \{\mu_1, \cdots, \mu_q)$, $N = \sum N_j$, and $L = \operatorname{diag}(N_j l_j)$.

Bayesian inference from the likelihood function (4.1) is of interest, for example in discrimination problems. Stein (1963) has posed the question of Bayesian inference about $l = (l_1, \cdots, l_q)$ when N is small.

We suppose the prior conditional distribution of H given M and $l = (l_1, \cdots, l_q)$ to be formal $W(D^{-1}, \nu)$,

$$p_{H|M,l} \propto [\Delta(D)]^{\nu/2}|H|^{(\nu-p-1)/2} \operatorname{etr}(-\tfrac{1}{2}DH),$$
$$\begin{aligned}\Delta(D) &= |D| && \text{if } |D| > 0\\ &= 1 && \text{if } |D| = 0,\end{aligned}$$

where D could be a function of M and l. Such prior distributions in the context of an unknown covariance matrix have been considered by Savage (1961), Geisser and Cornfield (1963), Dempster (1963), Stone (1964), Ando and Kaufman (1965).

Given $B, S \geqq 0$, let

$$D = B + (M - C)S(M - C)'$$

(B, S possibly functions of l), and suppose

$$p_{M\mid l} \propto [\Delta(B)]^{(\nu-q)/2}[\Delta(S)]^{p/2}\cdot[\Delta(D)]^{-\nu/2},$$

so

$$p_{H,M\mid l} \propto [\Delta(B)]^{(\nu-q)/2}[\Delta(S)]^{p/2}|H|^{(\nu-p-1)/2} \operatorname{etr}(-\tfrac{1}{2}DH).$$

Then we have the formal prior conditional distribution of H and M *given* l:

$$M \sim T(B^{-1}, S^{-1}, C, \nu);$$

the column vectors σ_j of $MS^{\frac{1}{2}}$ are independent given H, and

$$\sigma_j \mid H \sim N(\tau_j, H^{-1}), \qquad CS^{\frac{1}{2}} = (\tau_1, \cdots, \tau_q);$$
$$H \sim W(B^{-1}, \nu - q); \qquad H \mid M \sim W(D^{-1}, \nu).$$

The posterior density of H, M, l is proportional to the product of $\rho_{H,M,l}$ and the likelihood (4.1),

$$\rho_{H,M,l \mid x\text{'s}} \propto |H|^{(\nu+N-p-1)/2} \operatorname{etr}(-\tfrac{1}{2}D_1H)[\Delta(B)]^{(\nu-q)/2}[\Delta(S)]^{p/2}(\Pi_j l_j^{N_j})^{p/2} \cdot \rho_l$$

where

$$D_1 = B_1 + (M - C_1)S_1(M - C_1)',$$

with

$$S_1 = S + L,$$
$$C_1 = S_1^{-1}(SC + L\bar{X}),$$
$$B_1 = B + \sum l_j A_j + (C - \bar{X})'SS_1^{-1}L(C - \bar{X}).$$

$\rho_{H,M \mid l,x\text{'s}}$ is of the same form in H, M as $\rho_{H,M \mid l}$ with B, S, C, ν replaced by B_1, S_1, C_1, $\nu + N$, respectively. Hence $\rho_{H,M \mid l}$ is a conjugate prior distribution in the sense of Raiffa and Schlaifer (1961). It possesses the same lack of flexibility in the stochastic dependence between H and the μ_j as do the conjugate prior distributions of Raiffa and Schlaifer and of Ando and Kaufman for a multivariate normal process.

The posterior density of l is given by the proportionality,

$$\rho_{l \mid x\text{'s}} \propto ([\Delta(B)]^{(\nu-q)/2}/|B_1|^{(\nu+N-q)/2})(\Delta(S)/|S_1|)^{p/2}(\Pi_j l_j^{N_j})^{p/2}\rho_l,$$

under the assmuption that $\nu + N > p + q - 1$ and $B_1, S_1 > 0$.

5. The matrix R.

THEOREM 5.1. *Let R and R_0 be the random $p \times q$ matrices,*

$$R = (P^{\frac{1}{2}\prime})^{-1}R_0Q^{\frac{1}{2}\prime}, \tag{5.1}$$

$$R_0 = [(U_0 + X_0X_0')^{\frac{1}{2}}]^{-1}X_0 \tag{5.2}$$

where, as in equation (3.5), the entries of X_0 are independent standard normal variables, distributed independently of $U_0 \sim W(I_p, m - q)$, $m > p + q - 1$. Then R has the density,

$$\begin{aligned}\rho_R &= [k(m,q,p)]^{-1}|P|^{(m-p-1)/2}|Q|^{-p/2}|P^{-1} - RQ^{-1}R'|^{(m-p-q-1)/2}, \\ &\qquad\qquad P^{-1} - RQ^{-1}R' > 0, \\ &= [k(m,p,q)]^{-1}|Q|^{-(m-q-1)/2}|P|^{q/2}|Q - R'PR|^{(m-p-q-1)/2}, \\ &\qquad\qquad Q - R'PR > 0.\end{aligned} \tag{5.3}$$

PROOF. It suffices to derive the distribution of R_0. Let $U = U_0 + X_0X_0'$, then successively obtain the densities of U given X_0, U and X_0, U and R_0, and R_0,

$$\rho_{R_0} = [k(m, q, p)]^{-1}|I_p - R_0R_0'|^{(m-p-q-1)/2}, \qquad I - R_0R_0' > 0.$$

The conditions, $I_p - R_0R_0' > 0$ and $I_q - R_0'R_0 > 0$, are both equivalent to the condition, $k_{ii}^2 < 1$, $i = 1, \cdots, \min(p, q)$, where $R_0 = O_pKO_q$, O_p and O_q each orthogonal, and $K = (k_{ij})$, $k_{ij} = 0$ for $i \neq j$.

Notice the absence of a prime on the square root in the definition of R_0 (5.2), in contrast to that in the definition of T_0 (3.5).

The following list of representations for R_0 in terms of random matrices defined in Section 3 can easily be verified and extended by the reader. A variety of representations for R and T are implicit.

By (5.3) and (5.2),

$$R_0 \sim Y_0[(V_0 + Y_0'Y_0)^{\frac{1}{2}'}]^{-1}, \tag{5.4}$$

with the entries of Y_0 standard normal variables, and $V_0 \sim W(I_q, m - p)$.

By (5.2) and (5.4)

$$\begin{aligned} R_0 &\sim [(I_p + T_0T_0')^{\frac{1}{2}}]^{-1}T_0 \\ &\sim T_0[(I_q + T_0'T_0)^{\frac{1}{2}'}]^{-1} \\ (T_0 &\sim [(I_p - R_0R_0')^{\frac{1}{2}}]^{-i}R_0 \\ &\sim R_0[(I_q - R_0'R_0)^{\frac{1}{2}'}]^{-1}). \end{aligned} \tag{5.5}$$

By (3.8), (3.9), and (5.5), if $m > p + 2q - 1$

$$\begin{aligned} R_0 &\sim [(S_{11}^2 + S_{12}S_{21})^{\frac{1}{2}}]^{-1}S_{12} \\ &\sim S_{11}^{-1}S_{12}[(I_q + S_{21}S_{11}^{-2}S_{12})^{\frac{1}{2}'}]^{-1} \\ &\sim [(I_p + S_{12}S_{22}^{-2}S_{21})^{\frac{1}{2}}]^{-1}S_{12}S_{22}^{-1} \\ &\sim S_{12}[(S_{22}^2 + S_{21}S_{12})^{\frac{1}{2}'}]^{-1}, \end{aligned}$$

where $(p + q) \times (p + q)$ $S \sim W(I_{p+q}, m - q)$.

We have the analogue of (3.7),

$$\begin{aligned} R_0R_0' &\sim [(U_0 + W_0)^{\frac{1}{2}}]^{-1}W_0[(U_0 + W_0)^{\frac{1}{2}'}]^{-1} \\ &\sim Y_1(V_1 + Y_1'Y_1)^{-1}Y_1'. \end{aligned}$$

If $q \geqq p$, R_0R_0' has the density,

$$\rho_{R_0R_0'} = [B_p((m - q)/2, q/2)]^{-1}|R_0R_0'|^{(q-p-1)/2}|I - R_0R_0'|^{(m-p-q-1)/2}, \qquad I - R_0R_0' > 0. \tag{5.6}$$

The density (5.6) is that of a particular multivariate beta distribution, generalizing the beta distribution. It was obtained as the density of $[(U_0 + W_0)^{\frac{1}{2}}]^{-1} \cdot W_0[(U_0 + W_0)^{\frac{1}{2}'}]^{-1}$ by Olkin and Rubin (1964).

Theorems completely analogous to Theorems 3.2 and 3.3 can be derived for the marginal and conditional distributions of submatrices of R.

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Note Added in Proof: C. G. Khatri (*Ann. Math. Statist.* **37** 468–479, Theorem 4) has obtained the density of TT', thereby generalizing (3.7b).

REFERENCES

ANDERSON, T. W. (1958). *Introduction to Multivariate Statistical Analysis*. Wiley, New York.

ANDO, ALBERT and G. W. KAUFMAN (1965). Bayesian analysis of the independent multinormal process—neither mean nor precision known. *J. Amer. Statist. Assoc.* **60** 347–358.

CORNISH, E. A. (1954). The multivariate t-distribution. *Australian J. Physics.* **7** 531–42.

DEMPSTER, A. P. (1963). On a paradox concerning inference about a covariance matrix. *Ann. Math. Statist.* **34** 1414–8.

GEISSER, SEYMOUR (1965). Bayesian estimation in multivariate analysis. *Ann. Math. Statist.* **36** 150–159.

GEISSER, SEYMOUR and JEROME CORNFIELD (1963). Posterior distributions for multivariate normal parameters. *J. Roy. Statist. Soc. Ser. B.* **25** 368–76.

KSHIRSAGAR, A. M. (1960). Some extensions of the multivariate t-distribution and the multivariate generalization of the distribution of the regression coefficient. *Proc. Cambridge Phil. Soc.* **57** 80–85.

KIEFER, J. and R. SCHWARZ (1965). Admissible Bayes character of T^2-, R^2-, and other fully invariant tests for classical multivariate normal problems. *Ann. Math. Statist.* **36** 747–770.

OLKIN, INGRAM and HERMAN RUBIN (1964). Multivariate beta distributions and independence properties of the Wishart distribution. *Ann. Math. Statist.* **35** 261–269.

RAIFFA, HOWARD and ROBERT SCHLAIFER (1961). *Applied Statistical Decision Theory*. Harvard Univ. Press, Boston.

SAVAGE, LEONARD J. (1961). *The Subjective Basis of Statistical Practice* (unpublished notes). Univ. of Michigan.

STEIN, C. M. (1963). Personal communication.

STONE, M. (1964). Comments on a posterior distribution of Geisser and Cornfield. *J. Roy. Statist. Soc. Ser. B.* **26** 274–76.

TIAO, GEORGE C. and ARNOLD ZELLNER (1964). On the Bayesian estimation of multivariate regression. *J. Roy. Statist. Soc. Ser. B.* **26** 277–285.