MULTISTAGE SAMPLING PROCEDURES BASED ON PRIOR DISTRIBUTIONS AND COSTS

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1. Introduction. The subject of this investigation is a problem of quality control: An article is produced in large lots; by means of a sampling procedure, distinguishing between "effective" and "defective" items, the decision upon acceptance or rejection of the lot is made.

The most general form of such a sampling procedure is a multistage test with variable sample sizes. The maximum number of stages (k) may be any integer between 1 and the lot size (N). Then the k-stage sampling procedure consists of the following instructions: Firstly, a fixed number between 0 and N is determined to be the size of the first sample. Secondly, for each stage j ($j = 1, \cdots, k - 1$) of the procedure, the size of the $(j + 1)$st sample, as a function of the outcome of the first j samples, is given. Thirdly, for the final stage k and for all those cases where on stage j the size of the $(j + 1)$st sample is zero, it is prescribed (in terms of the sampling outcome thus far obtained) whether the lot has to be accepted or rejected. If all sample sizes are fixed in advance, then it is merely necessary to determine whether the procedure on stage j is to be continued and, if not, what terminal decision is to be made.

We intend to take economic considerations as a basis for the construction of such sampling procedures. We thus assume the costs and the prior distribution of the number of defective items in the lot to be given. Then among all k-stage sampling procedures the one with the lowest total expected costs is considered optimal.

2. Optimal k-stage sampling procedures. To the ith item in a lot of size N the random variable x_i ($i = 1, \cdots, N$) is assigned, asuming the value 0 or 1 if the item is effective or defective, respectively. Then the sampling space is

$$(1) \qquad S = \{x = (x_1, \cdots, x_N): x_i = 0 \text{ or } 1, i = 1, \cdots, N\}.$$

More generally, S can be thought of as the cartesian product of some sets $S_1, \cdots, S_N$, i.e. $S = S_1 \times \cdots \times S_N$ and $x = (x_1, \cdots, x_N)$, with $x_i \varepsilon S_i$ ($i = 1, \cdots, N$). Let F be the σ-field of events over S, let F_i be the σ-field of events which can be described by $(x_1, \cdots, x_i)$, and let P be a probability measure over F. Then $F_0 \subset F_1 \subset \cdots \subset F_N$ with $F_0 = \{\varnothing, S\}$ and $F_N = F$. Let the symbol E_i denote the conditional expectation of a random variable depending on x, given F_i ($i = 0, 1, \cdots, N$). Thus E_0 or E means the unconditional expectation.

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Our first aim is a sub-division of S into subsets upon which terminal decisions are made at stage 0, 1, 2, $\cdots$. By means of a decision function these subsets will be divided into regions of acceptance and rejection, so that the sampling procedures are uniquely described.

Let $n_1, \cdots, n_k$ be the sizes of the samples on the different stages of the sampling procedure and put $n_1 + \cdots + n_j = v_j$ $(j = 1, \cdots, k)$. Let the size of the jth sample depend on the outcomes of the foregoing ones. Then we get a system of functions $v = \{v_j(x)\}$ which evidently satisfies the following conditions:

(a) $v_j(x)$ is measurable on $F_{v_{j-1}}$ $(j = 2, \cdots, k)$;
$v_1(x) = \text{const}$;

(2) (b) $v_j(x)$ is an integer with $0 \leqq v_j \leqq N, j = 1, \cdots, k$;

(c) $v_j(x) \geqq v_{j-1}(x), j = 1, \cdots, k$;

(d) from $v_j(x) = v_{j-1}(x)$ follows $v_i(x) = v_{j-1}(x)$ for $i = j, \cdots, k$ and $j = 1, \cdots, k$.

For convenience we define

(2) (e) $v_0(x) \equiv 0, v_{k+1}(x) \equiv v_k(x)$.

Now we assume such a system $v = \{v_j(x)\}$ to be given. Then we denote by $G_j(t)$ the set of all points $x \varepsilon S$ for which $v_j(x)$ has the value t,

$$G_j(t) = \{x \varepsilon S: v_j(x) = t\}, \qquad j = 0, 1, \cdots, k; t = 0, 1, \cdots, N.$$

Note that $G_0(0) = S$. With

$$T_j(v) = \textstyle\sum_{t_0<t_1<\cdots<t_j} G_0(t_0)G_1(t_1) \cdots G_j(t_j)G_{j+1}(t_j) \qquad (j = 0, 1, \cdots, k; t_0 = 0) \tag{3}$$

we have

$$T_0(v) + T_1(v) + \cdots + T_k(v) = S. \tag{4}$$

For in

$$\textstyle\sum_{j=0}^{k} T_j(v) = \sum_{j=0}^{k} \sum_{t_0<\cdots<t_j} G_0(t_0) \cdots G_j(t_j)G_{j+1}(t_j)$$

we may write the intersection $G_0(t_0)G_1(t_1) \cdots G_j(t_j)G_{j+1}(t_j)$ because of (2)(d), as $G_0(t_0) \cdots G_j(t_j)G_{j+1}(t_j)G_{j+2}(t_j) \cdots G_k(t_j)G_{k+1}(t_j)$, getting

$$\textstyle\sum_{j=0}^{k} T_j(v) = \sum_{t_0 \leqq t_1 \leqq \cdots \leqq t_k} G_0(t_0)G_1(t_1) \cdots G_k(t_k)G_{k+1}(t_k).$$

Of this, equation (4) is an immediate consequence, because for each point $x \varepsilon S$ the system of functions v is defined and takes on a system of values of type $t_0 \leqq t_1 \leqq \cdots \leqq t_k$.

Thus each system of functions v satisfying the conditions (2) leads to a subdivision $T(v) = \{T_j(v)\}$ of S into $k + 1$ disjoint subsets, where T_j means the event of a terminal decision on stage j of the sampling procedure. T_0 is either equal to the empty set $\varnothing$ or to S depending on whether a sampling procedure is undertaken or not.

The description of a k-stage sampling procedure is completed by the definition of a decision function $D(T(v))$, which attaches to each point $x \in T_j(v)$ a value out of a set Δ of possible terminal decisions. (If we denote by A the acceptance and by R the rejection of the lot, we have in our case $\Delta = \{A, R\}$). However, with each point $x \in T_j(v)$ all other points with the same initial coordinates $x_1, \cdots, x_{v_j(x)}$ also lie in $T_j(v)$; consequently the decision function becomes independent of $T(v)$, if we write it in the form $D = \{d_0, \cdots, d_N\}$ where $d_i = d_i(x)$ is measurable on F_i and $d_i \in \Delta$, $i = 0, 1, \cdots, N$. Thus having constructed $T(v)$ and having observed $x \in T_j(v)$ with $v_{j+1}(x) = v_j(x) = i$, we arrive at the terminal decision $d_i(x)$.

The basic quantity for the following considerations is the conditional expectation of the costs caused by this terminal decision, given $x_1, \cdots, x_i$. This quantity will be denoted $g_i(x, d_i(x))$. It is an F_i-measurable function which arises from the underlying model of costs and from the probability measure P on F. This might be given in terms of a distribution $f_N(Y)$ of the number Y of defectives in a lot of size N. Then the probability of y_i defectives in a sample of size i is given by

$$P_i(y_i) = \sum_{Y=y_i}^{N-i+y_i} [\binom{i}{y_i}\binom{N-i}{Y-y_i}/\binom{N}{Y}] f_N(Y).$$

Now the conditional expectation $p_i(y_i)$ of the fraction of defective items in the uninspected part of the lot, given the sample (i, y_i), can easily be computed as

$$p_i(y_i) = [(y_i + 1)/(i + 1)] \cdot P_{i+1}(y_i + 1)/P_i(y_i),$$

(cf. Hald [2], p. 293, formula 48).

As a model of costs one can take, for example, that of Hald [2] which takes into account the costs (k_s) of sampling inspection per item inspected, the costs (k_r) per item arising from the rejection of a lot (e.g. costs of sorting), and the loss caused by the acceptance of a defective item (taken as unity). Then the function g_i, depending only on the sum $y_i = x_1 + \cdots + x_i$ of the first i coordinates of x, has the form

$$\begin{aligned} g_i &= ik_s + (N - i)p_i(y_i) \qquad \text{for } d_i = A \\ &= ik_s + (N - i)k_r \qquad \text{for } d_i = R. \end{aligned}$$

But whatever the special form of g_i, the expectation $K(T(v), D)$ of the total costs of the sampling procedure, corresponding to the sub-division $T(v)$ of S and to the decision function D, is given by

$$K(T(v), D) = \sum_{j=0}^{k} \sum_{t_0 < \cdots < t_j} \int_{G_0(t_0) \cdots G_{j+1}(t_j)} g_{t_j}(x, d_{t_j}(x)) \, dP.$$

It is clear that an optimal decision function $D^* = \{d_i^*\}$ will be obtained by determining d_i^* such that

(5) $$g_i(x, d_i^*) = \min_{d_i \varepsilon \Delta} g_i(x, d_i), \qquad i = 0, 1, \cdots, N.$$

For then $g_i(x, d_i^*) \leqq g_i(x, d_i)$ holds for all d_i, consequently $K(T(v), D^*) \leqq K(T(v), D)$ for all D. We may write, for abbreviation, $K(T(v), D^*) = K(T(v))$ and $g_i(x, d_i^*) = g_i(x)$.

Now a system of functions v^* has to be found such that $K(T(v^*)) \leqq K(T(v))$ holds for all admissible v.

This system may be obtained by means of the following

THEOREM. *Let the F_i-measurable and P-integrable functions g_i $(i = 0, 1, \cdots, N)$ be defined over the set $S = S_1 \times \cdots \times S_N$ with elements $x \varepsilon S$. Let $k = 0, 1, \cdots, N$ be arbitrary, but fixed. For each system of functions $v = \{v_j(x)\}$ satisfying the conditions (2) let $T(v) = \{T_j(v)\}$ be the sub-division of S defined by (3). Let*

(6) $$K(T(v)) = \sum_{j=0}^{k} \sum_{t_0 < \cdots < t_j} \int_{G_0(t_0) \cdots G_{j+1}(t_j)} g_{t_j} \, dP.$$

Furthermore, define the F_i-measurable and P-integrable functions $w_{j,i}$ $(j = 1, \cdots, k;$ $i = j, \cdots, N; j = i = 0)$ according to the recursion formula

$$w_{k,i} = g_i \qquad \text{for } i = k, \cdots, N;$$

$$w_{j,i} = \min \{g_i, E_i w_{j+1,t}(t = i + 1, \cdots, N)\}$$

$$\text{for } j = 1, \cdots, k - 1, i = j, \cdots, N \text{ and for } j = i = 0.$$

Then $K(T(v)$ is minimized by the system $v^* = \{v_j^*(x)\}$ with

$$v_j^*(x) = v_{j-1}^*(x) = i \quad \text{for } x \varepsilon \{x: w_{j-1,i} = g_i\} \quad \text{and}$$

$$v_j^*(x) = t > v_{j-1}^*(x) = i$$

$$\text{for } x \{x: w_{j-1,i} = E_i w_{j,t}, \varepsilon w_{j-1,i} < \min [g_i, E_i w_{j,\tau} (\tau < t)]\}$$

$$(j = 1, \cdots, k; i = 0, 1, \cdots, N).$$

This system of functions yields $K(T(v^*)) = w_{0,0}$.

3. Proof of the theorem. The proof proceeds by complete induction with respect to k. If $k = 1$ we have

$$K(T(v)) = \int_{G_0(0) G_1(0)} g_0 \, dP + \sum_{t_1=1}^{N} \int_{G_0(0) G_1(t_1) G_2(t_1)} g_{t_1} \, dP$$

whence by $G_0(0) = S$, $v_1(x) = n_1 = \text{const}$, $v_2(x) \equiv v_1(x)$ follows

$$K(T(v)) = \int_S g_{n_1} \, dP = E g_{n_1},$$

regardless whether $n_1 = 0$ or $n_1 > 0$. Forming

$$w_{1,i} = g_i \qquad \text{for } i = 1, \cdots, N \quad \text{and}$$

$$w_{0,0} = \min \{g_0, E w_{1,t} (t = 1, \cdots, N)\}$$

$$= \min \{E g_t : t = 0, 1, \cdots, N\}$$

according to the conditions of the theorem we have to determine v_1^* from

$$w_{0,0} = Eg_{v_1^*} = \min \{Eg_t : t = 0, 1, \cdots, N\}.$$

Then $K(T(v^*)) = Eg_{v_1^*} \leqq Eg_{n_1} = K(T(v))$ for all v. Note that $K(T(v^*)) = w_{0,0}$, thus proving the propositions for $k = 1$.

In the general case

$$G_0(t_0)G_1(t_1) \cdots G_{k-1}(t_{k-1})G_k(t_k) \,\varepsilon\, F_{t_{k-1}},$$

because for $x \,\varepsilon\, G_{k-1}(t_{k-1})$ $v_k(x)$ it already determined by the coordinates $x_1, \cdots, x_{t_{k-1}}$. Putting $g_{t_k} = w_{k,t_k}$ we have

$$\int_{G_0(t_0)\cdots G_k(t_k)} g_{t_k}\, dP = \int_{G_0(t_0)\cdots G_k(t_k)} E_{t_{k-1}} w_{k,t_k}\, dP.$$

Hence

$$\begin{aligned} K(T(v)) = {} & \sum_{j=0}^{k-2} \sum_{t_0<\cdots<t_j} \int_{G_0(t_0)\cdots G_{j+1}(t_j)} g_{t_j}\, dP \\ & + \sum_{t_0<\cdots<t_{k-1}} \int_{G_0(t_0)\cdots G_k(t_{k-1})} g_{t_{k-1}}\, dP \\ & + \sum_{t_0<\cdots<t_k} \int_{G_0(t_0)\cdots G_k(t_k)} E_{t_{k-1}} w_{k,t_k}\, dP. \end{aligned} \tag{7}$$

Assume the functions $v_1(x), \cdots, v_{k-1}(x)$ to be given and $v_k(x)$ to be determined in such a way that $K(T(v))$ becomes minimal; i.e., the sum

$$\begin{aligned} & \sum_{t_0<\cdots<t_{k-1}} \int_{G_0(t_0)\cdots G_k(t_{k-1})} g_{t_{k-1}}\, dP + \sum_{t_0<\cdots<t_k} \int_{G_0(t_0)\cdots G_k(t_k)} E_{t_{k-1}} w_{k,t_k}\, dP \\ & = \sum_{t_0<\cdots<t_{k-1}} \Big[\int_{G_0(t_0)\cdots G_{k-1}(t_{k-1}) G_k(t_{k-1})} g_{t_{k-1}}\, dP \\ & \quad + \int_{G_0(t_0)\cdots G_{k-1}(t_{k-1}) G_{k-1}(t+1)} E_{t_{k-1}} w_{k,t_{k-1}+1}\, dP \\ & \quad + \cdots + \int_{G_0(t_0)\cdots G_{k-1}(t_{k-1}) G_k(N)} E_{t_{k-1}} w_{k,N}\, dP\Big] \end{aligned}$$

is to be minimized with respect to $v_k(x)$. Obviously the minimum is obtained by splitting the set $G_0(t_0) \cdots G_{k-1}(t_{k-1})$ for all $t_0, \cdots, t_{k-1}$ into $N - t_{k-1} + 1$ subsets in such a way that on the union of these subsets just the function $w_{k-1,t_{k-1}}$ is integrated. Thus to the first subset (on which $v_k = t_{k-1}$) we attribute all those points x for which

$$g_{t_{k-1}} = \min \{g_{t_{k-1}}, E_{t_{k-1}} w_{k,t_k}(t_k = t_{k-1} + 1, \cdots, N)\},$$

to the second (on which $v_k = t_{k-1} + 1$) all those points x for which

$$E_{t_{k-1}} w_{k,t_{k-1}+1} = \min \{g_{t_{k-1}}, E_{t_{k-1}} w_{k,t_k}(t_k = t_{k-1} + 1, \cdots, N)\},$$

and so on. Those points x for which the minimum is assumed simultaneously by several functions may be arbitrarily assigned to the corresponding sets. For example, a unique sub-division is obtained by postulating

$$E_{t_{k-1}} w_{k,t_{k-1}+i} < \min \{g_{t_{k-1}}, E_{t_{k-1}} w_{k,t_{k-1}+j}(j < i)\}, \qquad i = 1, \cdots, N - t_{k-1}.$$

Then the optimal value $v_k^*(x)$ of $v_k(x)$ is given by

$$\begin{aligned} & v_k^*(x) = t_{k-1} \quad \text{if} \quad x \,\varepsilon\, \{x \colon w_{k-1,t_{k-1}} = g_{t_{k-1}}\}, \\ & v_k^*(x) = t_{k-1} + i \quad \text{if} \quad x \,\varepsilon\, \{x \colon w_{k-1,t_{k-1}} = E_{t_{k-1}} w_{k,t_{k-1}+i}, \\ & \qquad w_{k-1,t_{k-1}} < \min [g_{t_{k-1}}, E_{t_{k-1}} w_{k,t_{k-1}+j}(j < i)]\}, \quad i = 1, \cdots, N - t_{k-1}. \end{aligned}$$

Carrying out this process for all admissible $t_0, \cdots, t_{k-1}$ we have determined $v_k^*(x)$ for given $v_1(x), \cdots, v_{k-1}(x)$.

Instead of the $(k + 1)$-stage sum (7) we now have to minimize the following k-stage one:

$$(8)\quad K(T(v')) = \sum_{j=0}^{k-2} \sum_{t_0<\cdots<t_j} \int_{G_0(t_0)\cdots G_{j+1}(t_j)} g_{t_j}\, dP + \sum_{t_0<\cdots<t_{k-1}} \int_{G_0(t_0)\cdots G_{k-1}(t_{k-1})} w_{k-1,t_{k-1}}\, dP.$$

However, according to the induction assumption, (8) can be minimized. This defines the functions

$$w_{j,i} = \min \{g_i, E_i w_{j+1,t}\ (t = j + 1, \cdots, N)\},$$
$$j = 1, \cdots, k - 2, \quad i = j, \cdots, N \quad \text{and} \quad j = i = 0,$$

and hence $v_1^*(x), \cdots, v_{k-1}^*(x)$, according to the conditions of the theorem. These values are used in the above expressions for $v_k^*(x)$, given $v_1(x), \cdots, v_{k-1}(x)$. Thus the minimization of $K(T(v))$ by the system $v^* = \{v_j^*(x)\}$ is achieved as stated in the theorem.

Moreover, using $(v')^* = \{v_1^*, \cdots, v_{k-1}^*\}$ instead of v' in (8), we obtain the relation $K(T(v'^*)) = w_{0,0}$ already proved for $k = 1$. However, as shown above, the sum remains unchanged on passing to the system $v^* = \{v_1^*, \cdots, v_k^*\}$; $K(T(v'^*)) = K(T(v^*))$. Hence $K(T(v^*)) = w_{0,0}$ which completes the proof.

4. The case of fixed sample sizes. Consider the case where $n_1, \cdots, n_k$ are given in advance as fixed integers. Then the function $v_j(x)$ can assume the fixed values $m_0, m_1, \cdots, m_j$ only, with $m_t = n_0 + n_1 + \cdots + n_t$ $(t = 0, 1, \cdots, j;$ $j = 0, 1, \cdots, k; n_0 = 0)$. However, the essential question is whether $v_j(x) = m_j$ or $v_j(x) < m_j$ holds. Defining the sets

$$G_j = \{x: v_j(x) = m_j\}$$

and

$$G_j' = \{x: v_j(x) < m_j\} = S - G_j$$

$(j = 0, 1, \cdots, k + 1; m_{k+1} > m_k$ arbitrary), we have because of (3)

$$T_j(v) = G_0 G_1 \cdots G_j G_{j+1}' \qquad (j = 0, 1, \cdots, k).$$

As $T_j(v)\ \varepsilon\ F_{m_j}$ we are able to write the sum in question as

$$K(T(v)) = \sum_{j=0}^{k} \int_{T_j(v)} g_{m_j}\, dP.$$

It is easily seen that for this special case our theorem can be formulated without reference to the system of functions v:

COROLLARY. *Let $k = 0, 1, \cdots, N$ be arbitrary, but fixed. Let $m_0, m_1, \cdots, m_k$ be given integers with $0 = m_0 < m_1 < \cdots < m_k$. Let F_{m_j}-measurable and P-integrable functions g_{m_j} $(j = 0, 1, \cdots, k)$ be defined over $S = S_1 \times \cdots \times S_N$. For each subdivision $T = \{T_j, j = 0, 1, \cdots, k\}$ of S with $T_j\ \varepsilon\ F_{m_j}$, form the sum*

$$K(T) = \sum_{j=0}^{k} \int_{T_j} g_{m_j}\, dP.$$

Define the F_{m_j}-measurable and P-integrable functions w_{m_j} according to the recursion formula

$$w_{m_k} = g_{m_k} ,$$
$$w_{m_j} = \min \{g_{m_j} , E_{m_j} w_{m_{j+1}}\} \quad (j = 0, 1, \cdots , k - 1)$$

and thereby the sets

$$H_{m_j} = \{x: w_{m_j} < g_{m_j}\} \quad \text{and} \quad H'_{m_j} = S - H_{m_j} \qquad (j = 0, 1, \cdots , k).$$

Then $K(T)$ is minimized by the sub-division

$$T^* = \{T_j^*\} \quad \text{with} \quad T_j^* = H_{m_0} H_{m_1} \cdots H_{m_{j-1}} H'_{m_j} , \qquad (j = 0, 1, \cdots , k).$$

This result could also be derived from a theorem of Richter ([3], p. 29). His idea for a proof was extended to prove the above theorem.

The optimal sequential sampling procedure is obtained from the corollary by putting $k = N$ and $m_j = j$ $(j = 0, 1, \cdots , N)$. This special case was investigated earlier by Arrow, Blackwell and Girshick ([1], p. 218).

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