# A NOTE ON SUMS OF INDEPENDENT RANDOM VARIABLES WITH INFINITE FIRST MOMENT

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**1. Introduction.** Let $X$ be a non-negative random variable with distribution function $F(x)$ such that $E(X) = \infty$. Put $S_n = X_1 + \cdots + X_n$ where $\{X_i\}$ is a sequence of mutually independent random variables each following the distribution of $X$. We consider the problem of finding positive monotonic sequences $\{a_n\}$ and $\{b_n\}$ such that

$$P(S_n > a_n \text{ i.o.}) = 0, \tag{1.1}$$

$$P(S_n < b_n \text{ i.o.}) = 0. \tag{1.2}$$

Such sequences form ultimate upper and lower bounds for the sequence $\{S_n\}$ of sums in the sense that

$$P(b_n \leqq S_n \leqq a_n \text{ for all sufficiently large } n) = 1.$$

We require the bounds not to be too crude. If $X$ has a finite variance then the law of the iterated logarithm provides the required sharp bounds.

Feller (1946) has shown that if $E(|X|) = \infty$ (here $X$ is not necessarily non-negative) and if $a_n/n$ is non-decreasing, then

$$P(|S_n| < a_n \text{ for all sufficiently large } n) = 1 \tag{1.3}$$

if and only if $\sum_n P(|X_n| > a_n) < \infty$. However, for a very asymmetrical $X$ the statement from (1.3) that $P(S_n < -a_n \text{ i.o.}) = 0$ may be rather crude. In particular, for non-negative $X$, Feller's result enables us to find sequences $\{a_n\}$ satisfying (1.1) but not sequences $\{b_n\}$ satisfying (1.2). Other work in this connection is by Chow and Robbins (1961) who show that if $E(|X|) = \infty$ in the general case, then for any sequence $\{c_n\}$ either

$$P(\limsup_{n\to\infty} |S_n/c_n| < \infty) = 1, \tag{1.4}$$

or

$$P(\liminf_{n\to\infty} |S_n/c_n| > 0) = 1, \tag{1.5}$$

but not both of (1.4) and (1.5) can hold.

A corollary is that if $E(|X|) = \infty$, then there is no sequence $\{c_n\}$ for which $\lim (S_n/c_n) = 1$ with probability 1.

In the present paper we outline a method for determining sequences $\{b_n\}$ satisfying (1.2) for the case of non-negative $X_i$, but to obtain explicit results it is necessary to assume something explicit about $1 - F(x)$ as $x \to \infty$. The

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general solution, if it exists in the form of a neat condition such as Feller's for the upper bound, eludes us. We examine the cases where as $x \to \infty$, $1 - F(x)$ varies regularly with exponent $-\alpha$ $(0 < \alpha \leqq 1)$, i.e.

$$1 - F(x) = L(x)/x^{\alpha} \qquad (x \to \infty; 0 < \alpha \leqq 1), \tag{1.6}$$

where $L(x)$ is slowly varying as $x \to \infty$, i.e. $L(cx)/L(x) \to 1$ as $x \to \infty$ for each fixed $c > 0$. Regularly varying functions were studied systematically by Karamata (1930); a more recent and accessible exposition is given by Feller (1966), p. 268.

An alternative way of expressing (1.2) is

$$P(\liminf_{n\to\infty} S_n/b_n \geqq 1) = 1. \tag{1.7}$$

Ideally we should like to obtain a precise lower bounding sequence $\{b_n\}$ satisfying

$$P(\liminf_{n\to\infty} S_n/b_n = 1) = 1. \tag{1.8}$$

However we are only able to do this for the case $1 - F(x) \sim L(x)/x$ and even then only provided $L(x)$ does not grow too rapidly (Theorem 2).

In a final section we examine some conditions for convergence of $S_n/n$ to $+\infty$ in the strong law of large numbers ($X$ is no longer non-negative). Previous work in this connection is by Derman and Robbins (1955).

**2. Preliminary results.** One method, used for example by Derman and Robbins (1955), for determining a sequence $\{b_n\}$ satisfying (1.2) is provided by the fact that

$$P(S_n < b_n) \leqq P\{\max(X_1, \cdots, X_n) < b_n\} = \{F(b_n)\}^n. \tag{2.1}$$

If $\{b_n\}$ is chosen so that $\sum \{F(b_n)\}^n$ converges (which is equivalent to the convergence of $\sum \exp[-n\{1 - F(b_n)\}]$), then by the Borel-Cantelli lemma, $P(S_n < b_n \text{ i.o.}) = 0$. For the case $1 - F(x) \sim x^{-1}$ say, this method gives a very crude lower bounding sequence $\{b_n\}$. We can do better by using the strong law of large numbers according to which $P(S_n < Kn \text{ i.o.}) = 0$ for any fixed $K > 0$. It appears in general that the "closer" $F(x)$ is to having a first moment, the cruder is the sequence $\{b_n\}$ obtained from this method.

An alternative method is the following. Let

$$U_1(x) = \int_0^x t\, dF(t), \qquad U_2(x) = \int_0^x t^2\, dF(t). \tag{2.2}$$

Define the truncated variables $X_n'$ by

$$\begin{aligned} X_n' &= X_n && (0 \leqq X_n \leqq \rho_n) \\ &= 0 && (X > \rho_n). \end{aligned} \tag{2.3}$$

Then $E(X_n') = U_1(\rho_n)$ and $E(X_n'^2) = U_2(\rho_n)$. Define

$$Y_n = \{E(X_n')\}^{-1} X_n'. \tag{2.4}$$

Then $E(Y_n) = 1$. If we choose $\rho_n \uparrow$ so that

$$\sum n^{-2}\, V(Y_n) < \infty, \tag{2.5}$$

then $n^{-1}(\sum_{k=1}^{n} Y_k - n) \to 0$ with probability 1 by Kolmogorov's theorem. Thus $P(\sum_{k=1}^{n} Y_k \sim n) = 1$, and if $k_n = o(n)$, then

$$P(\textstyle\sum_{k=k_n}^{n} Y_k \sim n) = 1. \tag{2.6}$$

Hence for each $\epsilon > 0$,

$$P\{\textstyle\sum_{k=k_n}^{n} Y_k > n(1 - \epsilon) \text{ for all } n \text{ sufficiently large}\} = 1. \tag{2.7}$$

Now with probability 1,

$$S_n \geqq \textstyle\sum_{k=1}^{n} X_k' \geqq \sum_{k=k_n}^{n} X_k' \geqq E(X_{k_n}') \sum_{k=k_n}^{n} Y_k,$$

and so for each $\epsilon > 0$ we have

$$P\{S_n > (1 - \epsilon)nU_1(\rho_{k_n}) \text{ for all } n \text{ sufficiently large}\} = 1. \tag{2.8}$$

We have thus proved the following:

LEMMA 1. *If* $\rho_n \uparrow$ *satisfies* (2.5), i.e. *if*

$$\textstyle\sum_{n=1}^{\infty} U_2(\rho_n)/n^2\{U_1(\rho_n)\}^2 < \infty \tag{2.9}$$

*and if* $k_n = o(n)$, *then*

$$P(\liminf_{n\to\infty} S_n/nU_1(\rho_{k_n}) \geqq 1) = 1. \tag{2.10}$$

Thus for each $\epsilon > 0$, the sequence $b_n = (1 - \epsilon)nU_1(\rho_{k_n})$ satisfies (1.2). It is possible in some cases to choose $k_n$ so that $U_1(\rho_{k_n}) \sim U_1(\rho_n)$ and we then have the tidier result than $b_n = (1 - \epsilon)nU_1(\rho_n)$ satisfies (1.2) for each $\epsilon > 0$.

LEMMA 2. *If* $\{\sigma_n\}$ *satisfies the two conditions*

$$0 \leqq \liminf_{n\to\infty} n\{1 - F(\sigma_n)\} < 1; \tag{2.11}$$

$$\lim_{n\to\infty} U_2(\sigma_n)/n\{U_1(\sigma_n)\}^2 = 0, \tag{2.12}$$

*then*

$$P(\liminf_{n\to\infty} S_n/nU_1(\sigma_n) \leqq 1) = 1. \tag{2.13}$$

PROOF. We have, following Feller (1966), p. 231, equation (7.5),

$$P\{|S_n/nU_1(\sigma_n) - 1| > \epsilon\} \leqq U_2(\sigma_n)/\epsilon^2 n\{U_1(\sigma_n)\}^2 + n\{1 - F(\sigma_n)\}.$$

If (2.11) and (2.12) hold, then for each $\epsilon > 0$

$$\liminf_{n\to\infty} P(S_n/nU_1(\sigma_n) < 1 + \epsilon) > 0,$$

from which (2.13) follows.

LEMMA 3. *Let* $L(x) > 0$ *be slowly varying as* $x \to \infty$. *Then there exists a positive function* $H(x)$ *satisfying*

$$H(x) \to \infty, \qquad H(x) = o(x) \qquad (x \to \infty) \tag{2.14}$$

*such that as* $x \to \infty$,

$$L\{x/H(x)\} \sim L(x). \tag{2.15}$$

PROOF. *If* $L(x) \to c$ $(0 < c < \infty)$ then the result is trivial. Otherwise, $L(x)$ may be expressed in the form

$$L(x) = c(x) \exp \{ \textstyle\int_1^x [\epsilon(t)/t] dt \} \tag{2.16}$$

where as $x \to \infty$, $c(x)$ tends to a finite non-zero limit and $\epsilon(x) \to 0$ and does not vanish outside a finite interval. It suffices to show that we can find $H(x)$ satisfying (2.14) such that as $x \to \infty$,

$$\textstyle\int_{x\{H(x)\}^{-1}}^x \{\epsilon(t)/t\} dt \to 0. \tag{2.17}$$

Define

$$\alpha(x) = \sup_{t \geqq x^{\frac{1}{2}}} |\epsilon(t)|.$$

Now provided $0 < x\{H(x)\}^{-1} < x$, the integral in (2.16) is bounded in modulus by

$$\{\sup_{xH^{-1} \leqq t \leqq x} |\epsilon(t)|\} \log H(x).$$

Some calculation will now demonstrate that if $H$ is defined by

$$\begin{aligned} \log H(x) &= \{\alpha(x)\}^{\frac{1}{2}} & (\alpha(x) \geqq 2/\log x) \\ &= (\log x)^{\frac{1}{2}} & (\alpha(x) < 2/\log x), \end{aligned}$$

then (2.17) holds and the requirements of the lemma are met.

**3. The case** $\alpha = 1$. Suppose $E(X) = \infty$ and

$$1 - F(x) = L(x)/x, \tag{3.1}$$

where $L(x)$ is slowly varying as $x \to \infty$, a property that is unaffected by how $L(x)$ behaves in any bounded interval. From the theory of slowly varying functions the following two relations hold as $x \to \infty$:

$$L(x)/\textstyle\int_0^x t^{-1}L(t)dt \to 0, \tag{3.2}$$

$$xL(x)/\textstyle\int_0^x L(t)dt \to 1. \tag{3.3}$$

It follows from (3.2) and (3.3) that as $x \to \infty$,

$$\textstyle\int_0^x t dF(t) \sim \int_0^x \{1 - F(t)\}dt = \int_0^x t^{-1}L(t)dt, \tag{3.4}$$

$$\textstyle\int_0^x t^2 dF(t) \sim x^2\{1 - F(x)\}. \tag{3.5}$$

We now take $\rho_n = n$ in Lemma 1. The series (2.9) behaves like

$$\textstyle\sum_{n=1}^{\infty} \{1 - F(n)\}/[\int_0^n \{1 - F(x)\}dx]^2. \tag{3.6}$$

The terms of (3.6) are monotonic and since the integral

$$\textstyle\int^{\infty} \{1 - F(x)\}/[\int_0^x \{1 - F(t)\}dt]^2 \, dx \tag{3.7}$$

is convergent, so is the series (3.6). Thus from Lemma 1, provided $k_n = o(n)$,

$$P\{\liminf_{n\to\infty} S_n/n \textstyle\int_0^{k_n} \{1 - F(x)\}dx \geqq 1\} = 1. \tag{3.8}$$

Now the slow variation of $L(x)$ implies that of $\int_0^x \{1 - F(t)\}dt$. By Lemma 3 we may choose therefore $k_n$ so that

$$\int_0^{k_n} \{1 - F(x)\}dx \sim \int_0^n \{1 - F(x)\}dx.$$

We have thus proved the following result.

THEOREM 1. *If $E(X) = \infty$ and if $1 - F(x) = x^{-1}L(x)$ where $L(x)$ is slowly varying as $x \to \infty$ then*

$$P\{\liminf_{n\to\infty} S_n/n\int_0^n xdF(x) \geqq 1\} = 1. \tag{3.9}$$

In the next theorem we show that if we impose a restriction (in the form (3.11) below) on the growth of $L(x)$ in Theorem 1 then we obtain equality inside the braces in (3.9). In a later remark we show that some such restriction is also necessary for equality.

THEOREM 2. *Suppose that in addition to the hypothesis of Theorem* 1,

$$x\{1 - F(x)\}/\int_0^x tdF(t) = o([\log\{\int_0^x tdF(t)\}]^{-1}), \tag{3.10}$$

*or equivalently,*

$$L(x) \log\{\int_0^x t^{-1}L(t)dt\}/\int_0^x t^{-1}L(t)dt \to 0 \qquad (x \to \infty), \tag{3.11}$$

*then*

$$P\{\liminf_{n\to\infty} S_n/n\int_0^n tdF(t) = 1\} = 1. \tag{3.12}$$

PROOF. Let

$$a(x) = x\{1 - F(x)\}/\int_0^x \{1 - F(t)\}dt. \tag{3.13}$$

Now $a(x) \to 0$ by (3.2) and (3.4). Since the expression on the right hand side of (3.10) is non-increasing it follows that

$$\sup_{t\geqq x}\{a(t)\} = o(1/\log[\int_0^x \{1 - F(t)\}dt]) \qquad (x \to \infty). \tag{3.14}$$

Next let

$$b_n = n\int_0^n t^{-1}L(t)dt. \tag{3.15}$$

We have, from integrating (3.13),

$$\int_0^{b_n} (1 - F)dt/\int_0^n (1 - F)dt = \exp\{\int_n^{b_n} t^{-1}a(t)dt\} \tag{3.16}$$

and

$$|\int_n^{b_n} t^{-1}a(t)dt| \leqq \{\sup_{t\geqq n} a(t)\} \log b_n/n \to 0$$

in virtue of (3.10) and (3.15). Thus by (3.4), as $n \to \infty$,

$$\int_0^{b_n} t^{-1}L(t)dt \sim \int_0^n t^{-1}L(t)dt. \tag{3.17}$$

We now show that the conditions of Lemma 2 are satisfied with $\sigma_n = b_n$. For

$$\int_n^{b_n} t^{-1}L(t)dt \geqq \{L(b_n)/b_n\}\,(b_n - n),$$

so that

$$(3.18)\quad n\{1 - F(b_n)\} = nL(b_n)/b_n \leqq \int_n^{b_n} t^{-1}L(t)dt/\{(b_n/n) - 1\}$$
$$= \int_n^{b_n} t^{-1}L(t)dt/\int_0^n t^{-1}L(t)dt - 1 \to 0 \ (n \to \infty)$$

in virtue of (3.17). Finally,

$$(3.19)\quad b_nL(b_n)/n\{\int_0^{b_n} t^{-1}L(t)dt\}^2 \leqq [b_n/n\int_0^n t^{-1}L(t)dt]\cdot[L(b_n)/\int_0^{b_n} t^{-1}L(t)dt]$$
$$= L(b_n)/\int_0^{b_n} t^{-1}L(t)dt \to 0$$

as $n \to \infty$, by (3.2). Hence because of (3.17), the results (3.18) and (3.19) show from Lemma 2 that

$$P\{\liminf_{n\to\infty} S_n/n\int_0^n t^{-1}L(t)dt \leqq 1\} = 1.$$

The proof is concluded by applying (3.4) and Theorem 1.

COROLLARY. *If in addition to the hypothesis of Theorem* 1,

$$(3.20)\qquad 0 \leqq \liminf_{n\to\infty} L(n) < 1,$$

*then*

$$P\{\liminf_{n\to\infty} S_n/n\int_0^n xdF(x) = 1\} = 1.$$

PROOF. The result follows from Theorem 1 and the fact that Lemma 2 holds with $\sigma_n = n$ if (3.20) is true.

REMARK. If, for example, $1 - F(x) \sim x^{-1} (\log x)^k$ as $x \to \infty$ for any fixed positive $k$ then the conditions of Theorem 2 can easily be shown to hold. We then have

$$P\{\liminf_{n\to\infty} S_n/n[(\log n)/(k + 1)]^{k+1} = 1\} = 1.$$

However, there are slowly varying functions which exceed in magnitude any power of $\log x$, for example $\exp\{(\log x)/(\log\log x)\}$. If $L(x)$ is such that

$$(3.21)\qquad \int_0^x t^{-1}L(t)dt \sim \exp(\log x/\log\log x),$$

(take $a(x)$ in (3.13) to be $(\log\log x)^{-1} - (\log\log x)^{-2}$ for all large $x$) then the condition (3.11) of Theorem 2 does not hold. Nor in this case does the conclusion of Theorem 2 hold. To see this we note that

$$\rho_n = [n/(\log n)(\log\log n)] \exp(\log n/\log\log n)$$

satisfies (2.9) of Lemma 1. If we take $k_n = [n/\log\log n]$, then $\rho_{k_n} \sim \rho_n/\log\log n$ and

$$P(\liminf_{n\to\infty} S_n/nU_1(\rho_n/\log\log n) \geqq 1).$$

Some calculation will show that $U_1(n) = o\{U_1(\rho_n/\log\log n)\}$ and hence, by (2.2) and (3.4),

$$P(\liminf_{n\to\infty} S_n/n\int_0^n tdF(t) = \infty) = 1.$$

It follows that some condition such as (3.10) is necessary for the conclusion of Theorem 2 to hold.

**4. The case $0 < \alpha < 1$.** Suppose now that

$$1 - F(x) = x^{-\alpha}L(x) \qquad (0 < \alpha < 1) \tag{4.1}$$

where $L(x)$ is again slowly varying as $x \to \infty$. The relations corresponding to (3.4) and (3.5) are

$$U_1(x) = \int_0^x t\,dF(t) \sim [\alpha/(1-\alpha)]\, x^{1-\alpha}L(x), \tag{4.2}$$

$$U_2(x) = \int_0^x t^2 dF(t) \sim [\alpha/(2-\alpha)]\, x^{2-\alpha}L(x). \tag{4.3}$$

These results follow from Feller (1966), Theorem 2, p. 275. From Lemma 1, we then obtain the following:

THEOREM 3. *If* (4.1) *holds, if* $\rho_n$ *satisfies*

$$\sum_{n=1}^{\infty} \rho_n^{\alpha}/n^2 L(\rho_n) < \infty \tag{4.4}$$

*and if* $k_n = o(n)$, *then* for any $A > 0$,

$$P\{\liminf_{n\to\infty} S_n/An\rho_{k_n}^{1-\alpha}L(\rho_{k_n}) \geqq 1\} = 1. \tag{4.5}$$

It is worth noting that when (4.1) holds, the method based on (2.1) for obtaining a sequence $\{b_n\}$ satisfying (1.2) gives reasonably good results in special cases, e.g. $L(x) \to 1$; but in such cases Theorem 3 gives better results. On the other hand when $\alpha = 0$, the opposite appears to be true, e.g. in the special case $1 - F(x) \sim (\log x)^{-1}$, the method based on (2.1) gives better results than Lemma 1.

**5. The strong law of large numbers.** Suppose now that $X$ can take positive and negative values. We examine some conditions under which $S_n/n \to +\infty$ with probability 1. Let

$$X^+ = \max(0, X), \qquad X^- = -\min(0, X).$$

Derman and Robbins (1955) showed that if for some constants $0 < \alpha < \beta < 1$ and $C > 0$,

$$1 - F(x) \geqq C/x^{\alpha}, \tag{5.1}$$

$$E\{(X^-)^{\beta}\} = \infty \tag{5.2}$$

then $S_n/n \to \infty$ with probability 1.

We can use the results of Sections 3 and 4 to show that even if, in a certain sense, the positive and negative tails of the distribution of $X$ are much more alike in order of magnitude than is implied by (5.1) and (5.2), we may still obtain convergence to $+\infty$ in the strong law of large numbers.

We write $S_n = T_n - W_n$ where $T_n = X_1^+ + \cdots + X_n^+$ and $W_n = X_1^- + \cdots + X_n^-$. Suppose for example that as $x \to \infty$, $1 - F(x) \sim x^{-\alpha}\log x$, while $F(-x) \sim x^{-\alpha}/\log x$ $(0 < \alpha < 1)$. Using Theorem 3, we can show that for $b_n = n^{1/\alpha}(\log n)^{1-\epsilon}$ (for any fixed $\epsilon > 0$) we have $P(\liminf T_n/b_n \geqq 1) = 1$ while from Feller's (1946) Theorem 2 we have $P(W_n/b_n \to 0) = 1$. It follows that $P(\liminf S_n/b_n \geqq 1) = 1$ and so $S_n/n \to +\infty$ with probability 1. When $\alpha = 1$, special cases may be approached in a similar way. Thus for example, if as $x \to \infty$, $1 - F(x) \sim x^{-1}$ while $F(-x) \sim \{x(\log \log x)^{1+\epsilon}\}^{-1}$ for some fixed $\epsilon > 0$, then $S_n/n \to +\infty$ with probability 1. It seems difficult to make any simple general statement along these lines.

As a last remark, we make a rather crude strong limiting statement for non-negative $X$. Suppose

$$1 - F(x) = x^{-\alpha}L(x) \qquad (0 < \alpha \leqq 1), \tag{5.3}$$

where $L(x)$ is slowly varying (we assume $E(X) = \infty$ if $\alpha = 1$). Since $L(x) = o(x^{\epsilon})$ for any $\epsilon > 0$, it follows from (2.1) that for any $\epsilon > 0$,

$$P(S_n > n^{1/\alpha-\epsilon} \text{ for all } n \text{ sufficiently large}) = 1,$$

while from Feller's (1946) Theorem 2 we have

$$P(S_n < n^{1/\alpha+\epsilon} \text{ for all } n \text{ sufficiently large}) = 1.$$

Hence it follows that if (5.3) holds, then

$$P(\lim_{n\to\infty} \log S_n/\alpha^{-1} \log n = 1) = 1.$$

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