ON A QUICKEST DETECTION PROBLEM[1]

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1. Introduction and summary. In a recent paper A. N. Shiryaev [1] discusses the problem of detecting the arrival of a "disorder" in an observed stochastic process, as quickly as possible subject to a limitation on the number of false alarms. He considers two versions of a simple model. In the first, the disorder arrives at a discrete instant θ according to a geometric distribution. The process disturbed by this event consists of a sequence of independent observations $\{\xi_t\}$, such that $\xi_1, \xi_2, \cdots, \xi_{\theta-1}$ arise from a certain distribution F_0, whereas $\xi_\theta, \xi_{\theta+1}, \cdots$ come from a different distribution F_1. In the continuous time version of the model, the *a-priori* distribution of θ is exponential:

$$P(\theta > t) = e^{-\lambda t} \qquad (t \geqq 0)$$

and the disorder is represented by a change in the mean drift of an observed Wiener process $\{\eta(t)\}$. More precisely, for any given value of θ, this process has independent normal increments $\delta\eta = \eta(t + \delta t) - \eta(t)$, with

$$E(\delta\eta) = 0 \qquad (0 \leqq t < \theta),$$

$$E(\delta\eta) = \delta t \qquad (t \geqq \theta),$$

$$\operatorname{Var}(\delta\eta) = \delta t \qquad (t \geqq 0).$$

In both versions it may be decided at any instant t to carry out a detailed inspection in order to ascertain whether or not the disturbance has occurred. Then, if it is found that $\theta < t$ the process terminates but observation must be resumed immediately after a false alarm. Within these rules it is required to find a decision procedure which determines the instants at which a thorough inspection is worthwhile.

Assuming that N, the expected number of false alarms, is specified in advance Shiryaev establishes the general form of policy which minimizes τ, the expected delay in verifying the arrival of the disorder. The *a posteriori* distribution of θ at any time, does not depend on anything which took place before the last false alarm. For example, in continuous time

$$p(t) = P(\theta \leqq t \mid \eta(t'), 0 \leqq t' \leqq t) = P(\theta \leqq t \mid \eta(t'), s \leqq t' \leqq t),$$

where s is the instant of the most recent false alarm. The geometric and exponential distributions have the useful property that

$$P(\theta > t + s \mid \theta > s) = P(\theta > t \mid \theta > 0).$$

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He deduces that the optimal policy for the period following any false alarm must correspond exactly with the procedure applied initially, before the first inspection. In addition, he proves the existence of a critical level $p^* = p^*(N)$ such that, in general, observations should continue so long as $0 \leqq p(t) < p^*$ with an immediate inspection as soon as $p(t) \geqq p^*$.

These and other similar results are established first for the discrete time model and then extended to the continuous time version. For the latter, the paper also gives more explicit calculations: the evaluation of τ in terms of p^*, for example. But no attempt is made to determine the critical level $p^*(N)$ for the optimal policy. In fact, as we shall see, a very simple relation holds:

$$p^*(N) = (N + 1)^{-1}.$$

However, the aim here is to show how the optimal policy can be found for a more realistic specification of the minimization problem, involving given delay and inspection costs. We shall concentrate entirely on the continuous time model and suppose that each inspection incurs an instantaneous cost $K > 0$, not depending on its outcome, whereas any delay in detecting the arrival of the disorder leads to a cost $c > 0$ per unit time. Hence the total expected cost is $K(N + 1) + c\tau$, which depends both on the decision procedure and on the initial condition $p(0) = 0$. The minimization will be based on the calculation of the minimum expected future cost $f_*(\varphi)$, as a function of the current state $\varphi(t) = p(t)/(1 - p(t))$, by solving a certain differential equation with special boundary conditions.

A heuristic argument, in which one simply assumes that $f_*(\varphi)$ is suitably differentiable, can be given without much difficulty. But, strictly speaking, it is not clear that risk functions such as $f_*(\varphi)$, each of whose values is defined as the infimum of a class of expectations, are sufficiently well behaved. This difficulty is often encountered in statistical applications of dynamic programming to processes in continuous time. Typically, it is extremely difficult to establish the required differentiability properties directly and it is necessary to seek an indirect justification by means of existence and uniqueness theorems. In our case the formal solution can be produced explicitly and, because of this, its justification is much easier. Nevertheless, the approach is complicated by the need to establish several preliminary results, and the discussion of these special properties is limited to a brief indication of the main steps, in the hope that the essential structure of the argument will be more generally useful.

Section 2 is concerned with the information process $\{\varphi(t)\}$. Its relation to the observed process $\{\eta(t)\}$ is described and certain properties of its increments are collected for later use. The main argument begins in Section 3 with a discussion of sub-optimal decision procedures defined by specifying an open continuation set $\mathcal{C}$ within the space $[0, \infty)$ of possible "initial" states φ. It is shown that any risk function $f(\varphi)$ can be determined for each sub-interval of the corresponding set $\mathcal{C}$ by solving the basic differential equation appropriately. Sections 4 and 5 consider the special solution $f_*(\varphi)$ which represents the optimal decision procedure, and give the required verification that $f_*(\varphi)$ is uniformly minimal. The final section contains a brief analysis of the operating characteristics of the optimal policy and indicates the importance of evaluating the particular minimum risk $f_*(0)$.

2. The information process. For the moment, let us disregard the possibility of stopping for an inspection and consider the way in which the process $\{\varphi(t)\}$ of *a posteriori* probability ratios is determined by the observations on $\{\eta(t)\}$. It is convenient to replace these always by

$$\zeta(t) = \eta(t) + (\lambda - \tfrac{1}{2})t.$$

Then the transition $\varphi(t) \to \varphi(t + h)$, generated by the realization of $\zeta(s)$ during an uninterrupted period $t \leqq s \leqq t + h$, is given by the following:

THEOREM 2.1.

$$\varphi(t + h) = \varphi(t) \exp \{\zeta(t + h) - \zeta(t)\} + \lambda \int_t^{t+h} \exp \{\zeta(t + h) - \zeta(s)\}\, ds.$$

PROOF. This formula follows directly from equation (26) of Shiryaev's paper.

We shall consider the implications of this result. It is immediately clear that, since the original process $\{\eta(t)\}$ is a.s. continuous, we may restrict attention to continuous realizations of $\{\varphi(t)\}$.

In what follows, t and the values $\zeta(t) = \zeta$, $\varphi(t) = \varphi \geqq 0$ are treated as fixed initial conditions, in considering the events of a short period of length δt. The process increments are denoted by $\delta\zeta$ and $\delta\varphi$. The symbols δ_M and δ_m refer to the corresponding maximum and minimum changes respectively. For example,

$$\delta_M\zeta = \max_{t \leqq s \leqq t+\delta t} \{\zeta(s) - \zeta(t)\}.$$

For some purposes, Theorem 2.1 can be replaced by crude inequalities. Since

$$0 \leqq \int_t^{t+\delta t} \exp \{\delta\zeta - (\zeta(s) - \zeta(t))\}\, ds \leqq \exp \{\delta\zeta - \delta_m\zeta\}\, \delta t,$$

we have

$$\varphi \exp(\delta\zeta) \leqq \varphi + \delta\varphi \leqq \varphi \exp(\delta\zeta) + \lambda\delta t \exp(\delta\zeta - \delta_m\zeta). \tag{2.1}$$

We need to calculate various expectations with respect to the distribution of $\delta\zeta$. It will be enough to carry terms of order δt. The random variable $\delta\zeta$ is unbounded, but we can avoid this difficulty by introducing suitable restrictions of the range of integration. Let E_0 denote expectation with the integration restricted to those paths $\{\zeta(s); t \leqq s \leqq t + \delta t\}$ which satisfy $|\delta\zeta| \leqq (\delta t)^{\frac{1}{4}}$. Similarly, let E_1 refer to the smaller class of paths such that $-(\delta t)^{\frac{1}{4}} \leqq \delta_m\zeta \leqq \delta_M\zeta \leqq (\delta t)^{\frac{1}{4}}$. Thus

$$E_1(1) < E_0(1) < 1. \tag{2.2}$$

Most of the results collected here are consequences of the fact that the distribution of $\delta\zeta$, given that $\varphi(t) = \varphi$, can be represented accurately up to terms of order δt as a mixture of two normal distributions. Thus, if $\theta \leqq t$ the distribution denoted by $N(\delta t(\lambda + \frac{1}{2}), \delta t)$ is applicable and if $\theta > t$ (or strictly speaking, when $\theta \geqq t + \delta t$) this becomes $N(\delta t(\lambda - \frac{1}{2}), \delta t)$. Since $p(t) = \varphi(1 + \varphi)^{-1}$, the corresponding weights must be $\varphi(1+\varphi)^{-1}$ and $(1+\varphi)^{-1}$, respectively. Then the following expressions can be obtained by direct calculation.

$$E_0\{\exp(\delta\zeta)\} = 1 + \{\lambda + \varphi(1+\varphi)^{-1}\}\delta t + o(\delta t), \tag{2.3}$$

$$E_0\{\exp(2\delta\zeta)\} = 1 + \{2\lambda + (1+3\varphi)(1+\varphi)^{-1}\}\delta t + o(\delta t). \tag{2.4}$$

In each case it is easily verified that the term $o(\delta t)$ applies uniformly in $\varphi \geqq 0$.

The extreme increments $\delta_M\zeta$ and $\delta_m\zeta$ can be bounded by normal random variables. For example,

$$0 \leqq \delta_M\zeta \leqq \max_{t\leqq s\leqq t+\delta t}\{(\lambda+\tfrac{1}{2})(s-t) + w(s) - w(t)\} \leqq (\lambda+\tfrac{1}{2})\delta t + \delta_M w,$$

where $\{w(s)\}$ is a standard Wiener process and, in particular, δw is distributed as $N(0, \delta t)$. It is not difficult to deduce from the symmetry of this process, that $\delta_M w$ has the same distribution as $|\delta w|$. Thus $\delta_M\zeta$ and similarly $\delta_m\zeta$ have the same order of magnitude as $|\delta w|$ and clearly

$$E_1(1) = 1 + o(\delta t) \tag{2.5}$$

uniformly in φ.

We return now to the information process. Under the restrictions corresponding to E_1, it follows from Theorem (2.1) that

$$\delta\varphi = \varphi\{\exp(\delta\zeta) - 1\} + \lambda\delta t + o(\delta t), \tag{2.6}$$

which again holds uniformly in φ. Again, if φ is bounded; $\varphi \leqq b$ say, then (2.1) indicates that under E_1

$$-(b+1)(\delta t)^{\frac{1}{4}} \leqq \delta_m\varphi \leqq \delta_M\varphi \leqq (b+1)(\delta t)^{\frac{1}{4}}, \tag{2.7}$$

provided that δt is small. Then $\delta_M\varphi$ and $\delta_m\varphi$ are constrained in the same way as $\delta_M\zeta$ and $\delta_m\zeta$. By (2.5), the probability of the event specified by the above inequalities is $1 + o(\delta t)$, uniformly in $\varphi \leqq b$.

The main purpose of all these preliminaries can now be achieved. The restricted moments $E_1(\delta\varphi)$ and $E_1(\delta\varphi^2)$ are essential to the investigation of our decision problem. In the first case, relation (2.6) shows that

$$E_1(\delta\varphi) = \varphi E_1\{\exp(\delta\zeta)\} - \varphi + \lambda\delta t + o(\delta t).$$

Since the integrand on the right is bounded under E_0 and since $E_0(1) - E_1(1) = o(\delta t)$, it follows that we can evaluate the expression by means of (2.3). The term $o(\delta t)$ still applies uniformly when E_1 is replaced by E_0, provided that $\varphi \leqq b$. Then we obtain

$$E_1(\delta\varphi) = \{\lambda(1+\varphi) + \varphi^2(1+\varphi)^{-1}\}\delta t + o(\delta t). \tag{2.8}$$

A similar argument, which also makes use of (2.4), shows that

$$E_1(\delta\varphi^2) = \varphi^2\delta t + o(\delta t). \tag{2.9}$$

Both these formulae apply uniformly in $\varphi \leqq b$.

Finally, we mention two results concerning the general variability of the process $\{\varphi(t)\}$. We note that $\delta_M\zeta$ and $\delta_m\zeta$ can be replaced by the corresponding increments of a standard Wiener process apart from terms of order δt. It follows by considering the inequalities (2.1), that $\delta_M\varphi$ and $\delta_m\varphi$ have similar properties, at least when $\varphi > 0$. In particular, the events $\{\delta_M\varphi > 0\}$ and $\{\delta_m\varphi < 0\}$ have probability 1 and we can interpret this as follows. For any fixed $\delta t > 0$, not necessarily small,

$$\lim_{\beta\to 0+} P(\delta_M\varphi \geqq \beta \mid \varphi(t) = \varphi) = 1, \tag{2.10}$$
$$\lim_{\beta\to 0+} P(\delta_m\varphi \leqq -\beta \mid \varphi(t) = \varphi) = 1.$$

The latter holds provided that $\varphi > 0$.

Suppose now that we place an absorbing barrier at the level b and that $T(\varphi)$ is the expected time to absorption, starting at the point $\varphi \leqq b$. Thus

$$T(\varphi) = E[\inf\{h > 0 : \varphi(t + h) \geqq b\} \mid \varphi(t) = \varphi],$$

which does not depend on t. We shall make use of the fact that $T(\varphi)$ is bounded for all $\varphi \leqq b$. This can be established without difficulty by a simple argument. It is enough to consider the independent increments of $\{\varphi(t)\}$ over a series of regular intervals of time. The discrete time period h can be selected so that

$$P(\varphi(t + h) \geqq \varphi(t) + b \mid \varphi(t) = \varphi) \geqq \epsilon,$$

for some $\epsilon > 0$ and all $\varphi \geqq 0$. Then $T(\varphi)$ is dominated for every $\varphi \leqq b$ by the expectation of a geometric random variable with parameter ϵ. It follows that

$$T(\varphi) \leqq h/\epsilon \qquad (\varphi \leqq b). \tag{2.11}$$

3. Sub-optimal policies. We shall consider the class of decision procedures which depend only on the quantity φ, since a knowledge of $\varphi(t)$ at any time contains all the information relevant to the minimization of future costs. More precisely, the decision maker is concerned with the behaviour of a homogeneous Markov process $\{j(t), \eta(t)\}$, where $j(t)$ is the mean drift of the observed Wiener process which may be 0 or 1. This component has only one possible transition, $0 \to 1$ which occurs at a fixed random rate λ. Furthermore, the delay and inspection costs to be considered by the decision maker, depend only on the state of the above Markov process and his decisions cannot affect costs incurred previously. Since

$$P(j(t) = 0 \mid \eta(t'), 0 \leqq t' \leqq t) = (1 + \varphi(t))^{-1},$$

it follows from the Markov property that, once the values $\varphi(t) = \varphi$ and $\eta(t) = \eta$ are recorded, any events depending only on the future behaviour of the process are conditionally independent of the previous observations $\eta(t')$, $0 \leqq t' \leqq t$. It is also clear that the decision maker may choose a new origin for his future observations, so that the minimization problem does not depend on the particular value of η. In other words, the decision problem specified for the period $[t, \infty)$ by the observations $\eta(t'), 0 \leqq t' \leqq t$, with $\varphi(t) = \varphi$, is identical with that specified for the period $[0, \infty)$ by the initial condition $\varphi(0) = \varphi$.

In view of these remarks, we shall seek an optimal policy within the class of decision procedures covered by the following:

DEFINITION 3.1. A policy is specified by an open set $\mathcal{C} \subset [0, \infty)$, which contains 0 but does not include a neighbourhood of ∞. At each instant t, the procedure must be to continue observation if $\varphi(t) \in \mathcal{C}$ and to stop for an immediate inspection if $\varphi(t) \notin \mathcal{C}$.

The restriction to open continuation sets is possible without loss of generality because, according to (2.10), any limit point of the stopping set is effectively a stopping point. We suppose that $0 \in \mathcal{C}$, since otherwise observation could not begin after a false alarm. Finally, if $[\varphi, \infty) \subset \mathcal{C}$ for some initial state φ, then it can be shown that the expected period of observation is infinite and hence that the total expected cost is infinite.

Consider an arbitrary policy with continuation region $\mathcal{C}$ which must consist of an interval $[0, b)$, together with a countable family of disjoint intervals (a_i, b_i). It follows from the continuity of the process $\{\varphi(t)\}$ that no state $\varphi > b$ can occur after the first inspection and the effects of the policy are largely determined by the critical level b. We associate a risk function $f(\varphi)$ with the policy by defining $f(\varphi)$ as the total expected future cost, given the present state $\varphi(t) = \varphi$ and using the procedure determined by $\mathcal{C}$ throughout. Thus, in the notation of Section 1,

$$f(0) = K(N + 1) + c\tau. \tag{3.1}$$

By (2.11), the expected length $T(0)$ of any run of the process $\{\varphi(t)\}$ which begins in the state 0 and ends with the first occurrence of the state b, is finite. But, having reached b, an inspection occurs and with probability $(1 + b)^{-1}$ this leads to a renewal of the original state 0 and a second run of the process, independent of the first. It follows that the total number of runs needed, including the last during which the disturbance occurs, can be treated as a geometric random variable. Hence

$$N = 1/b \tag{3.2}$$

and $f(0) \leqq (N + 1)(cT(0) + K)$, which is clearly finite.

It is instructive to study the risk $f(\varphi)$ for an arbitrary starting point φ. In particular, if φ is a stopping point, there is an immediate cost K and with probability $(1 + \varphi)^{-1}$, the original state is replaced by 0. Thus

$$f(\varphi) = K + (1 + \varphi)^{-1}f(0) \qquad (\varphi \notin \mathcal{C}). \tag{3.3}$$

It follows that $f(\varphi)$ is bounded and continuous on the stopping set. Our next step is to establish its continuity in general.

LEMMA 3.2. *The risk function $f(\varphi)$ for any policy, is continuous in $\varphi \geqq 0$.*

PROOF. We already know that $f(\varphi)$ is continuous when $\varphi \notin \mathcal{C}$ and each of the component intervals of $\mathcal{C}$ can be treated separately. For convenience of notation let us suppose that $0 \leqq \varphi \leqq b$. In this case, the first run of observations leads to an expected cost not exceeding $cT(\varphi) + K$, where $T(\varphi)$ is the expected time to absorption at b, and hence

$$0 \leqq f(\varphi) \leqq cT(\varphi) + K + (1 + b)^{-1}f(0) \leqq B \qquad (\varphi \leqq b). \tag{3.4}$$

Here the upper bound, provided by (2.11), may depend on b. We can now apply the properties (2.10) to show that $f(\varphi)$ is continuous. Suppose for example, that $0 \leqq \varphi < \varphi' < b$ and consider whether the information process, starting at $\varphi(t) = \varphi$, reaches the level φ' during the period $[t, t + \delta t]$. In the notation of Section 2, we have

$$\begin{aligned} f(\varphi) = E\{f(\varphi + \delta\varphi) \mid \delta_M\varphi < \varphi' - \varphi, \varphi(t) = \varphi\}P(\delta_M\varphi < \varphi' - \varphi \mid \varphi(t) = \varphi) \\ + f(\varphi')P(\delta_M\varphi \geqq \varphi' - \varphi \mid \varphi(t) = \varphi) + O(\delta t). \end{aligned}$$

The last term here represents the possible delay cost and even if $\theta < t$, this cannot exceed $c\delta t$. Since the risk is bounded, it follows that

$$|f(\varphi) - f(\varphi')| \leqq 2BP(\delta_M\varphi < \varphi' - \varphi \mid \varphi(t) = \varphi) + c\delta t.$$

Then, by letting $\varphi' \to \varphi+$ and using the first part of (2.10), we obtain

$$\limsup_{\varphi' \to \varphi+} |f(\varphi) - f(\varphi')| \leqq c\delta t.$$

But δt can be made arbitrarily small and hence the risk is right-continuous at φ. This holds whenever $0 \leqq \varphi < b$ and a similar analysis, relying on the second part of (2.10), shows that $f(\varphi)$ is left-continuous when $0 < \varphi \leqq b$. This completes the proof.

In addition to (3.3), the risk function satisfies a fundamental integral equation in the continuation region. This can be established by considering the possible trajectories $\{\varphi(s), t \leqq s \leqq t + \delta t\}$ and the costs incurred over a short period. Suppose first that $\varphi(t) = \varphi \leqq \beta < b$ and consider the paths included in the restricted expectation E_1. For these paths, the inequalities (2.7) are valid and δt may be chosen sufficiently small to ensure that no inspection occurs during the period. In this case, the cost is easily evaluated up to terms of order δt. For any other path, the total cost is bounded and (2.5) shows that the effect of such paths is negligible. Hence

$$f(\varphi) = E_1\{\varphi(1 + \varphi)^{-1}c\delta t + f(\varphi + \delta\varphi)\} + o(\delta t)$$

holds uniformly in $\varphi \leqq \beta$. Finally, using (2.5) and the fact that the risk function is bounded, we can replace E_1 by the unrestricted expectation. The argument extends without difficulty to show that in general,

$$f(\varphi) = E\{f(\varphi + \delta\varphi) \mid \varphi(t) = \varphi\} + \varphi(1 + \varphi)^{-1}c\delta t + o(\delta t) \qquad (\varphi \in \mathcal{C}), \tag{3.5}$$

where the term $o(\delta t)$ applies uniformly in any closed subset of $\mathcal{C}$. The uniform validity of this term is important in establishing the following uniqueness property.

THEOREM 3.3. *The risk function for any policy, can be determined as the unique continuous solution of equations* (3.3) *and* (3.5).

PROOF. Since $f(\varphi)$ is continuous and satisfies both equations, it is enough to prove the uniqueness. For simplicity, it will be assumed that $\mathfrak{C} = [0, b)$. Let $d(\varphi)$ be the difference between any two continuous solutions of (3.3) and (3.5). Then we have

$$d(\varphi) = (1 + \varphi)^{-1} d(0) \qquad (\varphi \geqq b), \tag{3.6}$$

$$d(\varphi) = E\{d(\varphi + \delta\varphi) \mid \varphi(t) = \varphi\} + o(\delta t) \qquad (0 \leqq \varphi \leqq b). \tag{3.7}$$

We now choose a particular closed interval $[0, \beta]$ within which the discrepancy $o(\delta t)$ applies uniformly. Suppose for example, that $d(0) \geqq 0$ and let $D = \max_{\varphi \geqq 0} \{d(\varphi)\} > 0$. Since $d(\varphi) \leqq (1 + b)^{-1} d(0)$ when $\varphi \geqq b$, this maximum is attained at some point $\varphi_0 < b$. Let ρ be any number such that $(1 + b)^{-1} < \rho < 1$. Then, using the continuity of $d(\varphi)$ as $\varphi \to b-$, we can select β so that $d(\varphi) \leqq \rho D$ whenever $\varphi \geqq \beta$ and $\varphi_0 < \beta < b$. Equation (3.7) holds uniformly in $\varphi \in [0, \beta]$. Thus, given $\epsilon > 0$, there exists a $\Delta > 0$ such that

$$d(\varphi) \leqq E\{d(\varphi + \delta\varphi) \mid \varphi(t) = \varphi\} + \epsilon\delta t \qquad (0 \leqq \varphi \leqq \beta), \tag{3.8}$$

for all $\delta t \leqq \Delta$. We shall interpret this result in the following way. Let h be a fixed large number and for each positive integer r, let us choose $\epsilon = \epsilon(r)$ as small as possible so that (3.8) holds with $\delta t = h/2^r$. Then the sequence $\{\epsilon(r)\}$ is non-increasing and has the limit zero as $r \to \infty$.

For the moment, let us regard r together with $\delta t = h/2^r$, $\epsilon = \epsilon(r)$ as fixed quantities and consider the random walk $\{\varphi_n, n = 0, 1, 2, \cdots\}$ where $\varphi_n = \varphi(n\delta t)$. Let

$$q_n = P(\varphi_1 < \beta, \varphi_2 < \beta, \cdots, \varphi_{n-1} < \beta, \varphi_n \geqq \beta), \qquad Q = \textstyle\sum_{n=1}^{2^r} q_n .$$

The initial condition $\varphi(0) = \varphi_0$ is omitted here to simplify the notation. Thus $Q = Q(r)$ is the probability that the discrete time process, starting at φ_0, is absorbed at some level $\varphi \geqq \beta$ before the time $h = 2^r\delta t$. For the first step in the random walk, relation (3.8) and our choice of β indicate that

$$D = d(\varphi_0) \leqq \epsilon\delta t + \rho D q_1 + E\{d(\varphi_1); \varphi_1 < \beta\}.$$

It follows by induction that in general,

$$D \leqq n\epsilon\delta t + \rho D(q_1 + q_2 + \cdots + q_n) + E\{d(\varphi_n); \varphi_1 < \beta, \varphi_2 < \beta, \cdots, \varphi_n < \beta\}.$$

If we now set $n = 2^r$ and bound the last term, the result is $D \leqq h\epsilon + \rho DQ + D(1 - Q)$. This can be rewritten to exhibit its dependence on r, as follows:

$$(1 - \rho)DQ(r) \leqq h\epsilon(r) \qquad (r = 1, 2, \cdots). \tag{3.9}$$

It is easy to see that the inequality (3.9) contradicts our assumption that $D > 0$. The sequence $\{Q(r)\}$ of absorption probabilities, associated with a period of fixed but arbitrary length h, is clearly non-decreasing. On the other hand $\epsilon(r) \to 0$ as $r \to \infty$. Thus, if $D > 0$ we must have $Q(r) = 0$ for every r and in particular, $\lim_{r\to\infty} Q(r) = 0$. But this conclusion is inconsistent with the continuity of the original process $\{\varphi(t)\}$ and our previous result (2.11), which shows that the time taken for the process to reach absorption at the level $b > \beta$ has finite expectation $T(\varphi_0)$. It follows that $\max_{\varphi \geqq 0} \{d(\varphi)\} = 0$, since we began by assuming without loss of generality that $d(0) \geqq 0$. Then of course, $d(0) = 0$ and the whole argument can be applied again to the function $-d(\varphi)$. Hence $\min_{\varphi \geqq 0} \{d(\varphi)\} = 0$ and this completes the proof.

THEOREM 3.4. *Let $g(\varphi)$ be a continuous function with the following properties: $g(\varphi)$ satisfies relation (3.3) when $\varphi \not\varepsilon \mathcal{C}$ and has continuous derivatives $g'(\varphi)$ and $g''(\varphi)$ when $\varphi \varepsilon \mathcal{C}$;*

$$\tfrac{1}{2}\varphi^2(1 + \varphi)g''(\varphi) + \{\lambda(1 + \varphi)^2 + \varphi^2\}g'(\varphi) + c\varphi = 0 \qquad (\varphi \varepsilon \mathcal{C}). \tag{3.10}$$

If the continuous region is bounded, then $g(\varphi)$ satisfies equation (3.5) and hence

$$g(\varphi) = f(\varphi) \qquad (\varphi \geqq 0).$$

PROOF. We can verify that $g(\varphi)$ is a solution of (3.5) by making use of the results of Section 2. Since $g(\varphi)$ is continuous and satisfies (3.3) when φ is large, it is bounded for all φ. Then by (2.5),

$$E\{g(\varphi + \delta\varphi) | \varphi(t) = \varphi\} = E_1\{g(\varphi + \delta\varphi)\} + o(\delta t) \qquad (\varphi \varepsilon \mathcal{C}).$$

Let $[\alpha, \beta]$ be any closed interval contained in $\mathcal{C}$. We can choose a slightly larger interval in which $g''(\varphi)$ is uniformly continuous, so that

$$g(\varphi + \delta\varphi) = g(\varphi) + \delta\varphi g'(\varphi) + \tfrac{1}{2}\delta\varphi^2\{g''(\varphi) + o(1)\}$$

as $\delta\varphi \to 0$. This holds uniformly in $\varphi \varepsilon [\alpha, \beta]$. We now substitute this expansion in the previous equation and apply (2.8) and (2.9). It follows from (2.7) that $E_1\{o(1)\delta\varphi^2\} = o(\delta t)$ and we obtain

$$E\{g(\varphi + \delta\varphi) | \varphi(t) = \varphi\}$$
$$= g(\varphi) + [\tfrac{1}{2}\varphi^2 g''(\varphi) + \{\lambda(1 + \varphi) + \varphi^2(1 + \varphi)^{-1}\}g'(\varphi)]\delta t + o(\delta t).$$

Then equation (3.10) shows that

$$g(\varphi) = E\{g(\varphi + \delta\varphi) | \varphi(t) = \varphi\} + \varphi(1 + \varphi)^{-1}c\delta t + o(\delta t),$$

where $o(\delta t)$ applies uniformly, $\varphi \varepsilon [\alpha, \beta]$. It follows from Theorem 3.3 that $g(\varphi)$ coincides with the risk function.

REMARK 3.5. It is now clear that the risk function $f(\varphi)$ can be determined, at least if $\mathcal{C}$ is bounded, provided that we can solve equation (3.10) subject to boundary conditions obtained from (3.3). A consideration of the differential equation as $\varphi \to 0$ shows that

$$g'(0) = 0. \tag{3.11}$$

This provides one boundary condition for the first interval $[0, b)$ of $\mathcal{C}$ and the other is

$$g(b) = K + (1 + b)^{-1}g(0). \tag{3.12}$$

In principle, we can evaluate the risk for $\varphi \leqq b$ and then treat any other parts of

$\mathcal{C}$ separately, using (3.3) and the value of $g(0)$ already found, to provide two boundary conditions for each interval. In view of this, our assumption that $\mathcal{C}$ is bounded can be eliminated.

4. The optimal policy. In this section, we calculate a special risk function $g_*(\varphi)$ and a constant $a > 0$ which determines the corresponding continuation region $\mathcal{C}_* = [0, a)$. The following conditions will be needed:

$$(4.1) \qquad (d/d\varphi)\{(1+\varphi)^2\varphi^{2\lambda}e^{-2\lambda/\varphi}g_*'(\varphi)\} = -2c(1+\varphi)\varphi^{2\lambda-1}e^{-2\lambda/\varphi} \qquad (\varphi \leqq a),$$

$$(4.2) \qquad g_*(\varphi) = K + (1+\varphi)^{-1}g_*(0) \qquad (\varphi \geqq a),$$

$$(4.3) \qquad g_*'(0) = 0,$$

$$(4.4) \qquad g_*'(a) = -(1+a)^{-2}g_*(0).$$

Equation (4.1) is equivalent to (3.10) and the only extra condition here is (4.4), which means that $g'(\varphi)$ is continuous at $\varphi = a$. The smoothing which this implies is characteristic of optimal stopping procedures. As we shall see, (4.4) is the necessary and sufficient optimality condition, which enables us to find a. This will appear later when we verify that $g_*(\varphi)$ coincies with the minimum risk for each $\varphi \geqq 0$.

It follows from (4.1) and (4.3) that

$$g_*'(\varphi) = -2c(1+\varphi)^{-2}\varphi^{-2\lambda}e^{2\lambda/\varphi}\int_0^\varphi (1+x)x^{2\lambda-1}e^{-2\lambda/x}\,dx.$$

Then we can express the integrand as $(2\lambda)^{-1}x\,d(x^{2\lambda}e^{-2\lambda/x})$ and integrate by parts to obtain

$$(4.5) \qquad g_*'(\varphi) = -c\lambda^{-1}(1+\varphi)^{-2}\{\varphi - \varphi^{-2\lambda}e^{2\lambda/\varphi}I(\varphi)\} \qquad (\varphi \leqq a),$$

where

$$(4.6) \qquad I(\varphi) = \int_0^\varphi x^{2\lambda}e^{-2\lambda/x}\,dx.$$

Condition (4.4) now shows that

$$(4.7) \qquad g_*(0) = c\lambda^{-1}\{a - a^{-2\lambda}e^{2\lambda/a}I(a)\}.$$

Finally, we can apply relation (4.2) when $\varphi \geqq a$. Hence

$$K = a(1+a)^{-1}g_*(0) + \int_0^a g_*'(\varphi)\,d\varphi.$$

On substituting the expressions (4.5) and (4.7) here, we obtain

$$(4.8) \qquad K\lambda/c = a - \log(1+a) - (1+a)^{-1}a^{1-2\lambda}e^{2\lambda/a}I(a) + \int_0^a (1+\varphi)^{-2}\varphi^{-2\lambda}e^{2\lambda/\varphi}I(\varphi)\,d\varphi.$$

This formula can be simplified by observing that the derivative of the terms on the right reduces to $2\lambda a^{-2\lambda-1}e^{2\lambda/a}I(a)$. Hence

$$(4.9) \qquad K/c = 2\int_0^a x^{-2\lambda-1}e^{2\lambda/x}I(x)\,dx.$$

Equation (4.9) defines a uniquely in terms of λ and K/c. This follows since the double integral increases through all positive values as a increases from zero. Then $g_*(0)$ which is the most important risk value, is given in terms of a by (4.7). These formulae will provide the basis for our later study.

5. Verification. In order to distinguish between particular solutions of the basic differential equation (3.10) and risk functions, we shall adopt a slightly different notation in what follows. Let $g(\varphi)$ denote a solution of the differential equation for all $\varphi \geqq 0$, which may or may not coincide with some risk function $f(\varphi)$ over part of its range. Then $g_*(\varphi)$ represents the risk associated with the special policy of Section 4, within the continuation region $\mathcal{C}_* = [0, a)$ and at the boundary point, but not when $\varphi > a$. In general, the risk for this policy is denoted by $f_*(\varphi)$. With this definition, we aim to prove the following:

THEOREM 5.1.

$$f_*(\varphi) = \inf \{f(\varphi)\} \qquad (\varphi \geqq 0),$$

where the infimum is taken with respect to the class of all policies.

PROOF. In order to see this, it is necessary to consider various solutions $g(\varphi)$ of equation (3.10). It is possible to find relations of the form $g_1(\varphi_1) < f_*(\varphi_1)$ for particular values of φ, but we shall establish that in such cases, $g_1(\varphi)$ cannot form part of a risk function in any interval containing the point φ_1. The proof of Theorem 5.1 will be developed in three stages which cover the cases $\varphi = 0$, $0 < \varphi \leqq a$, $\varphi > a$, respectively. But first we need a few preliminaries.

For any solution $g(\varphi)$ of equation (3.10), let us define the function

$$\gamma_g(\varphi) = K + (1 + \varphi)^{-1} g(0) - g(\varphi). \tag{5.1}$$

Then, writing $\gamma_*(\varphi)$ for $\gamma_{g_*}(\varphi)$, we have

$$(d/d\varphi)\{(1 + \varphi)^2 \varphi^{2\lambda} e^{-2\lambda/\varphi} \gamma_*'(\varphi)\} = 2(1 + \varphi)\varphi^{2\lambda-2} e^{-2\lambda/\varphi}\{c\varphi - \lambda g_*(0)\} \tag{5.2}$$

since $g_*(\varphi)$ satisfies equation (4.1). Now $\gamma_*(a) = \gamma_*'(a) = 0$ by our construction of $g_*(\varphi)$. Also, (4.7) shows that

$$a > \lambda g_*(0)/c. \tag{5.3}$$

Since the expression on the right of (5.2) changes sign only once at $\varphi = \lambda g_*(0)/c$ and since $\gamma_*'(0) = -g_*(0) < 0$ by (4.3), it follows that $\gamma_*'(\varphi)$ has a unique zero at $\varphi = a$. Hence $\gamma_*(\varphi)$ is decreasing in φ when $\varphi < a$ and increasing when $\varphi > a$. Thus

$$\gamma_*(\varphi) \geqq 0 \qquad (\varphi \geqq 0), \tag{5.4}$$

with equality only when $\varphi = a$.

CASE (i). We now prove that $f_*(0)$ is minimal. Since every continuation region must contain the state zero, it is enough to show that $\gamma_g(\varphi) > 0$ for all $\varphi \geqq 0$ whenever $g(0) < g_*(0)$. Then the corresponding function $g(\varphi)$ cannot form part of any risk function near $\varphi = 0$. Equation (5.2) can be applied to $\gamma_g'(\varphi)$, with

$g_*(0)$ replaced by $g(0)$. Hence

$$(5.5)\quad (d/d\varphi)[(1+\varphi)^2\varphi^{2\lambda}e^{-2\lambda/\varphi}\{\gamma_g'(\varphi) - \gamma_*'(\varphi)\}] = 2\lambda(1+\varphi)\varphi^{2\lambda-2}e^{-2\lambda/\varphi}\{\gamma_g'(0) - \gamma_*'(0)\},$$

where $\gamma_g'(0) - \gamma_*'(0) = g_*(0) - g(0)$.

If $g_*(0) > g(0)$, it follows from equation (5.5) that $\gamma_g'(\varphi) > \gamma_*'(\varphi)$ for all $\varphi \geqq 0$ and since $\gamma_g(0) = \gamma_*(0) = K$, we have

$$(5.6)\quad \gamma_g(\varphi) > \gamma_*(\varphi) \geqq 0 \qquad (\varphi > 0).$$

Hence $\gamma_g(\varphi) > 0$ for all $\varphi \geqq 0$, as required.

We remark that, since $f_*(0)$ is minimal, it follows from relation (3.3) that, in general, the risk $f_*(\varphi)$ cannot be reduced by stopping. It remains to consider the possible risks which can be achieved by continuation from the point φ.

Case (ii). Let $g_1(\varphi_1) < g_*(\varphi_1)$ at some point $\varphi_1 \geqq 0$. We can express the difference

$$g_*(\varphi) - g_1(\varphi) = (1+\varphi)^{-1}\{g_*(0) - g_1(0)\} + \{\gamma_{g1}(\varphi) - \gamma_*(\varphi)\}.$$

But, according to relation (5.6), the two terms on the right here must always have the same sign. If $g_1(\varphi_1) < g_*(\varphi_1)$, then $g_1(\varphi) < g_*(\varphi)$ for all values of φ. Making use of (5.4), we have

$$(5.7)\quad g_1(\varphi) < K + (1+\varphi)^{-1}f_*(0) \qquad (\varphi \geqq 0).$$

This shows that $g_1(\varphi)$ never attains the minimum stopping risk, so $g_1(\varphi)$ cannot be interpreted as a risk for any state φ. In particular, if $\varphi_1 \leqq a$ and $g_1(\varphi_1) < f_*(\varphi_1)$, then $g_1(\varphi_1)$ is not an attainable risk at the point φ_1. Hence $f_*(\varphi)$ is minimal when $\varphi \leqq a$.

Case (iii). Finally, suppose that $g_2(\varphi_2) < f_*(\varphi_2)$ for some $\varphi_2 > a$. The preliminary argument for case (ii) still applies and if $g_2(\varphi_2) < g_*(\varphi_2)$, then $g_2(\varphi_2)$ cannot be regarded as a risk. Hence $g_2(\varphi_2) \geqq g_*(\varphi_2)$ and it follows, as before, that $g_2(\varphi) \geqq g_*(\varphi)$ always. Then there is a point φ_3 with $a \leqq \varphi_3 < \varphi_2$, such that $g_2(\varphi_3) = f_*(\varphi_3)$. Now consider the function

$$\sigma(\varphi) = K + (1+\varphi)^{-1}f_*(0) - g_2(\varphi) \qquad (\varphi \geqq \varphi_3).$$

This satisfies the differential equation

$$(5.8)\quad (d/d\varphi)\{(1+\varphi)^2\varphi^{2\lambda}e^{-2\lambda/\varphi}\sigma'(\varphi)\} = 2(1+\varphi)\varphi^{2\lambda-2}e^{-2\lambda/\varphi}\{c\varphi - \lambda f_*(0)\}.$$

By (5.3), the right hand side is strictly positive for all $\varphi \geqq \varphi_3$. We have $\sigma(\varphi_3) = 0$ and it may be assumed that $\sigma'(\varphi_3) \geqq 0$. It follows that $\sigma'(\varphi) > 0$ and hence $\sigma(\varphi) > 0$, whenever $\varphi > \varphi_3$. This eliminates the possibility that $g_2(\varphi_2)$ is a proper risk and the proof is complete.

6. Operating characteristics. We have established that the optimal decision procedure is determined by the rule: stop for an inspection if and only if the current information level $\varphi(t) \geqq a$. This policy alone attains the minimum risk $f_*(\varphi)$, for each possible starting point φ. The critical level a, determined in terms of the cost and process parameters by equation (4.9) and the particular risk value $f_*(0) = g_*(0)$ given by (4.7) together provide the main characteristics of the optimal policy. In conclusion, let us consider some of these characteristics.

Suppose first that a run of observations on the process $\{\varphi(t)\}$ starts from $\varphi(0) = \varphi < a$. Then, since the run must terminate at the level a, the probability that it ends with a false alarm is simply $(1 + a)^{-1}$, as we remarked at the beginning of Section 3. It follows almost immediately that the expected number of false alarms is $N(\varphi) = a^{-1}$. Similarly if the starting point is $\varphi \geqq a$, there is an immediate false alarm, leading to the state zero, with probability $(1 + \varphi)^{-1}$ and hence $N(\varphi) = (1 + \varphi)^{-1}(1 + N(0))$. Collecting these results, we have

$$N(\varphi) = \min \{1, (1 + a)(1 + \varphi)^{-1}\}a^{-1} \qquad (\varphi \geqq 0). \tag{6.1}$$

Thus $N(0) = a^{-1}$ and if we fix $N(0) = N$ in advance, without reference to any costs, the critical level for the *a posteriori* probability $p(t) = \varphi(t)(1 + \varphi(t))^{-1}$ is

$$p^*(N) = (N + 1)^{-1}. \tag{6.2}$$

Any risk function involves both the expected number of inspections and the expected delay between the arrival of the disorder and its detection. In particular the expected delay $\tau(\varphi)$ for the optimal policy can be obtained from a knowledge of $f_*(\varphi)$. We have

$$f_*(\varphi) = K(N(\varphi) + 1) + c\tau(\varphi). \tag{6.3}$$

Then equation (6.1) shows, for example, that

$$\tau(0) = c^{-1}\{f_*(0) - K(1 + a^{-1})\}. \tag{6.4}$$

The mean period occupied by any run of observations is also easy to evaluate. We can define the stopping time S, conditional on $\varphi(0) = 0$, as the first instant s at which $\varphi(s) = a$. Let Σ be the total time which elapses during the succession of runs needed before the disturbance is finally detected. By isolating the first run, we have

$$\begin{aligned} E(\Sigma) &= E(S) + E(\Sigma)(1 + a)^{-1}, \\ E(S) &= E(\Sigma)a(1 + a)^{-1}. \end{aligned} \tag{6.5}$$

On the other hand, the total delay is simply $\Sigma - \theta$ and since $E(\theta) = \lambda^{-1}$, it follows that

$$E(\Sigma) = \lambda^{-1} + \tau(0). \tag{6.6}$$

Then relations (6.4)–(6.6) provide a formula for the mean run length,

$$E(S) = \{c^{-1}f_*(0) + \lambda^{-1}\}a(1 + a)^{-1} - c^{-1}K. \tag{6.7}$$

Finally, we remark that this expectation can be split into two components, conditional on the presence or absence of the disorder.

$$E(S) = E(S \mid \theta > S)(1 + a)^{-1} + E(S \mid \theta \leqq S)a(1 + a)^{-1}. \tag{6.8}$$

One of Shiryaev's results [1], Lemma 2, is a formula which refers to $E(S \mid \theta > S)$ and from this, it is possible to calculate the second component which represents the expected length of the terminal run of observations.

REFERENCE

[1] SHIRYAEV, A. N. (1963). On optimum methods in quickest detection problems. *Theory Prob.* **8** 22–46.