THE CONSTRUCTION OF SATURATED 2_R^{k-p} DESIGNS

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0. Summary. If a 2_R^{k-p} design, of fixed resolution R and specified number of runs 2^q, accommodates the maximum possible number of variables, we say that it is *saturated*. In this paper, we develop a method for constructing saturated designs and apply it to an example.

We first show that when R is odd, the set of all distinct 2_R^{k-p} designs (where $q = k - p$ is specified) can be obtained easily from a particular class of $2_{R+1}^{(k+1)-p}$ designs. We then develop a stage by stage method for constructing this "parent" class of designs of (even) resolution $R + 1$. This class is shown, incidentally, to contain a saturated design. The complete set of 2_R^{k-p} designs, which naturally includes all saturated 2_R^{k-p} designs, can then be obtained at once. The problem of arranging the designs constructed into blocks of runs, so that the blocked designs have certain desirable confounding properties, is also investigated, and a method for obtaining optimal blocking arrangements is given. As an important part of our method, a "sequential conjecture" procedure is developed and utilized to test the equivalence of any two designs.

These procedures have been programmed for the computer, and are illustrated by the example $R = 5$, $q = 7$.

1. Introduction.

1.1. *2^{k-p} fractional factorial designs.* The construction of 2^{k-p} fractional factorial designs and the study of their confounding properties has been approached from several closely related points of view, e.g., geometrically (Kempthorne (1947)), as a special case of an orthogonal array (Rao (1947)), Bose and Bush (1952)), and through the theory of groups (Fisher (1942)). (For a review of these and other approaches to the construction of 2^{k-p} fractional factorial designs, see Addelman (1963).)

Box and Hunter (1961a), (1961b) have distilled the essential results and presented straightforward techniques for constructing, blocking, and analyzing 2^{k-p} fractional factorial designs of resolutions III, IV, and V. Throughout this paper we shall often refer to this work.

We shall assume that the variables of a 2^{k-p} design are labeled $(1, 2, \cdots, k)$. From any subset of these variables, or *letters*, we can form a *word*, e.g., 1357 is a word composed of the letters 1, 3, 5, and 7. Associated with every 2^{k-p} design

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is a set of p words, $W_1, W_2, \cdots, W_p$, called *generators*. If we define the *product* of two words X and Y to be that word which contains the letters appearing in X *or* Y, but not in both, then the set of words which is composed of all possible products involving the p generators appears in the *defining relation*:

$$
\begin{aligned}
I &= W_1 = W_2 = \cdots = W_p \\
&= W_1W_2 = \cdots = W_{p-1}W_p && \text{(all products of } 2W\text{'s)} \qquad (1.1.1)\\
&= W_1W_2W_3 = \cdots = W_{p-2}W_{p-1}W_p && \text{(all products of } 3W\text{'s)}\\
&\;\vdots \\
&= W_1W_2 \cdots W_p && \text{(the product of } pW\text{'s).}
\end{aligned}
$$

(I is called the *identity* and is such that $IX = XI = X$ for all words X.)

The length (i.e., the number of letters) of the shortest word in the defining relation is called the *resolution* (R) of the design and is used to classify it. In a design of resolution R all main effects are confounded with interactions involving $(R - 1)$ or more factors, all two-factor interactions are confounded with interactions involving $(R - 2)$ or more factors, and so on. If the experimenter tentatively assumes that the importance of J-factor interactions diminishes as J increases, then the higher the value of R, the more satisfactory the design is with respect to the principle of confounding the "important" effects with "unimportant" effects. Of course, given sufficient runs we can always make R suitably large. In practice, when the number of factors k is specified, we may want R to be as large as possible for some *given* number of runs. Or, if R is specified, we may wish to minimize the number of runs necessary to examine k factors in a 2_R^{k-p} design. Both of these problems can be solved if we have solved, in an appropriate number of cases, the equivalent problem of accommodating the largest number of factors in a 2_R^{k-p} design of given resolution and given number of runs. It is this latter problem which we shall consider.

1.2. *Saturated* 2^{k-p} *designs of resolution* R. For designs of resolution III, Box and Hunter (1961a) used the word *saturated* to describe the two-level resolution III designs which incorporate $N - 1$ variables in N runs. This number $(N - 1)$ is the maximum number of variables possible. We shall extend the use of the word saturated to two-level fractional factorial designs of general resolution R as follows:

Suppose the number of runs, 2^q, and the resolution R, of a two-level fractional factorial design are both specified. A 2_R^{k-p} design (where $k - p = q$) which contains the maximum possible number of variables k will be called a *saturated resolution* R *design in* 2^q *runs* or, simply, a *saturated design.*

It can be shown that for resolutions III and IV (Box and Hunter (1961a)), the number of variables accommodated in a saturated design in 2^q runs is $2^q - 1$ and 2^{q-1}, respectively.

For designs of resolution V the situation is not as straightforward. Box and Hunter (1961b) summarize the solutions of the problem for the cases $q = 4$, 5, 6, and 7 as shown in the following tabulation:

	q	4	5	6	7
(1.2.1)	no. of runs ($= 2^q$)	16	32	64	128
	max. no. of variables which can be accommodated in a resolution V design	5	6	8	11

When $q \geqq 8$, the maximum number of variables which can be accommodated in a resolution V design has not previously been determined. A specific 2_{V}^{17-9} design was given by Addelman (1965) who concluded that "... it is unlikely that more than 17 factors can be accommodated in such a plan."

For resolutions >5, no results have been published, though the simpler cases can readily be solved by extending the methods applied by Box and Hunter (1961b) to some resolution V examples.

In this paper we develop a general method for constructing saturated designs of resolutions R and $R + 1$, where R is odd. The procedure, which has been programmed for the computer, is illustrated in the case $R = 5$, $q = 7$. A general method for blocking 2^{k-p} designs in such a way that the maximum possible number of blocks are attained is also given. (The more extensive application of these methods to the cases (i) $R = 5$, $q = 8$ and (ii) $R = 5$, $q = 9$ will appear in subsequent papers.)

2. Development.

2.1. *Preliminary definitions.* We shall say that two designs D_1 and D_2 are *equivalent* if and only if one may be obtained from the other by a relabeling of the variables. A more precise definition is the following:

DEFINITION. Designs D_1 and D_2, each of which incorporates the variables $(1, 2, \cdots, k)$, are equivalent (denoted by $D_1 \equiv D_2$) if and only if there is a permutation of the variables $(1, 2, \cdots, k)$ which creates a one to one mapping of the words of D_1 into the words of D_2. (Note: For the sake of brevity, we shall often use the expression "words of D" to mean "words of the defining relation of D.")

We shall say that two designs are *distinct* if and only if they are not equivalent.

It will be convenient to classify designs as *even* or *odd*, according to the following definition.

DEFINITION. An *even* design is one whose defining relation consists entirely of words of even length. (The identity I is considered to be a word of even length.) An *odd* design is one whose defining relation contains at least one word of odd length.

2.2. *Preliminary remarks.* We note that obtaining a 2_R^{k-p} design which maximizes $k = q + p$ for given R and fixed q is equivalent to obtaining a 2_R^{k-p} design which maximizes p for given R and fixed q. Since p is the number of generators of the design, we may consider a saturated design as one which has the greatest number of generators (p) for fixed $q = k - p$, these generators satisfying, of course, the restrictions imposed by the resolution R. We shall call a set of such generators a *maximal set* for fixed q and R.

Our procedure for building up a maximal set of generators for fixed q and R is based on a particular form of construction, which we obtain as a consequence of the following observations.

REMARK 1. The defining relation of a 2^{k-p} fractional factorial design contains 2^p words (including I). Of these, either

(i) half are of even length and half are of odd length, or

(ii) all are of even length,

where I is counted as an even word. In particular, the defining relation of every 2^{k-p} design of odd resolution R, which by definition includes at least one word of length R, must be composed of 2^{p-1} odd words and 2^{p-1} even words (including I). (Note: An equivalent form of Remark 1 can also be found in Brownlee, Kelly, and Loraine (1948).)

REMARK 2. Given the defining relation D of an arbitrary 2_R^{k-p} design, where R is odd, we can "attach" an extra variable $(k + 1)$ to each of the 2^{p-1} odd words of D. Then the resulting expression, which contains only words of even length, is the defining relation, E say, of some $2_{R+1}^{(k+1)-p}$ design. (Note: E is actually the defining relation of the design obtained by associating the variable $(k + 1)$ with the I column of the design matrix of D, then "folding over" this design. (See Box and Hunter (1961a)).)

We note that, in Remark 2, D can be recovered from E simply by *erasing*, i.e., removing from each word, the variable $(k + 1)$. Remark 2 implies, therefore, that the defining relation D of any 2_R^{k-p} design, where R is odd, can be obtained from the defining relation E of some even $2_{R+1}^{(k+1)-p}$ design, by erasing a particular variable wherever it appears in E.

Letting $q = k - p$, we shall write the generators of E in the form:

$$\begin{aligned} W_1 &= K_1(q + 2), \\ W_2 &= K_2(q + 3), \\ &\vdots \\ W_{p-1} &= K_{p-1}(q + p), \\ W_p &= K_p(q + p + 1) = K_p(k + 1), \end{aligned} \tag{2.2.1}$$

where, for each $i = 1, 2, \cdots, p$, the variable $q + 1 + i$ appears in one and only one generator, namely W_i.

The expression of a set of generators in a form such as (2.2.1), in which each of p variables is isolated in one of the p generators, was introduced by Box and Hunter (1961b). We shall call this form a *standard form* and shall refer to the p isolated variables as *indicator variables*.

It can easily be shown that every defining relation has a set of generators which can be written in a standard form. Every even $2_{R+1}^{(k+1)-p}$ design, where R is odd, is therefore equivalent to a design whose generators are written as in (2.2.1).

The set $\{K_i\}$ satisfies the conditions:

$$\begin{aligned}
&\text{(i)}\quad K_i \text{ is composed of letters of the set } (1, 2, \cdots, q+1);\\
&\text{(ii)}\quad K_i \text{ is of odd length i.e. } l(K_i) \text{ is odd};\\
(2.2.2)\quad &\text{(iii)}\quad l(K_i) \geqq R,\\
&\qquad l(K_iK_j) \geqq R - 1,\\
&\qquad \vdots\\
&\qquad l(K_iK_jK_k \cdots) \geqq 1 \quad \{R \text{ factors}\} \qquad (i \neq j \neq k \neq \cdots)
\end{aligned}$$

where l(word) is the *length*, i.e., the number of letters, of the word. Condition (i) is a consequence of the requirement that the generators be in standard form, condition (ii) ensures that the design is even, and condition (iii) is necessary if the resolution is to equal $R + 1$.

Our procedure will involve the construction of the complete set of distinct designs whose generators are written in the form (2.2.1), where the set $\{K_i\}$ satisfies the conditions (2.2.2). This set, which includes the set of even $2_{R+1}^{(k+1)-p}$ designs for specified $q = k - p$ and odd R, is actually the set of distinct even $2_S^{(k+1)-p}$ designs, where the resolution S is even and equals or exceeds $R + 1$. The set of distinct 2_R^{k-p} designs can then be obtained if we erase, in every possible way, one variable from each design in the set of even $2_{R+1}^{(k+1)-p}$ designs.

This approach may, at first, appear to complicate, rather than simplify, the investigation. The contrary is true, however, since for each value of p, the number of distinct even $2_{R+1}^{(k+1)-p}$ designs (for specified $q = k - p$ and odd R) never exceeds, and is generally less than, the number of distinct 2_R^{k-p} designs. Thus, by dealing with designs of the former type, as we build a maximal set of $\{K_i\}$ which satisfy (2.2.2), we shall reduce substantially the number of distinct designs which need to be considered at each stage.

Another advantage to this approach is derived from the fact that the set of *even* $2_{R+1}^{(k+1)-p}$ designs, which we use to obtain the set of 2_R^{k-p} designs, always contains a saturated design of resolution $R + 1$ in 2^{q+1} runs (Mitchell (1966)). This close relationship between saturated designs of odd resolution R and saturated designs of even resolution $R + 1$, will allow us to construct saturated designs for the two resolutions R (in 2^q runs) and $R + 1$ (in 2^{q+1} runs) simultaneously.

2.3. *Stage by stage construction of saturated designs.* For a given odd resolution R, our object will be to construct the set of *all* distinct even $2_{R+1}^{(k+1)-p}$ designs for specified $q = k - p$ and all possible values of p. (We shall take $p = 1, 2, \cdots, p^*$ up to the saturation point $p = p^*$.) From the saturated designs of this set we can then easily obtain the saturated 2_R^{k-p} designs.

We shall construct, stage by stage, the set of even $2^{(k+1)-p}$ designs of resolution $\geqq R + 1$ in the form (2.2.1), where the set $\{K_i\}$ must, at each stage, satisfy the conditions (2.2.2).

At the rth stage, i.e., for $p = r$, we construct a typical new design by adding to the set of generators of one of the distinct designs $\{D((r-1).i)\}$, $i = 1, 2, \cdots, J_{r-1}$, which have been found at the $(r-1)$st stage, a generator of form $W_r = K_r(q + r + 1)$ which is *compatible* with (i.e., whose presence does not violate the resolution conditions (2.2.2)) the $(r-1)$ generators already present. All possible *candidates* K_r are incorporated in a generator $W_r = K_r(q + r + 1)$ and tested for compatibility with $D((r-1).i)$. Hence, for the *parent design* $D((r-1).i)$, there may be several new designs which can be formed, each corresponding to a particular W_r which is compatible with the generators of $D((r-1).i)$. We consider, in turn, each possible parent design $D((r-1).i)$, $i = 1, 2, \cdots, J_{r-1}$, and obtain the set of new rth stage designs which are derived from it. We then select one design from each set of equivalent rth stage designs. The selected designs are distinct, and are denoted $\{D(r.i)\}$, $i = 1, 2, \cdots, J_r$. The designs $\{D(r.i)\}$ are then used as parent designs for the next stage $(p = r + 1)$.

At each stage r, therefore, we obtain a set of distinct even $2^{(q+r+1)-r}$ designs, of resolution $\geqq R + 1$, in 2^{q+1} runs. We now show, by induction, that every possible even $2^{(q+r+1)-r}$ design of resolution $\geqq R + 1$ is equivalent to a design in this set.

Let us assume (for the purposes of induction) that every even $2^{(q+r)-(r-1)}$ design of resolution $\geqq R + 1$ is equivalent to a design in the set $\{D((r-1).i)\}$, $i = 1, 2, \cdots, J_{r-1}$, which has been obtained by the procedures described above. Now suppose we are given an arbitrary $2^{(q+r+1)-r}$ design E which is even and has resolution $\geqq R + 1$. When the generators of E are in standard form (2.2.1), it is obvious that the first $(r-1)$ of them are the generators of some even $2^{(q+r)-(r-1)}$ design E' of resolution $\geqq R + 1$, where E' is, by our assumption, equivalent to a design, D' say, in the set $\{D((r-1).i)\}$. Let the rth generator of E be denoted $W_r = K_r(q + r + 1)$ and let $L_r = P(K_r)$ where P is the permutation of the variables which transforms the defining relation of E' into the defining relation of D'. Since the generator $K_r(q + r + 1)$ is compatible with the set of generators $(W_1, W_2, \cdots, W_{r-1})$ of E', the word $L_r(q + r + 1)$ will be compatible with the set of generators of D'. If we include the word $L_r(q + r + 1)$ with the set of generators of D', therefore, the resulting set will be a set of generators which define some $2^{(q+r+1)-r}$ design D of resolution $\geqq R + 1$. (Note that $D \equiv E$, since $P(D') = E'$ and $P(W_r) = L_r(q + r + 1)$.) Now we need only show that design D is indeed produced in our stage by stage procedure. This is seen to be the case if we replace the generator $L_r(q + r + 1)$ of D with $M_r(q + r + 1)$, where M_r is the product of L_r with the word of D' which contains that subset of the indicator variables $(q + 2, \cdots, q + r)$ appearing in L_r. Since M_r is composed of the variables $(1, 2, \cdots, q + 1)$ and is of odd length, it will arise as a candidate in the stage by stage procedure, with the result that design D is produced. Hence D is equivalent to a design in $\{D(r.i)\}$ and so E (which is equivalent to D) is also equivalent to a design in $\{D(r.i)\}$. We have therefore shown that, if every even $2^{(q+r)-(r-1)}$ design of resolution $\geqq R + 1$ is equivalent to a design in

$\{D((r-1).i)\}$, $i = 1, 2, \cdots, J_{r-1}$, every even $2^{(q+r+1)-r}$ design of resolution $\geqq R+1$ is equivalent to a design in $\{D(r.i)\}$, $i = 1, 2, \cdots, J_r$.

To complete our inductive argument, we need only state the obvious fact that the result holds true when $r = 1$, that is, that every $2^{(q+2)-1}$ design, of resolution $\geqq R+1$, whose (single) generator has even length, is equivalent to a design in the set $\{D(1.i)\}$, where the $\{D(1.i)\}$ are chosen in sequence according to their even word length (greater than or equal to $R+1$, of course).

We can therefore proceed, knowing that, at every stage r, each set $\{D(r.i)\}$, $i = 1, 2, \cdots, J_r$, contains *all* the distinct even $2^{(q+r+1)-r}$ designs of resolution $\geqq R+1$ which exist at that stage. Our procedure will stop only when we reach the stage p^*+1, say, when *no* candidate K_{p^*+1} is compatible with the generators of any design in the set $\{D(p^*.i)\}$. The set $\{D(p^*.i)\}$ will therefore be the set of distinct saturated even designs of resolution $R+1$ in 2^{q+1} runs. The set of all distinct saturated resolution R designs in 2^q runs can then be obtained from the set $\{D(p^*.i)\}$ as indicated in Section 2.2.

2.4. *Blocking designs of resolutions R and R + 1.* In blocking any given 2^{k-p} fractional factorial design, one can associate "blocking generators" $B_1, B_2, \cdots, B_t$, say, with any t *independent* columns in the estimation matrix of the design. (See Box and Hunter (1961a).) The choice of t blocking generators provides 2^t blocks, each containing 2^{k-p-t} runs.

The effects which are confounded with block effects for a given design can be determined very simply as follows. We multiply through the defining relation of the design by the product of any subset of the words $(B_1, B_2, \cdots, B_t)$. If we do this for every possible subset of the $\{B_i\}$, then the resulting expressions list all the effects which are confounded with block effects.

In designs of odd resolution R, the effects which are tentatively assumed to be the important effects are the main effects and the interactions of $(R-1)/2$ or fewer variables. We want to ensure that such effects are not confounded with blocks. The blocking generators $B_1, B_2, \cdots, B_t$, together with the generators $W_1, W_2, \cdots, W_p$ of the 2_R^{k-p} design to be blocked (called the *base design*) must therefore generate a defining relation which is of resolution R' not less than $(R+1)/2$. Such a blocked design will be denoted as a $2_{R;R'}^{k-p-t}$ design.

It can be shown that every $2_{R;R'}^{k-p-t}$ design (where R is odd) can be obtained through the erasure of a variable from some even $2_{R+1;S'}^{(k+1)-p-t}$ design, where $S' = R'$ if R' is even and $S' = R'+1$ if R' is odd, i.e., S' is even and $S' \geqq (R+1)/2$. (The argument is analogous to that suggested by Remark 2 of Section 2.2 to show that every 2_R^{k-p} design can be obtained from an even $2_{R+1}^{(k+1)-p}$ design.)

We can now rely on our stage by stage procedure to construct the set of distinct generating relations associated with even $2_{R+1;S'}^{(k+1)-p-t}$ designs having a given base design. The object of the procedure will be to add as many blocking generators as possible, in order to obtain the maximum number of blocks.

The form of construction of the generators is as follows:

$$(2.4.1)\quad W_1 = K_1(q+2),\quad W_2 = K_2(q+3), \cdots, W_p = K_p(q+p+1),$$
$$B_1, B_2, \cdots, B_t.$$

In other words, we first write the generators $(W_1, W_2, \cdots, W_p)$ of the base design just as they are obtained from the stage by stage construction of Section 2.3, and then complete the set of generators (2.4.1) with a set of words $(B_1, B_2, \cdots, B_t)$ which are independent of each other and of the W's. Without loss of generality, we can insist that no word in the set $\{B_i\}$ contain any of the indicator letters $q + 2, q + 3, \cdots, q + p + 1$. For if one of the B's—B_u, say—originally contains some subset of these indicator variables, we can replace B_u by a new generator—the product of B_u with the particular product of the W's which contains that subset of the indicator variables. In (2.4.1) we can therefore take the words in the set $\{B_i\}$, $i = 1, 2, \cdots, t$, to be composed of letters from the set $(1, 2, \cdots, q + 1)$. Since we are interested in even designs, each B_i is of even length. The generators $(W_1, W_2, \cdots, W_p, B_1, B_2, \cdots, B_t)$ must, of course, generate a defining relation whose shortest word is not less than $(R + 1)/2$, to satisfy the resolution conditions.

In the stage by stage procedure which adds blocking generators to a given base design, we discard, at each stage, any defining relation which is equivalent to a defining relation already found at that stage. The argument which shows that we obtain, in this way, the complete set of distinct defining relations associated with even designs of type $2^{(k+1)-p-t}_{R+1;S'}$, is analogous to that used in Section 2.3 in the stage by stage construction of $2^{(k+1)-p}$ designs.

We should remark here that, although the defining relations of two blocked designs may be equivalent, the designs themselves are equivalent if and only if the transforming permutation also connects the defining relations of the base designs. However, we shall not "lose" any designs by considering only the "complete" defining relations (without regard to the labeling of the generators), since we can always recover a design which has been discarded simply by relabeling the generators of a design which has been retained, in addition to the usual permuting of the variables.

The even $2^{(k+1)-p-t}_{R+1;S'}$ designs (where $S' \geqq (R + 1)/2$), which we use to obtain the $2^{k-p-t}_{R;R'}$ designs (where $R' \geqq (R + 1)/2$), are themselves of interest. Among blocked designs of resolution $(R + 1; S')$, the importance of these even designs is indicated by the fact that if we are given an arbitrary $2^{k-p-t}_{R+1;S'}$ design F, where k, p, and t are specified, then there exists an even $2^{k-p-t}_{R+1;S'}$ design E having the same values of k, p, and t. If one is interested in using a blocked design of resolution $(R + 1; S')$, and one's criteria for selection of a design involve only the number of variables, runs, and blocks, one can thus restrict attention to the even designs. This fact lends additional importance to this class of designs, which was introduced for another purpose, namely that of constructing blocked resolution $(R; R')$ designs.

2.5. *Examining the possible equivalence of two designs.* At the rth stage of the procedure outlined in Section 2.3, we wish to construct a set of designs $\{D(r.i)\}$, $i = 1, 2, \cdots, J_r$, which are *distinct*. In practice, we ensure that all members of this set are distinct by refusing to accept, at the rth stage, any designs which are equivalent to a design already found at this stage. A necessary requirement of this procedure is that we be able to recognize whether or not two specified designs are equivalent.

Suppose we are given two 2^{k-p} designs A and B, and we wish to determine whether or not $A \equiv B$. That is, we wish to investigate whether there is a relabeling of the variables which will transform A into B. If such a relabeling, P say, exists, the vector of variables $(1, 2, \cdots, k)$ in design A is transformed by P into the vector $(P(1), P(2), \cdots, P(k))$ in such a way that the words of the defining relation of A are transformed into the words of the defining relation of B.

We shall adopt the convention that the variables of both designs A and B are labeled $1, 2, \cdots, k$. Hence the relabeling P will simply be a *permutation* of the variables $(1, 2, \cdots, k)$. There may be several such transforming permutations which take design A into design B. The discovery of any one of these will suffice to show that the designs A and B are equivalent.

Suppose design A is such that there are α_t words of length t in the defining relation of A, where $t = 1, 2 \cdots, k$. The vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ will be called the *word length pattern* of A. Similarly, we can define the word length pattern of design B.

We note at once that two designs A and B which have different word length patterns cannot be equivalent, since a transforming permutation P, if it existed, could not change the word lengths. If the word length patterns of A and B are the same, however, further investigation is necessary to determine whether or not $A \equiv B$.

Our approach will involve making a sequence of "conjectures" about the nature of a possible transforming permutation P. Each conjecture will then be "rejected" or "not rejected" on the basis of an examination of the defining relations of A and B.

We shall define a *conjecture at the* rth *stage* to be a tentative assumption that there does exist a transforming permutation P which is such that $P(i_1, i_2, \cdots, i_r) = (j_1, j_2, \cdots, j_r)$, where $(i_1, i_2, \cdots, i_r)$ is a subset of the variables of A and $(j_1, j_2, \cdots, j_r)$ is a subset of the variables of B. The effect of P on the remaining variables of A is left unspecified by the conjecture.

In order to develop a test which will allow us to reject certain conjectures, we first note that any conjecture may be used to map sets of words in the defining relation of A into sets of words in the defining relation of B.

Suppose we are given the conjecture: $P(i_1, i_2, \cdots, i_r) = (j_1, j_2, \cdots, j_r)$. Each word of the defining relation of A contains a particular subset of the variables $(i_1, i_2, \cdots, i_r)$. We may use this fact to divide the words of A into distinct sets; two words will belong to the same set if and only if they both contain precisely the same subset of the variables $(i_1, i_2, \cdots, i_r)$. There are 2^r such possible subsets of $(i_1, \cdots, i_r)$, including the one which contains *none* of these variables. The sets of words of the defining relation of A which are induced in this way will be denoted by $\{A_i\}$, $i = 1, 2, \cdots, 2^r$. Similarly, we can use the variables $(j_1, j_2, \cdots, j_r)$ to divide the words of the defining relation of B into sets $\{B_i\}$,

$i = 1, 2, \cdots, 2^r$, where, for each i, the words in B_i include that subset of the variables which is chosen from $(j_1, j_2, \cdots, j_r)$ in exactly the same way as the subset of the variables associated with A_i is chosen from $(i_1, i_2, \cdots, i_r)$.

We now define the mapping M, denoted by $M(A_i) = B_i$, $i = 1, 2, \cdots, 2^r$, to be the mapping *induced by* the conjecture $P(i_1, i_2, \cdots, i_r) = (j_1, j_2, \cdots, j_r)$. It is important to note that the elements involved in the mapping M are *sets* of words and not the words themselves. If the conjecture is true, i.e., if there does exist a transforming permutation P such that $P(i_1, i_2, \cdots, i_r) = (j_1, j_2, \cdots, j_r)$, then the following properties hold for the words of the sets $\{A_i\}$ and $\{B_i\}$:

(2.5.1)
(i) The number of words in A_i is equal to the number of words in $B_i = M(A_i)$, $i = 1, 2, \cdots, 2^r$.
(ii) The word lengths of the words in A_i are equal to the word lengths of the words in $B_i = M(A_i)$, $i = 1, 2, \cdots, 2^r$.

The conditions (2.5.1) are necessary but not sufficient for the truth of the conjecture. We can, therefore, use them to eliminate many, but not all, false conjectures.

If a conjecture induces a mapping M which satisfies (2.5.1), then we shall say that the conjecture is *consistent*. We now show that if a conjecture involving all of the p indicator variables of A is consistent, then A and B are equivalent. We first note that no two words of the defining relation of A contain the same subset of the indicator variables, since each word is formed by a different product of the generators. A conjecture which involves the indicator variables $(i_1, i_2, \cdots, i_p)$ therefore divides the words of A into sets $\{A_i\}$, each of which contains one and only one word.

(2.5.2)

i_1	i_2	·	·	·	i_p	induced sets	elements of induced sets
−	−	·	·	·	−	A_1	I
+	−	·	·	·	−	A_2	W_1
−	+	·	·	·	−	A_3	W_2
+	+	·	·	·	−	A_4	W_1W_2
⋮	⋮				⋮	⋮	⋮
+	+	·	·	·	+	A_{2^p}	$W_1W_2 \cdots W_p$

(The plus signs in the ith row of this array correspond to the particular subset of the indicator variables which is contained in the word of A_i. For convenience, we have included the identity I as a word of the defining relation, namely the word which includes *none* of the variables $(i_1, i_2, \cdots, i_p)$.)

Now suppose the conjecture $P(i_1, i_2, \cdots, i_p) = (j_1, j_2, \cdots, j_p)$, where $(j_1, j_2, \cdots, j_p)$ are variables of design B, is consistent. This implies that there is one and only one word in each of the subsets of B induced by the conjecture. In particular, for each i, there is one and only one word of the defining relation of B which contains j_i and none of the other j's. Denoting this word by V_i, and letting

$i = 1, 2, \cdots, p$, we can write:

$$\begin{aligned} V_1 &= L_1(j_1), \\ V_2 &= L_2(j_2), \\ \vdots &\quad \vdots \\ V_p &= L_p(j_p), \end{aligned} \tag{2.5.3}$$

where no element of the set $(j_1, j_2, \cdots, j_p)$ appears in any of the words $\{L_i\}$. The set $(V_1, V_2, \cdots, V_p)$ is, clearly, a set of generators of B.

We could, at this point, construct a table for design B, similar to the table (2.5.2) already available for design A, replacing the i's by j's, the A's by B's, and the W's by V's. The consistency of the conjecture then implies that, for each i, the length of the word in set A_i must be equal to the length of the word in set B_i. That is,

$$l(W_1) = l(V_1), \qquad l(W_2) = l(V_2), \\ l(W_1W_2) = l(V_1V_2), \qquad \cdots, \qquad l(W_1W_2 \cdots W_p) = l(V_1V_2 \cdots V_p). \tag{2.5.4}$$

Now we need only to show that, if the generators of A and B are named in such a way that the vector of word lengths, when written as

$$\begin{bmatrix} l(W_1) \\ l(W_2) \\ l(W_1W_2) \\ l(W_3) \\ \vdots \\ l(W_1W_2 \cdots W_p \end{bmatrix} \tag{2.5.5}$$

is the same for both designs, then the designs are equivalent. We can show this as follows.

We first note that the letter which corresponds to each variable of design A appears in a particular subset of the generators $(W_1, W_2, \cdots, W_p)$, including that "subset" which contains no generators. We shall represent this by the following tabulation:

(2.5.6)

W_1	W_2	·	·	W_p	
−	−	·	·	−	a_1
+	−	·	·	−	a_2
−	+	·	·	−	a_3
+	+	·	·	−	a_4
⋮	⋮			⋮	⋮
+	+	·	·	+	a_{2p}

where a_j, $j = 1, 2, 3, \cdots, 2^p$, is the number of letters which appear in that subset of the generators which is denoted by the *minus* signs in row j. (The use of minus signs rather than plus signs facilitates the argument leading up to (2.5.12).) We note that the sum of the elements in the vector $\mathbf{a}$, where $\mathbf{a}' = (a_1, a_2, \cdots, a_{2^p})$, is equal to the total number of variables (k) of the design. At this point, we also draw attention to a fact which we shall use later in this argument, namely that if we label the variables associated with each a_i, $i = 1, 2, 3, \cdots, 2^p$, in any specified manner, we can use (2.5.6) to construct a set of generators (W_1', W_2', $\cdots$, W_p', say) which are equivalent to the generators $W_1, W_2, \cdots, W_p$ of A. For this reason we shall call the vector $\mathbf{a}$ a *generating vector* of design A. If we have, in addition to $\mathbf{a}$, a labeling of the variables which correspond to each element of $\mathbf{a}$, we shall refer to $\mathbf{a}$ as a *labeled generating vector.*

Now let $\mathbf{M}$ be the matrix of $+1$'s and -1's which is derived by associating the value 1 with each sign in the array (2.5.6). In terms of its column vectors, $\mathbf{M}$ can be written

$$\mathbf{M} = [\mathbf{W}_1\, \mathbf{W}_2 \cdots \mathbf{W}_p]. \tag{2.5.7}$$

We shall define a new matrix $\mathbf{X}$ in terms of the *products* of the columns of $\mathbf{M}$, where we define the *product* $\mathbf{W}_i\mathbf{W}_j$ of two column vectors as follows.

DEFINITION. The product of two $(N \times 1)$ vectors $\mathbf{W}_i$ and $\mathbf{W}_j$ is the $(N \times 1)$ vector whose uth element is the product of the uth elements of $\mathbf{W}_i$ and $\mathbf{W}_j$, i.e., $(\mathbf{W}_i\mathbf{W}_j)_u = (\mathbf{W}_i)_u(\mathbf{W}_j)_u$, $u = 1, 2, \cdots, N$. The obvious extension of this definition to products of more than two vectors can be made, since associativity holds, e.g., $(\mathbf{W}_i\mathbf{W}_j\mathbf{W}_k) = (\mathbf{W}_i\mathbf{W}_j)(\mathbf{W}_k) = (\mathbf{W}_i)(\mathbf{W}_j\mathbf{W}_k)$.

We can now define the $(2^p \times 2^p)$ matrix $\mathbf{X}$, which is written in terms of its column vectors as:

$$\mathbf{X} = [\mathbf{I}\ \mathbf{W}_1\ \mathbf{W}_2\ \mathbf{W}_1\mathbf{W}_2\ \mathbf{W}_3 \cdots \mathbf{W}_1\mathbf{W}_2 \cdots \mathbf{W}_p] \tag{2.5.8}$$

where the first column $\mathbf{I}$ is a $(2^p \times 1)$ column of $+1$'s, and the remaining columns are formed by taking (in the order indicated) all possible products of the columns of $\mathbf{M}$. We observe that the columns of $\mathbf{X}$ are orthogonal vectors.

We now use $\mathbf{X}$ to introduce a linear transformation $L(\mathbf{a})$, which is defined as follows:

$$L(\mathbf{a}) = (k\mathbf{j} - \mathbf{X}'\mathbf{a})/2 \tag{2.5.9}$$

where $\mathbf{j}'$ is the (1×2^p) vector $(1, 1, 1, \cdots, 1)$. If we write $L(\mathbf{a})$ in terms of the elements of $\mathbf{a}$, we obtain

$$L(\mathbf{a}) = \left|\begin{matrix} 0 \\ a_1 + a_3 + a_5 + a_7 + \cdots \\ a_1 + a_2 + a_5 + a_6 + \cdots \\ a_2 + a_3 + a_6 + a_7 + \cdots \\ \vdots \end{matrix}\right| = \left|\begin{matrix} 0 \\ l(W_1) \\ l(W_2) \\ l(W_1W_2) \\ l(W_1W_2\cdots W_p) \end{matrix}\right| \tag{2.5.10}$$

i.e., L transforms the generating vector $\mathbf{a}$ into the vector of word lengths of the design.

Suppose that the generators of two designs A and B are named in such a way that the corresponding generating vectors, denoted $\mathbf{a}$ and $\mathbf{b}$ respectively, give rise to the same vector of word lengths, i.e.,

$$L(\mathbf{a}) = L(\mathbf{b}). \tag{2.5.11}$$

This means that $\mathbf{X}'\mathbf{a} = \mathbf{X}'\mathbf{b}$, so

$$\mathbf{a} = \mathbf{b}, \tag{2.5.12}$$

since $\mathbf{X}'$ is a nonsingular matrix. Suppose, for every j, $(j = 1, 2, \cdots, 2^p)$, we label the variables of B which correspond to b_j to be the same as the variables of A which correspond to a_j. Then the set of generators of design B which arise from the labeled generating vector $\mathbf{b}$ will be identical to the set of generators of design A which arise from the labeled generating vector $\mathbf{a}$. Therefore $A \equiv B$.

This is the result to which we have been led by the assumed consistency of the conjecture (which involves the specified indicator variables of A) and the particular choice of generators (2.5.3) of design B. We have therefore shown that, given A and B, if a conjecture involving all of the p indicator variables of A is consistent, then A and B are equivalent designs.

In order to establish the equivalence of two designs A and B, we shall attempt to formulate a consistent conjecture involving the indicator variables of design A. We first make a conjecture $P(i_1) = (j_1)$, involving only one of the indicator variables of A, and then test for consistency by inspecting the word lengths in the sets of A and B induced by the conjecture. If this conjecture is found to be inconsistent, a new conjecture involving i_1 is made and tested. We proceed in this way until we find a consistent conjecture $P(i_1) = (j_{1*})$. We then make a conjecture at the second stage, $P(i_1, i_2) = (j_{1*}, j_2)$, which is chosen to incorporate the consistent first stage conjecture. If this conjecture is inconsistent, we change j_2 and test again.

Continuing in this way: conjecture → test → conjecture → test, and so on, we attempt to find at each stage a consistent conjecture, which we then incorporate into a conjecture at the succeeding stage. If we obtain a consistent conjecture at the pth stage, we can conclude that designs A and B are equivalent.

In the course of this procedure, it is possible that at the rth stage, $r \leqq p$, none of the candidates for j_r give rise to a consistent conjecture. If this happens, we say that the consistent conjecture at the $(r - 1)$st stage has *failed* at the rth stage. We must therefore return to the $(r - 1)$st stage and try to find another consistent conjecture on which to base conjectures at the rth stage. (When we are forced in this way to go back to the $(r - 1)$st stage, the conjectures we select to test are, like all our conjectures, based on the consistent conjecture already found for the previous stage $(r - 2)$.)

We continue until one of two things happens. Either

(i) we find a consistent conjecture at the pth stage, in which case A and B are equivalent; or

(ii) every conjecture at the first stage is either inconsistent itself or fails at a succeeding stage, in which case A and B are not equivalent.

3. An example: $R = 5, q = 7$.

3.1. *Introduction.* The procedures described in Section 2 were programmed for the computer and run on the CDC 3600 located at the University of Wisconsin Computing Center. We shall now illustrate the results of the programmed procedures in the case: $R = 5, q = 7$, to find saturated designs of resolution V in 128 runs and of resolution VI in 256 runs.

3.2. *Even 256-run designs of resolution* $\geqq 6$. We first constructed the complete set of distinct even $2^{(k+1)-p}$ designs of resolution $\geqq 6$, where $k - p = q = 7$. These designs are listed in Table 3.1 together with their word length patterns.

TABLE 3.1

The even 256-run designs of resolution $\geqq 6$

Word Length Pattern

No.	v	6	8	10	12	Ref.	Delete
1.1	9	1	0	0	0	4.1	10, 11, 12
1.2	9	0	1	0	0	4.1	2, 5, 7
2.1	10	3	0	0	0	4.1	11, 12
2.2	10	2	1	0	0	4.1	9, 10
3.1	11	6	1	0	0	4.1	12
4.1	12	12	3	0	0	4.1	—

Generators of design 4.1:

$W_1 = 123459$ $W_2 = 12367(10)$ $W_3 = 12468(11)$ $W_4 = 13578(12)$

The number of each design in Table 3.1 is written in the form $(p \cdot a)$, where p is the number of generators and a is a number which orders those designs having the same value of p.

The column headed "v" ($= k + 1$, in our previous notation) in Table 3.1 gives the number of variables which are accommodated in each design. We see that the single design (4.1) which was found at the last stage accommodates 12 variables, i.e., 12 is the maximum number of variables which can be incorporated into a 256-run resolution VI design. This implies at once that the maximum number of variables which can be accommodated by a 128-run resolution V design is 11 (in agreement with Box and Hunter (1961b)).

If we examine the word length patterns of the designs of Table 3.1, we see that no two designs have identical word length patterns. Although this *distinct pattern property* is not true in general, it does hold for many sets of designs which are of interest, to the extent that it even merits consideration as a basis for testing the equivalence of designs. (A more thorough discussion of this point will be included in a subsequent paper, together with an example of two distinct designs whose word length patterns are identical.)

The five distinct 256-run even designs of resolution $\geqq 6$ which are *not* saturated can all be obtained from the saturated *reference design* 4.1 through the deletion of variables. (The deletion of a specified set of variables involves removing from the defining relation all words in which any of the specified variables appear. Note that this is not the same as the erasure of a variable, discussed above.) Table 3.1 gives, in each case, the appropriate variables to be deleted from design 4.1. These deletions are not, of course, unique and the same designs can be obtained from the saturated design through other deletion patterns.

We should remark that, although we see in this simple example that the complete set of designs of the type constructed can be expressed in terms of deletion of variables from a saturated design, this property is not true in general.

3.3. *Odd* 128-*run designs of resolution* $\geqq 5$. The set of distinct odd 128-run designs of resolution $\geqq 5$ can be obtained directly from the designs of Table 3.1 through the erasure of a variable. In order to ensure completeness, the erasure of each possible variable was performed on each design. During this procedure, designs which were found to be equivalent to any previously obtained design were discarded. The resulting set of designs, which is the complete set of distinct odd 128-run designs of resolution $\geqq 5$, is given in Table 3.2.

Each design in Table 3.2 is identified by means of a number written in the form $(p \cdot a/b)$. The meaning of this notation is that $(p \cdot a)$ is the design of Table 3.1 from which the design $(p \cdot a/b)$ is derived (through the erasure of a variable), and $((p - 1) \cdot b)$ is the even design of Table 3.1 which corresponds to the even words of $(p \cdot a/b)$.

We note that design 4.1/1 is the *unique* saturated resolution V design in 128 runs, i.e., every other 2_V^{11-4} design is equivalent to it. Design 4.1/1 is therefore equivalent to the 2_V^{11-4} designs given, for example, by Brownlee, Kelly, and

TABLE 3.2

The odd 128-run designs of resolution $\geqq 5$

		Word Length Pattern									
No.	k	5	6	7	8	9	10	11	Ref.	Delete	Erase
1.1/0	8	1	0	0	0	0	0	0	4.1	10, 11, 12	9
1.2/0	8	0	0	1	0	0	0	0	4.1	2, 5, 7	12
2.1/1	9	2	1	0	0	0	0	0	4.1	11, 12	10
2.2/1	9	1	1	1	0	0	0	0	4.1	9, 10	12
2.2/2	9	2	0	0	1	0	0	0	4.1	9, 10	8
3.1/1	10	3	3	1	0	0	0	0	4.1	12	11
3.1/2	10	4	2	0	1	0	0	0	4.1	12	6
4.1/1	11	6	6	2	1	0	0	0	4.1	—	12

Loraine (1948), National Bureau of Standards (1957), and Box and Hunter (1961b).

Each of the designs in Table 3.2 can be obtained from the reference design 4.1 through the deletion of a set of variables followed by the erasure of a single variable. Appropriate variables to be deleted and erased are given in Table 3.2 for each design.

3.4. *Blocking the designs constructed.* Two distinct optimum blocking arrangements, each involving 8 blocks, were found for the saturated 2_{VI}^{12-4} design 4.1, using the procedures described in Section 2.4, with $R = 5$. (We use the word "optimum" to refer to those arrangements which provide the maximum possible number of blocks.) The blocking generators for each arrangement are as follows:

(3.4.1) (i) $B_1 = 1238, \quad B_2 = 1478, \quad B_3 = 2456,$

(ii) $B_1 = 1258, \quad B_2 = 1368, \quad B_3 = 2467.$

The resulting blocked design is, in each case, of type $2_{\text{VI;IV}}^{12-4-3}$.

It is of interest to determine whether or not an optimum blocking arrangement for each of the other designs of Table 3.1 can be found by deleting variables from the optimally blocked saturated design. Since the deletion of variables does not affect the number of blocks, each of the designs of Table 3.1 can be obtained in 8 blocks by deleting the appropriate variables from design 4.1. We now show that 8 is indeed the maximum number of blocks which can be accommodated in such a design (which must be even and of type $2_{\text{VI};S'}^{(8+p)-p-t}$, where $S' \geqq 4$). We first note that each block of a $2_{\text{VI};S'}^{(q+1+p)-p-t}$ design is itself a $2_{S'}^{(q+1+p)-(p+t)}$ design and can therefore accommodate no more than 2^{q-t} variables, if $S' \geqq 4$. Hence $q + 1 + p \leqq 2^{q-t}$. When $q = 7$, as in this example, we have $8 + p \leqq 2^{7-t}$, or $8 < 2^{7-t}$, since $p > 0$. This implies that $7 - t > 3$, i.e., $t \leqq 3$. The number of blocks, 2^t, cannot, therefore, exceed 8. We have thus shown that any of the designs of Table 3.1 can be obtained, optimally blocked, by deleting variables from design 4.1, blocked according to arrangement (i) or arrangement (ii) of (3.4.1).

We can also block any of the designs of Table 3.2 optimally (in blocks of 8) as follows. Given the base design $(p \cdot a/b)$, say, we

(i) first write down the generators of the saturated $2_{\text{VI;IV}}^{12-4-3}$ design, using either set of blocking generators given in (3.4.1);

(ii) delete and then erase the appropriate variables (given in Table 3.2) to obtain the desired design $(p \cdot a/b)$.

In step (ii), the variables to be deleted should first be isolated as indicator variables in the generators of the base design. (To isolate the variable i, say, in generator W, replace each generator G which contains i (including the blocking generators but excluding W itself) with the product GW.) The erasure procedure follows, where the appropriate variable must be erased from all the generators, including the blocking generators.

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