ON THE THEORY OF RANK ORDER TESTS FOR LOCATION IN THE MULTIVARIATE ONE SAMPLE PROBLEM[1]

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0. Summary. In the multivariate one sample location problem, the theory of permutation distribution under sign-invariant transformations is extended to a class of rank order statistics, and this is utilized in the formulation of a genuinely distribution free class of rank order tests for location (based on Chernoff-Savage (1958) type of test-statistics). Asymptotic properties of these permutationally distribution free rank order tests are studied, and certain stochastic equivalence relations with a similar class of multivariate extensions of one sample Chernoff-Savage type of tests are derived. The power properties of these tests are studied.

1. Introduction. Let $\mathbf{X}_\alpha = (X_\alpha^{(1)}, \cdots, X_\alpha^{(p)})'$, $\alpha = 1, \cdots, N$ be N independent and identically distributed (vector valued) random variables (iidrv) distributed according to a continuous p-variate cumulative distribution function (cdf) $F(\mathbf{x}, \boldsymbol{\theta})$, where $\mathbf{x} = (x^{(1)}, \cdots, x^{(p)})'$, $\boldsymbol{\theta} = (\theta^{(1)}, \cdots, \theta^{(p)})'$ and p is a positive integer which in the sequel will be assumed to be greater than one. Let Ω be the set of all continuous p-variate cdf's and it is assumed that $F \varepsilon \Omega$. Now let ω be the set of all continuous p-variate cdf's which are symmetric about some known origin which we may without any loss of generality take to be $\mathbf{x} = \mathbf{0}$. the symmetry being defined by the invariance of the distribution $F(\mathbf{x})$ under simultaneous changes of signs of all the coordinates. If, in addition, $F(\mathbf{x})$ is absolutely continuous having a density function $f(\mathbf{x})$, the symmetry of $F(\mathbf{x})$ may also be defined by the invariance of the density function under simultaneous changes of signs of all the coordinate variates. For the convenience of terminology, this will be termed as the sign-invariance. (We shall drop the assumption of absolute continuity of $F(\mathbf{x})$, except in Sections 5 and 6.)

Thus, we may frame the null hypothesis H_0 of sign-invariance as

$$H_0 : F \varepsilon \omega \subset \Omega. \tag{1.1}$$

The two particular classes of alternatives in which we may usually be interested are

$$H_1 : F(\mathbf{x}) \quad \text{is symmetric about some} \quad \boldsymbol{\delta} \neq \mathbf{0}. \tag{1.2}$$

This will be the multivariate extension of the well known one sample location problem.

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(1.3) $H_2 : F(\mathbf{x})$ is asymmetric about $\mathbf{x} = \mathbf{0}$,

though it may have the location vector $\mathbf{0}$.

This is the multivariate extension of the univariate symmetry problem.

Multivariate extensions of the univariate sign test are due to Hodges (1955), Blumen (1958), Bennett (1962) and Chatterjee (1966), among others. The tests by Hodges, Blumen and Chatterjee (all for the bivariate case) are strictly distribution free while the sign test by Bennett is only asymptotically distribution free. For a comparative study of these sign tests, the reader is referred to Chatterjee (1966). Bickel (1965) has also considered an asymptotically distribution free test based on a quadratic form involving the coordinatewise Wilcoxon's (1949) signed-rank statistic. The object of the present investigation is to consider a general class of rank order tests based on one sample Chernoff-Savage (1958) type of test-statistics. These tests are shown to be permutationally distribution free. Moreover, they are also shown to be asymptotically power equivalent to a class of asymptotically distribution free tests based on the multivariate extensions of the univariate one sample Chernoff-Savage type of (unconditional) tests. Various efficiency results allied to these tests are studied and this includes the work of Bickel (1965) as a special case.

2. The basic permutation argument. Let us denote the sample point by

$$\mathbf{Z}_N = (\mathbf{X}_1, \cdots, \mathbf{X}_N), \qquad \mathbf{X}_\alpha' = (X_\alpha^{(1)}, \cdots, X_\alpha^{(p)}), \qquad \alpha = 1, \cdots, N, \tag{2.1}$$

and the sample space by $\mathcal{Z}_N$. Then under the null hypothesis to be tested the joint distribution of $\mathbf{Z}_N$ remains invariant under the following finite group G_N of transformations g_N given by

$$g_N\mathbf{Z}_N = ((-1)^{j_1}\mathbf{X}_1, \cdots, (-1)^{j_N}\mathbf{X}_N), \qquad j_i = 0, 1; \quad i = 1, \cdots, N, \tag{2.2}$$

where

$$(-1)\mathbf{X}_\alpha' = (-X_\alpha^{(1)}, \cdots, -X_\alpha^{(p)}), \qquad \alpha = 1, \cdots, N.$$

Hence, conditionally given $\mathbf{Z}_N$ all 2^N sample points generated by G_N are equiprobable, each having the conditional probability 2^{-N}. Let us denote this conditional probability measure defined over $\{g_N\mathbf{Z}_N, g_N \varepsilon G_N\}$ by P_N. Ranking the N elements in each row of $\mathbf{Z}_N$ in increasing order of their absolute values we get a $p \times N$ rank matrix

$$\mathbf{R}_N = \begin{pmatrix} R_1^{(1)} & \cdots & R_N^{(1)} \\ \cdots & \cdots & \cdots \\ R_1^{(p)} & \cdots & R_N^{(p)} \end{pmatrix}, \tag{2.3}$$

where by virtue of the assumed continuity of F, the possibility of ties is neglected, in probability. For every $i = 1, \cdots, p$, replacing the ranks $1, \cdots, N$ in the ith row of $\mathbf{R}_N$ by a set of general scores $\{E_{N,\alpha}^{(i)}, \alpha = 1, 2, \cdots, N\}$ which is a set of N real constants, we get a $p \times N$ matrix of general scores $\mathbf{E}_N$ corresponding to $\mathbf{R}_N$:

$$\mathbf{E}_N = \begin{pmatrix} E^{(1)}_{N,R_1^{(1)}} & \cdots & E^{(1)}_{N,R_N^{(1)}} \\ \cdots & \cdots & \cdots \\ E^{(p)}_{N,R_1^{(p)}} & \cdots & E^{(p)}_{N,R_N^{(p)}} \end{pmatrix}. \tag{2.4}$$

Later on, in Section 3, we shall specify certain conditions on $\mathbf{E}_N$. We consider now the usual univariate rank order statistics coordinatewise

$$T_N^{(j)} = \sum_{\alpha=1}^N E^{(j)}_{N,R_\alpha^{(j)}} C_\alpha^{(j)}, \qquad j = 1, 2, \cdots, p \tag{2.5}$$

and let $\mathbf{T}_N = (T_N^{(1)}, \cdots, T_N^{(p)})$, where $C_\alpha^{(j)} = +1$ or -1 according as $X_\alpha^{(j)} > 0$ or <0 respectively. $T_N^{(j)}$ is thus the difference of the sum of the scores $E^{(j)}_{N,\alpha}$ for which $X^{(j)}_{N,\alpha} > 0$ and the sum of those for which $X_\alpha^{(j)} < 0$.

To construct a test statistic we consider now the conditional mean and dispersion matrix under the conditional probability measure P_N. P_N attaches equal probability 2^{-N} to each of the 2^N possible sign changes of columns of $(C_\alpha^{(i)})$ under the group G_N of transformations, each of these transformations, however, leaves the matrix $\mathbf{E}_N$ unaltered. Clearly then under P_N, $C_\alpha^{(j)} = +1$ or -1 each with probability $\frac{1}{2}$, which makes $E(T_N^{(j)} \mid P_N) = 0$. Also under each $g_n \varepsilon G_N$ the product $C_\alpha^{(j)} C_\alpha^{(k)}$ remains constant and for $\alpha \neq \beta$, $C_\alpha^{(j)} C_\beta^{(k)} = +1$ or -1 each with probability $\frac{1}{2}$. This leads to

$$\begin{aligned} &\operatorname{Cov}(T_N^{(j)}, T_N^{(k)} \mid P_N) \\ &\quad = \sum_{\alpha=1}^N E^{(i)}_{N,R_\alpha^{(i)}} E^{(j)}_{N,R_\alpha^{(j)}} C_\alpha^{(i)} C_\alpha^{(j)} = N v_{N,jk} \quad \text{for} \quad j, k = 1, \cdots, p. \end{aligned} \tag{2.6}$$

Let then

$$\mathbf{V}_N = (v_{N,jk}), \qquad j, k = 1, \cdots, p, \tag{2.7}$$

and assume (for the time being) that $\mathbf{V}_N$ is positive definite. ($\mathbf{V}_N$ being a covariance matrix will be positive semi-definite at least. If $\mathbf{V}_N$ is singular, then we can work with the highest order non-singular minor of $\mathbf{V}_N$ and work with the corresponding variates). Later we will show that under certain conditions $\mathbf{V}_N$ will be positive definite, in probability. Thus, we consider the following positive definite quadratic form

$$S_N = (1/N)\{\mathbf{T}_N \mathbf{V}_N^{-1} \mathbf{T}_N'\} \tag{2.8}$$

where $\mathbf{V}_N^{-1}$ is the inverse matrix of $\mathbf{V}_N$. By virtue of S_N being positive definite, it seems natural to use the following test function.

$$\begin{aligned} \phi(\mathbf{z}_N) &= 1 && \text{if} \quad S_N > S_{N,\epsilon}(\mathbf{z}_N) \\ &= a_{N,\epsilon} && \text{if} \quad S_N = S_{N,\epsilon}(\mathbf{z}_N) \\ &= 0 && \text{if} \quad S_N < S_{N,\epsilon}(\mathbf{z}_N) \end{aligned} \tag{2.9}$$

where $S_{N,\epsilon}(\mathbf{z}_N)$ and $a_{N,\epsilon}$ are so chosen such that $E\{\phi(\mathbf{Z}_N) \mid P_N\} = \epsilon$, $0 < \epsilon < 1$. Thus, $\phi(\mathbf{Z}_N)$ will be a similar size ϵ test for H_0 in (1.1).

For small samples, $S_{N,\epsilon}(\mathbf{z}_N)$ and $a_{N,\epsilon}$ are to be determined from the actual permutational cdf of S_N, while for large samples, we shall simplify (2.9) considerably. This is done in the next section.

3. Properties of V_N and asymptotic permutation distribution of S_N. Let us denote the marginal cdf of $X_\alpha^{(j)}$ and of $(X_\alpha^{(j)}, X_\alpha^{(k)})$ by $F_j(x; \theta_j)$ and $F_{j,k}(x, y; \theta_j, \theta_k)$, respectively, and let

$$(3.1)\qquad H_j(x;\theta_j) = F_j(x;\theta_j) - F_j(-x;\theta_j); \qquad (x \geqq 0),$$

$$(3.2)\qquad H_{j,k}(x, y; \theta_j, \theta_k) = F_j(x, y; \theta_j, \theta_k) - F_{j,k}(x, -y; \theta_j, \theta_k) - F_{j,k}(-x, y; \theta_j, \theta_k) + F_{j,k}(-x, -y; \theta_j, \theta_k); \quad (0 \leqq x, y \leqq \infty),$$

$$(3.3)\qquad F_{N,j}(x) = (\text{number of } X_\alpha^{(j)} \leqq x)/N,$$

$$(3.4)\qquad F_{N,j,k}(x, y) = (\text{number of } (X_\alpha^{(j)}, X_\alpha^{(k)}) \leqq (x, y))/N,$$

$$(3.5)\qquad H_{N,j}(x) = F_{N,j}(x) - F_{N,j}(-x-),$$

$$(3.6)\qquad H_{N,j,k}(x, y) = F_{N,j,k}(x, y) - F_{N,j,k}(x, -y-) - F_{N,j,k}(-x, y) + F_{N,j,k}(-x-, -y).$$

Finally, as in Chernoff and Savage (1958), we write

$$(3.7)\qquad E_{N\alpha}^{(j)} = J_{N,j}(\alpha/(N+1)) = J_{N,j}((N/(N+1))H_{N,j}(x)); \quad H_{N,j}(x) = \alpha/N, \quad \alpha = 1, \cdots, N; \; j = 1, \cdots, p,$$

where $J_{N,j}$ though defined only at $1/(N+1), \cdots, N/(N+1)$ may have its domain of definition extended to $(0, 1)$ by letting it have constant value over $[\alpha/(N+1), (\alpha+1)/(N+1))$. Furthermore, we make the following assumptions:

Assumption 1. $\lim_{N\to\infty} J_{N,j}(u) = J_j(u)$ exists for $0 < u < 1$ and is not a constant; $J_j(0) = 0$

Assumption 2.

$$\int_0^{+\infty} [J_{N,j}((N/(N+1))H_{N,j}) - J_j((N/(N+1))H_{N,j})]\, dF_{N,j}(x) = o_p(N^{-\frac{1}{2}}).$$

Assumption 3. $J_j(u)$ is absolutely continuous, and

$$|J_j^{(i)}(u)| = |d^iJ_j(u)/du^i| \leqq K[u(1-u)]^{\delta - i - \frac{1}{2}}, \qquad i = 0, 1, 2$$

for some finite K and some $\delta > 0$.

Assumption 4.

$$\int_0^\infty \int_0^\infty [J_{N,j}((N/(N+1))H_{N,j})J_{N,k}((N/(N+1))H_{N,k}) - J_j((N/(N+1))H_{N,j})J_k((N/(N+1))H_{N,k})]\, dH_{N,j,k}(x, y) = o_p(1).$$

Let us now define

$$(3.8)\qquad \nu_{jk} = \int_0^\infty \int_0^\infty J_j[H_j(x;\theta_j)] \cdot J_k[H_k(y;\theta_k)]\, dH_{j,k}(x, y; \theta_j, \theta_k), \quad j, k = 1, \cdots, p;$$

$$(3.9)\qquad \boldsymbol{\nu} = (\nu_{jk}) \qquad j, k = 1, \cdots, p$$

Assumption 5. $\boldsymbol{\nu}$ is positive definite.

It may be noted that in the sequel often we will replace the cdf's $F(x; \theta_j)$ or $F(x, y; \theta_j, \theta_k)(j \neq k = 1, \cdots, p)$ by some sequences of cdf's which may depend on the sample size N in certain manner. As a result, it follows from (3.1), (3.2) and (3.8) that the matrix $\mathbf{\nu}$ in (3.9) may also depend on N. Thus, we would prefer in attaching a suffix N to ν to denote its possible dependence of N. However, when there is no confusion, the suffix N in $\mathbf{\nu}_N$ will be suppressed. We may remark that the Assumptions 1, 2 and 3 are needed for the proof of the joint asymptotic normality of the permutation distribution of $N^{\frac{1}{2}}\mathbf{T}_N$, defined by (2.5). Assumption 4 is required only for the asymptotic convergence of the permutation covariante matrix $\mathbf{V}_N$, defined by (2.6) and (2.7).

THEOREM 3.1. *Under assumptions* 1 *to* 4, $[\mathbf{V}_N - \mathbf{\nu}_N] \rightarrow_P \mathbf{0}$, *in probability, where* $\mathbf{0}$ *is a null matrix of order* $p \times p$ *and* $\mathbf{V}_N$ *and* $\mathbf{\nu}_N$ *are defined by* (2.7) *and* (3.9), *respectively.*

PROOF. Using (3.6), (3.7) and the Assumption 2, we obtain from (2.7) that

$$\begin{aligned} v_{N,j,k} &= \int_0^\infty \int_0^\infty J_{N,j}[(N/(N+1))H_{N,j}(x)]J_{N,k}[(N/(N+1))H_{N,k}(y)] \\ &\qquad \cdot dH_{N,j,k}(x, y), \\ &= \int_0^\infty \int_0^\infty J_j[(N/(N+1))H_{N,j}(x)]J_k[(N/(N+1))H_{N,k}(y)] \\ &\qquad \cdot dH_{N,j,k}(x, y) + o_p(1). \end{aligned} \tag{3.10}$$

Proceeding then exactly as in Puri and Sen (1966, Theorem 4.2), we find, on omitting the details of computations that,

$$v_{N,j,k} = \int_0^\infty \int_0^\infty J_j[H_j(x; \theta_j)]J_k[H_k(y; \theta_k)]\, dH_{j,k}(x, y; \theta_j, \theta_k) + o_p(1). \tag{3.11}$$

Hence, the theorem

REMARK. It follows from Theorem 3.1 that under the Assumptions 1–5, $\mathbf{V}_N$ is positive definite, in probability (as $N \rightarrow \infty$).

COROLLARY 3.1. *If*

(i) $F_j(x; \theta_j) = F_j(x - \theta_j/N^{-\frac{1}{2}})$,

$$F_{j,k}(x, y; \theta_j, \theta_k) = F_{j,k}(x - \theta_j/N^{-\frac{1}{2}}, y - \theta_k/N^{-\frac{1}{2}})$$

where F_j *is symmetric and* $F_{j,k}$ *diagonally symmetric about the origin, for* $j \neq k = 1, \cdots, p$;

(ii) *the assumptions of Theorem* 3.1 *are satisfied, then* $\mathbf{V}_N \rightarrow \mathbf{\nu}^*$, *in probability, as* $N \rightarrow \infty$, *where* $\mathbf{\nu}^* = (\nu_{jk}^*)$ *is given by*

$$\nu_{jk}^* = \int^{+\infty} \int^{+\infty} J_j[2F_j(x) - 1]J_k[2F_k(y) - 1]\, dH_{j,k}(x, y). \tag{3.12}$$

where $H_{j,k}(x, y)$ is defined by (3.2) with $\theta = \mathbf{0}$.

THEOREM 3.2. *Under the assumptions* 1 *to* 4, $N^{-\frac{1}{2}}\mathbf{T}_N$ *has* (*under* P_n) *asymptotically a p-variate normal distribution with mean vector zero and covariance matrix* $V_N = (v_{N,jk})$ *is given by* (2.7). *Hence, the limiting permutation distribution of* S_N *in* (2.8) *is a chi-square with p degrees of freedom.*

PROOF. To prove this theorem, it suffices to show that for any arbitrary constant vector $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_p)$, the distribution of $N^{-\frac{1}{2}}\boldsymbol{\delta}\mathbf{T}_N'$ is asymptotically normal under the permutational probability measure P_N. From (2.5) we notice that

$$(3.13) \quad \boldsymbol{\delta}\cdot\mathbf{T}_N' = \sum_{\alpha=1}^{N}\sum_{j=1}^{P}\delta_j E_{NR_\alpha(j)}^{(j)}\cdot C_\alpha^{(j)} = \sum_{\alpha=1}^{N} a_{N\alpha}(\mathbf{E}_N, \mathbf{C}_{(N)}, \boldsymbol{\delta})$$

where by definition $a_{N\alpha}(\mathbf{E}_N, \mathbf{C}_{(N)}, \boldsymbol{\delta})$ depends upon $\mathbf{E}_N$, $\mathbf{C}_N$ and $\boldsymbol{\delta}$. Let us now consider the permutation distribution generated by 2^N transformations g_N in G_N defined in (2.2). It then follows that $\mathbf{E}_N$ and $\boldsymbol{\delta}$ remain invariant under G_N on $\mathbf{Z}_N$, while $\mathbf{C}_\alpha = (C_\alpha^{(1)}, \cdots, C_\alpha^{(p)})$ may have permutationally two equal probable values, namely $\mathbf{C}_\alpha$ and $(-1)\mathbf{C}_\alpha$, for $\alpha = 1, \cdots, N$. Thus, we may rewrite (3.13) as

$$(3.14) \qquad \sum_{\alpha=1}^{N} |a_{N_\alpha}(\mathbf{E}_N, \mathbf{C}_{(N)}, \boldsymbol{\delta})|\, d_{N_\alpha},$$

where $\{d_{N_\alpha}\}$ are mutually independent (under P_n) and $d_{N_\alpha} = +1$ or -1 with equal probability $\frac{1}{2}$; and where conditionally on the given $\mathbf{Z}_N$, $|a_{N_\alpha}(\mathbf{E}_N, \mathbf{C}_{(N)}, \boldsymbol{\delta})|$, $\alpha = 1, \cdots, N$ are all fixed. It is also easily seen that the permutational average of (3.14) is zero, and the variance $N\boldsymbol{\delta}\mathbf{V}_N\boldsymbol{\delta}'$. So if we define $W_{N,\alpha} = |a_{N\alpha}(\mathbf{E}_N, \mathbf{C}_{(N)}, \boldsymbol{\delta})|\, d_{N,\alpha}$, then under P_N, $\{W_{N,\alpha}\}$ are independent random variables with means zero and $W_{N,\alpha}$ can assume only two values $\pm |W_{N,\alpha}|$ each with probability $\frac{1}{2}$. We will now apply the central limit theorem to the sequence $\{W_{N,\alpha} : \alpha = 1, \cdots, N\}$ under the Liapounoff's condition, [cf. Loeve (1962, p. 275)] and for this we require to show that for some $r > 2$,

$$(3.15) \quad \lim_{N\to\infty} N^{-(r-2)/2}\cdot(1/N)\sum_{\alpha=1}|a_{N,\alpha}(\mathbf{E}_N, \mathbf{C}_{(N)}, \boldsymbol{\delta})|^r \cdot\{(1/N)\sum_{\alpha=1}^{N} a_{N,\alpha}^2(\mathbf{E}_N, \mathbf{C}_{(N)}, \boldsymbol{\delta})\}^{-r/2} = 0.$$

The denominator of the second factor of (3.15) is $(\boldsymbol{\delta}\mathbf{V}_N\boldsymbol{\delta}')^{r/2}$, which by Theorem 3.1 converges to some positive quantity for any non-null $\boldsymbol{\delta}$ where $\boldsymbol{\nu}$ is assumed to be positive definite. So we require only to show that for some $r > 2$, the numerator of the second factor of (3.15) is bounded in probability. Since $a_{N,\alpha}(\mathbf{E}_N, \mathbf{C}_{(N)}, \boldsymbol{\delta})$ is a linear function of $\mathbf{E}_{N_\alpha}$ and $\boldsymbol{\delta}$ (with the sign of the coefficients being a minus or plus), on applying the well-known inequality that

$$|\sum_{i=1}^{p} a_i|^r \leqq p^{r-1}\sum_{i=1}^{p}|a_i|^r,$$

[cf. Loeve (1962, pp. 155], we get

$$(1/N)\sum_{\alpha=1}^{N}|a_{N,\alpha}(\mathbf{E}_N, \mathbf{C}_{(N)}, \boldsymbol{\delta})|^r \leqq p^{r-1}\sum_{i=1}^{p}|\delta_i|^r\cdot\{(1/N)\sum_{\alpha=1}^{N}|E_{N,\alpha}^{(i)}|^r\} < \infty,$$

by Assumptions 1 and 3. Thus, (3.15) converges, in probability, to zero. Hence, the first part of the theorem. The second part is an immediate consequence of the first part and hence the proof is omitted.

4. Asymptotic normality of $\mathbf{T}_N$ for arbitrary F. In this section we shall prove that under a certain set of conditions the random vector $\mathbf{T}_N = (T_N^{(1)}, \cdots, T_N^{(p)})$ has a joint normal distribution in the limit. For the sake of convenience we shall consider the statistic

$$\mathbf{T}_N^* = (T_{N,1}, \cdots, T_{N,p}) \tag{4.1}$$

where

$$T_{N,j} = (1/N) \sum_{i=1}^{N} E_{N,i}^{(j)} Z_{N,i}^{(j)} \tag{4.2}$$

where, $Z_{N,i}^{(j)} = 1$ if $X_i^{(j)} > 0$, and $Z_{N,i}^{(j)} = 0$ otherwise. $E_{N,i}^{(j)}$; $i = 1, \cdots, N$; $j = 1, \cdots, p$, are the constants satisfying the Assumptions (1) to (3) of the previous section. The reader may notice that the vector $\mathbf{T}_N^*$ is related to the vector $\mathbf{T}_N$ by the relation $\mathbf{T}_N = 2N\mathbf{T}_N^* - N\bar{\mathbf{E}}_N$ where

$$\bar{\mathbf{E}}_N = (1/N)\left(\sum_{i=1}^{N} E_{N,i}^{(1)}, \cdots, \sum_{i=1}^{N} E_{N,i}^{(p)}\right),$$

and so it suffices to consider the equivalent statistic $\mathbf{T}_N^*$. The main theorem of this section is the following: For the particular case of $p = 1$, the reader is referred to [10] and [17].

THEOREM 4.1. *Under the assumptions* (1) *to* (3) *of Section* 3, *the random vector* $N^{\frac{1}{2}}(\mathbf{T}_N^* - \mu_N(\theta))$ *has asymptotically a p-variate normal distribution with mean vector zero, and covariance matrix* $(\sigma_{N,jk})$, *where*

$$\mu_N(\theta) = (\mu_{N,1}, \cdots, \mu_{N,p}), \tag{4.3}$$

$$\mu_{N,j} = \int_0^\infty J_j[H_j(x; \theta_j)]\, dF_j(x; \theta_j); \quad j = 1, \cdots, p,$$

and where $(\sigma_{N,jk})$ *is given by* (4.9) *and* (4.10) *respectively.*

PROOF. We can express $T_{N,j}$ (cf. [10]) as

$$T_{N,j} = \mu_{N,j} + B_{1N,j} + B_{2N,j} + \sum_{i=1}^{4} C_{iN,j} \tag{4.4}$$

where

$$\mu_{N,j} = \int_0^\infty J_j(H_j(x, \theta_j))\, dF_j(x, \theta_j), \tag{4.5}$$

$$B_{1N,j} = \int_0^\infty J_j(H_j(x, \theta_j))\, d(F_{N,j}(x) - F_j(x, \theta_j), \tag{4.6}$$

$$B_{2N,j} = \int_0^\infty (H_{N,j} - H_j) J_j'(H_j)\, dF_j(x), \tag{4.7}$$

and the C terms are all $o_p(N^{-\frac{1}{2}})$.

The difference $N^{\frac{1}{2}}(T_{N,j} - \mu_{N,j}) - N^{\frac{1}{2}}(B_{1N,j} + B_{2N,j})$ tends to zero in probability and so to prove this theorem, it suffices to show that for any real δ_i, $i = 1, \cdots, p$, not all zero, $N^{\frac{1}{2}} \sum_{j=1}^{p} \delta_j(B_{1N,j} + B_{2N,j})$ has normal distribution in the limit. Now proceeding as in [4], we can express $N^{\frac{1}{2}} \sum_{j=1}^{p} \delta_j(B_{1N,j} + B_{2N,j})$ as a sum of independent and identically distributed random variables having finite first two moments. The proof follows.

To compute the variance-covariance matrix of $N^{\frac{1}{2}}(B_{1N,j} + B_{2N,j})$, we note from (4.6) and (4.7) that

$$\begin{aligned} &N^{\frac{1}{2}}(B_{1N,j} + B_{2N,j}) \\ &\quad = N^{\frac{1}{2}} \int_0^\infty (F_{N,j}(x) - F_j(x; \theta_j)) J_j'(H_j(x; \theta_j))\, dF_j(-x; \theta_j) \\ &\qquad - N^{\frac{1}{2}} \int_0^\infty (F_{N,j}(-x) - F_j(-x; \theta_j)) J_j'(H_j(x; \theta_j))\, dF_j(x; \theta_j). \end{aligned} \tag{4.8}$$

This has mean zero, and variance

$$\begin{aligned}
\sigma_{N,jj} = {} & 2 \int\int_{0<x<y<\infty} F_j(x;\theta_j)[1 - F_j(y;\theta_j)]J_j'(H_j(x;\theta_j)) \\
& \cdot J_j'(H_j(y;\theta_j))\, dF_j(-x;\theta_j)\, dF_j(-y;\theta_j) \\
& + 2 \int\int_{0<x<y<\infty} F_j(-y;\theta_j)[1 - F_j(-x;\theta_j)]J_j'(H_j(x;\theta_j)) \\
& \cdot J_j'(H_j(y;\theta_j))\, dF_j(x;\theta_j)\, dF_j(y;\theta_j) \\
& - 2 \int_{x=0}^{\infty}\int_{y=0}^{\infty} F_j(-y;\theta_j)[1 - F_j(x;\theta_j)]J_j'(H_j(x;\theta_j)) \\
& \cdot J_j'(H_j(y;\theta_j))\, dF_j(-x;\theta_j)\, dF_j(y;\theta_j).
\end{aligned} \tag{4.9}$$

Similarly,

$$\begin{aligned}
\sigma_{N,jk} = {} & \int_{x=0}^{\infty}\int_{y=0}^{\infty} [F_{j,k}(x, y;\theta_j, \theta_k) - F_j(x;\theta_j)F_k(y;\theta_k)]J; \\
& \cdot J_j'[H_j(x;\theta_j)]J_k'[H_k(y;\theta_k)]\, dF_j(-x;\theta_j)\, dF_k(-y;\theta_k) \\
& - \int_{x=0}^{\infty}\int_{y=0}^{\infty} [F_{j,k}(x, -y;\theta_j, \theta_k) - F_j(x;\theta_j)F_k(-y;\theta_k)] \\
& \cdot J_j'[H_j(x;\theta_j)]J_k'[H_k(y;\theta_k)]\, dF_j(-x;\theta_j)\, dF_k(y;\theta_k) \\
& - \int_{x=0}^{\infty}\int_{y=0}^{\infty} [F_{j,k}(-x, y;\theta_j, \theta_k) - F_j(-x;\theta_j)F_k(y;\theta_k)] \\
& \cdot J_j'[H_j(x;\theta_j)]J_k'[H_k(y;\theta_k)]\, dF_j(x;\theta_j)\, dF_k(-y;\theta_k) \\
& + \int_{x=0}^{\infty}\int_{y=0}^{\infty} [F_{j,k}(-x, -y;\theta_j, \theta_k) - F_j(-x;\theta_j)F_k(-y;\theta_k)] \\
& \cdot J_j'[H_j(x;\theta_j)]J_k'[H_k(y;\theta_k)]\, dF_j(x;\theta_j)\, dF_k(y;\theta_k).
\end{aligned} \tag{4.10}$$

In what follows, we shall concern ourselves with a sequence of admissible alternative hypothesis $\{H_N^P\}$, which specifies that for each $j, k = 1, \cdots, p$,

$$\begin{aligned}
F_j(x;\theta_j) &= F_j(x - \theta_j/N^{\frac{1}{2}}); \\
F_{j,k}(x, y;\theta_j;\theta_k) &= F_{j,k}(x - \theta_j/N^{\frac{1}{2}}, y - \theta_k/N^{\frac{1}{2}})
\end{aligned} \tag{4.11}$$

where $F_j(x)$ and $F_{j,k}(x, n)$ are symmetric about zero, and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)'$ is unknown.

We shall also assume that the constant $E_{N,i}^{(j)}$, $i = 1, \cdots, N; j = 1, \cdots, p$, is the expected value of the ith order statistic of a sample of size N from a distribution function $\Psi_j(x)$ given by

$$\Psi_j(x) = \Psi_j^*(x) - \Psi_j^*(-x); \qquad x \geqq 0, \tag{4.12}$$

where $\Psi_j^*(x)$ is a distribution function either symmetric about zero or uniform over $(-1, 1)$. It may be noted that the above definition of $E_{N,i}^{(j)}$ implies that the function

$$J_j(x) = \Psi_j^{-1}(x) = \Psi_j^{*-1}((1 + x)/2). \tag{4.13}$$

The following corollary is then an immediate consequence of Theorem 4.1.

Corollary 4.1. *If, for every* $j, k = 1, \cdots, p$, (4.11) *holds and the conditions of*

Theorem 4.1 *are satisfied, then* $N^{\frac{1}{2}}(\mathbf{T}_N - \boldsymbol{\mu}_N)$ *has asymptotically a p-variate normal distribution with a null mean vector and a dispersion matrix* $\mathbf{T} = ((\tau_{jk}))$; *where*

$$\tau_{jk} = \tfrac{1}{4}\int_0^1 J_j^{*2}(x)\,dx, \qquad j = k; \quad J_j^* = \Psi_j^{*-1};$$

$$(4.14) \qquad = \tfrac{1}{4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} J_j^*(F_j(x)) \cdot J_k^*(F_k(y))\,dF_{j,k}(x, y), \qquad j \neq k, \quad J_j^* = \Psi_j^{*-1}.$$

It may be noted that the limiting distribution of $N^{\frac{1}{2}}(\mathbf{T}_N{}^* - \boldsymbol{\mu}_N)$ is nonsingular if and only if the functions J_j and F_j are such that the moment matrix of $\{J_j^*(F_j(x)), j = 1, \cdots, p\}$ is non-singular. Furthermore, the limiting distribution of $N^{\frac{1}{2}}(\mathbf{T}_N{}^* - \boldsymbol{\mu}_N)$ is singular, if and only if, a.s. F

$$(4.15) \quad J_j^*(F_j(x)) = \textstyle\sum_{k\neq j} \alpha_k J_k^*(F_k(y)) + \text{constant},$$

(where α_k's are some constants)

5. The proposed class of asymptotic tests and its limiting distribution. We now assume that the covariance matrix $\mathbf{T}$ defined by (4.14) is non-singular. Then for testing the hypothesis H_0 given by (1.1), we propose to consider the test statistics $S_N{}^*$ defined as

$$(5.1) \qquad S_N{}^* = N(\mathbf{T}_N{}^* - \boldsymbol{\mu}_N{}^{(0)})\hat{T}^{-1}(\mathbf{T}_N{}^* - \boldsymbol{\mu}_N(0))'$$

where $T_N{}^*$ is defined by (4.1) and (4.2); $\hat{\mathbf{T}}^{-1}$ is a consistent estimator of $\mathbf{T}^{-1}$, and

$$(5.2) \qquad \mu_N(0) = (\mu_{N,1}(0), \cdots, \mu_{N,p}(0))$$

where

$$(5.3) \qquad \mu_{N,j}(0) = \int_{x=0}^{\infty} J_j(2F_j(x) - 1)\,dF_j(x), \qquad J_j = \Psi_j^{-1}.$$

Then we have:

THEOREM 5.1. *If* (i) *the conditions of Corollary* 4.1 *are satisfied, and* (ii) *the assumptions of Lemma* 7.2 *of Puri* (1964) *hold for each* F_j *and* J_j; $j = 1, \cdots, p$, *then for* $N \to \infty$, *the limiting distribution of the statistics* $S_n{}^*$ *is non-central chi-square with p degrees of freedom and non-centrality parameter* $\Delta(S_N{}^*)$ *given by*

$$(5.4) \qquad \Delta(S_N{}^*) = \underset{\sim}{a}\mathbf{T}^{-1}\underset{\sim}{a}' = \boldsymbol{\Theta}\mathbf{T}^{*-1}\boldsymbol{\Theta}'$$

where

$$(5.5) \quad \underset{\sim}{a} = (c_1\theta_1, \cdots, c_p\theta_p),$$

$$c_j = -2\int_0^{\infty} J_j'(2F_j(x) - 1)f_j(x)\,dF_j(x), \quad j = 1, \cdots, p,$$

and $\mathbf{T}^* = (\tau_{jk}^*)$ *is given by*

$$(5.6) \qquad \tau_{jk}^* = \tau_{jk}/(c_j \cdot c_k) \quad \text{for} \quad j, k = 1, \cdots, p.$$

PROOF. The proof of this theorem is a consequence of the facts that under the given assumptions, $N^{\frac{1}{2}}(\mathbf{T}_N{}^* - \boldsymbol{\mu}_N(0))$ has asymptotically a p-variate normal distribution with mean vector $\underset{\sim}{a}$ and covariance matrix $\mathbf{T}$, and $\hat{\mathbf{T}}^{-1}$ is a consistent estimator of $\mathbf{T}^{-1}$.

From Theorem 5.1, it is clear that the choice of $\hat{\mathbf{T}}^{-1}$ is of no importance in the limit. Any consistent estimator of $\mathbf{T}^{-1}$ will preserve the asymptotic distribution of the statistic S_N^*. One such consistent estimator is provided by the permutational covariance elements in Section 3. We may also use a theorem by Bhuchongkul (1964) to propose a similar class of estimates. However here conditions are relatively more restrictive (since they relate to asymptotic normality) than the ones in Theorem 3.1, and for our purpose, we need not bother about the conditions as we simply require here the asymptotic convergence (not normality) of the estimates.

THEOREM 5.2. *The permutation test based on S_N given by (2.8) and the asymptotically non-parametric test based on S_N^* given by (5.1) are asymptotically power equivalent for the sequence of translation type of alternatives defined by (4.11).*

PROOF. From Corollary 4.1, it follows that for any such translation type of alternatives, the covariance matrix of the proposed vector of rank order statistics converges asymptotically to $\mathbf{T}$, of which $\hat{\mathbf{T}}$ is a consistent estimator. Again by Corollary 3.1, it readily follows that under any such sequence of translation type of alternatives, the permutation covariance matrix in Theorem 3.2 also converges to $\mathbf{T}$. Thus by looking at (2.8) and Theorem 5.1, we observe that under (4.11) $S_N \overset{P}{\ } S_N^*$, (where ^{P} means stochastically equivalent). The proof follows.

By virtue of the stochastic equivalence of the tests S_N and S_N^*, we shall only consider the asymptotic properties of the unconditional test based on S_N^* in the next section.

SPECIAL CASES. (a) Let J_j be the inverse of a chi-distribution with one degree of freedom. Then the S_N^* test reduces to the normal scores $S_N^*(\Phi_N)$ test which may be regarded as a p-variate version of the univariate one-sample normal scores test. Let us define

$$(5.7)\quad B_j(F;\Phi) = \int_{-\infty}^{\infty} f_j^2(x)\,dx/\phi[\Phi^{-1}(F_j(x))];$$

$$(5.8)\quad \nu_{jk}(F;\Phi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi^{-1}(F_j(x))\Phi^{-1}(F_k(y))\,dF_{j,k}(x,y), \qquad j \neq k$$
$$= 1, \qquad j = k;$$

$$(5.9)\quad \lambda_{jk} = \nu_{jk}(F;\Phi)/[B_j(F;\Phi)B_k(F;\Phi)], \qquad j,k = 1,\cdots,p.$$

Then for the normal score test, the non-centrality parameter (5.4) reduces to

$$(5.10)\quad \delta(\Phi) = \boldsymbol{\theta}\boldsymbol{\lambda}^{-1}\boldsymbol{\theta}' \quad \text{where} \quad \boldsymbol{\lambda} = ((\lambda_{jk})) \quad \text{is given by (5.9).}$$

(b) Let $J_j(u) = u$, then the S_N^* test reduces to the rank-sum $S_N^*(R)$ test (which may be regarded as a p-variase version of the univariate one sample test) considered in detail by Bickel (1965). In this case the non-centrality parameter (5.4) reduces to

$$(5.11)\quad \delta(R) = \boldsymbol{\theta}\Gamma^{-1}\boldsymbol{\theta}'$$

where $\mathbf{\Gamma} = (\gamma_{jk})$ is given by

$$\gamma_{jk} = 1/12B_j^2(F; R),$$

(5.12) $$= [B_j(F; R)B_k(F; R]^{-1} \cdot \{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F_j(x)F_k(y)\, dF_{j,k}(x, y) - \tfrac{1}{4}\}, \quad j \neq k;$$

for $j, k = 1, \cdots, p$, where

(5.13) $$B_j(F; R) = \int_{-\infty}^{\infty} f_j^2(x)\, dx, \quad \text{for} \quad j = 1, \cdots, p.$$

6. Asymptotic relative efficiency. It is well known [14] that in the situations we are considering the asymptotic efficiency of one statistic relative to another is equal to the ratio of their noncentrality parameters. Hence, denoting e_{T_N, T_N^*} as the asymptotic efficiency of a test T_N relative to T_N^*, we have

(6.1) $$e_{S_N^*, T_N 2} = (\boldsymbol{\theta}\mathbf{T}^{*-1}\boldsymbol{\theta}')/(\boldsymbol{\theta}\mathbf{\Pi}^{-1}\boldsymbol{\theta}')$$

where T_N^2 denotes the Hotelling's T_N^2 test, $\mathbf{T}^* = (\tau_{jk}^*)$ is given by (5.6) and $\mathbf{\Pi} = (\sigma_{jk})$ is the covariance matrix of F.

Special cases. (a) *Normal Scores and Rank Sum Tests.* From (6.1), we find that the efficiencies of the normal scores $S^*(\Phi)$ test and the rank sum $S_N^*(R)$ test relative to T_N^2-test are

(6.2) $$e_{S_N^*(\Phi), T_N 2} = (\boldsymbol{\theta}\boldsymbol{\lambda}^{-1}\boldsymbol{\theta}')/(\boldsymbol{\theta}\mathbf{\Pi}^{-1}\boldsymbol{\theta}')$$

where $\boldsymbol{\lambda} = (\lambda_{jk})$ is given by (5.9) and $\mathbf{\Pi} = (\sigma_{jk})$,

(6.3) $$e_{S_N^*(R), T_N 2} = (\boldsymbol{\theta}\mathbf{\Gamma}^{-1}\boldsymbol{\theta}')/(\boldsymbol{\theta}\mathbf{\Pi}^{-1}\boldsymbol{\theta}')$$

where $\mathbf{\Gamma} = (\gamma_{jk})$ is given by (5.13).

We may remark that the expression (6.3) is the same as the one obtained by Bickel (1965) for the p-variate one-sample problem, and by Chatterjee and Sen (1964) for the bivariate two-sample problem. For the study of the various aspects of the efficiency (6.3) in special situations the reader is referred to the interesting paper of Chatterjee and Sen (1964).

(b) *Totally symmetric case.* A bivariate random vector (X, Y) is said to be totally symmetric if (X, Y), $(X, -Y)$, $(-X, -Y)$ have the same distribution function. It can be shown following the lines of the argument of Bickel (1965), that a sufficient condition for the asymptotic independence of the components of $\mathbf{T}_N^*$ is the total symmetry of $(X_1^{(j)}, X_1^{(k)})$ for every pair (j, k). Thus in the event of totally symmetric case $\boldsymbol{\gamma}$, $\mathbf{\Pi}$, $\mathbf{\Gamma}$ are all diagonal matrices, and hence we have

(6.4) $$e_{S_N^*(\Phi), T_N 2} = \sum_{j=1}^{p} \theta_j^2(B(F; \Phi))^2/\sum_{j=1}^{p} \theta_j^2/\sigma_j^2,$$

(6.5) $$e_{S_N^*(R), T_N 2} = 12\sum_{j=1}^{p} \theta_j^2(B(F; R))^2/\sum_{j=1}^{p} \theta_j^2/\sigma_j^2,$$

(6.6) $$e_{S_N^*(\Phi), S_N^*(R)} = \sum_{j=1}^{p} \theta_j^2(B(F; \Phi))^2/12\sum_{j=1}^{p} \theta_j^2(B(F; R))^2,$$

where $B_j(F; \Phi)$ and $B_j(F; R)$ are defined by (5.7) and (5.13), respectively. Applying a theorem of courant, Bickel (1965) has proved that

(6.7) $$\inf_{F\varepsilon\mathfrak{F}} \inf_\theta e_{S_N^*(R), T_N^2} = 0.86$$

where $\mathfrak{F}$ is the family of all totally symmetric p-variate distributions whose marginal densities exist.

It may be noted that the efficiency factors (6.4) and (6.6) are the same as in the case of the corresponding tests for the multivariate two sample problem (cf. Puri and Sen [16]).

In passing we may remark that when the components of F are totally symmetric as well as identically distributed, then the expressions (6.4), (6.5) and (6.6) become independent of θ's, and the results are the same as in the case of the corresponding univariate one-sample problems.

(c) *Normal case.* Let us now assume that the underlying distribution function F is a non-singular p-variate normal with mean vector zero and covariance matrix $\Pi = (\sigma_{jk})$. Then it can easily be checked that

(6.8) $$e_{S_N^*(\Phi), T_N^2} = 1.$$

This means that in the case of normal distributions, the property of the univariate normal scores test relative to the student's t test is preserved in the multivariate case. This is interesting in the sense that the same is not the case with the multivariate rank sum test as Bickel (1965) has shown that

(6.9) $$\inf_{F\varepsilon\Phi} \inf_\theta e_{S_N^*(R), T_N^2} = 0 \quad \text{for} \quad p \geqq 3,$$

and

(6.10) $$\tfrac{3}{4} \leqq e_{S_N^*(R), T_N^2} \leqq 0.965 \quad \text{for} \quad p = 2.$$

From (6.8) and (6.9), it follows that

(6.11) $$\sup_{F\varepsilon\Phi} \sup_\theta e_{S_N^*(\Phi), S_N^*(R)} = \infty \quad \text{for} \quad p \geqq 3.$$

[Φ is the family of all nonsingular p-variate normal distributions.]

For the related study regarding the bounds of $e_{S_N^*(\Phi), S_N^*(R)}$ the reader is referred to Bhattacharyya (1966); and for the corresponding study of the estimation of location parameters, the reader is referred to Puri and Sen (1967a)

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