ON TWO K-SAMPLE RANK TESTS FOR CENSORED DATA[1]

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1. Summary and introduction. Let X_{ij} ($j = 1, 2, \cdots, n_i$; $i = 1, 2, \cdots, k$) be k independent samples of sizes $n_1, n_2, \cdots, n_k$ respectively from k populations with continuous cumulative distribution functions $F_1, F_2, \cdots, F_k$ respectively. We assume that the F_i's belong to a family $\mathfrak{F}$ of distribution functions indexed by a parameter θ. Let all the $N = \sum_{i=1}^{k} n_i$ observations be put together and ordered to form a single sequence and suppose that only the first r ordered observations are available. That is, let us have a combined (right) censored sample of total size r.

Such a censored sample occurs naturally in many physical situations as for example, in problems of life testing where we are interested in comparing the mean life of several physical systems, or in clinical trials or bio-assay problems where we want to compare the efficacy of several drugs but we cannot afford to wait indefinitely to get information on all the sampling units put on test. For details see Basu [2]. For facility of discussion, we shall use the terminology of life testing.

Any test based on the first r ordered observations (out of a combined sample of size N) will be termed an *r out of N* test. In this paper we propose two k-sample r out of N rank tests which generalize the rank tests proposed by Kruskal [7], Jonckhere [6] and Terpstra [10], [11].

In the first part of the paper we propose the statistic $B_r^{(N)}$ (large values being critical) to test the null hypothesis

$$H_0: F_1(x) = F_2(x) = \cdots = F_k(x) \tag{1.1}$$

(or equivalently, $H_0: \theta_1 = \theta_2 = \cdots = \theta_k = 0$ say, under location alternatives) against the alternative hypothesis

$$H_1: F_i(x) = F(x, \theta_i) \qquad (i = 1, 2, \cdots, k) \tag{1.2}$$

In Section 2 we define the statistic $B_r^{(N)}$ and show its relationship with other statistics. The mean and variance of $B_r^{(N)}$ under the null hypothesis is derived in Section 3. In Section 4 we find the asymptotic distribution of $B_r^{(N)}$ both under the null and the non-null case. The computation of $B_r^{(N)}$ has been illustrated by an example in Section 5.

In the second part of the paper we consider the statistic $V(N, r)$ (to be de-

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fined later) for testing the hypothesis (1.1) against the ordered alternative

$$H_2: F_1(x) < F_2(x) < \cdots < F_k(x) \tag{1.3}$$

for all x, where the $F_i(x)$ are labeled in such a way that (1.3) is the ordered alternative being considered. The second formulation is useful in life testing where the k cdf's might be associated with k different processes and the experimenter wishes to test whether the k processes give rise to units with the same life-time distributions against the alternative that the processes can be ordered in a particular manner, in the sense that the k life-time distributions can be ordered uniformly with respect to x, (that is, according to their reliabilities).

In Section 6 we define the statistic $V(N, r)$ and in Section 7 we have derived the mean and variance of $V(N, r)$ under H_0. Section 8 is devoted to investigate the extreme values of $V(N, r)$ and the asymptotic normality of $V(N, r)$ is proved in Section 9.

Both the statistics $B_r^{(N)}$ and $V)N, r)$ may be considered as k-sample extensions of the V_r statistic proposed by Sobel [9]. The definitions of $B_r^{(N)}$ and $V(N, r)$ are slightly different from what is given earlier in Basu [2].

2. Definition of $B_r^{(N)}$. The statistic $B_r^{(N)}$ is defined below. Let the combined N observations be ordered and define

$$\begin{aligned} Z_\alpha^{(i)} &= 1 \quad \text{if the } \alpha\text{th ordered observation is from the } i\text{th population} \\ &= 0 \quad \text{otherwise} \quad (\alpha = 1, 2, \cdots, N; i = 1, 2, \cdots, k). \end{aligned} \tag{2.1}$$

The statistic $B_r^{(N)}$ is then defined by

$$B_r^{(N)} = G \textstyle\sum_{i=1}^k (1/n_i)(S_i + rn_i/2N)^2 \tag{2.2}$$

which depends only on the first r ordered observations from the combined sample; here

$$S_i = \textstyle\sum_{\alpha=1}^r ((2\alpha - N - r - 1)/2N) Z_\alpha^{(i)} \qquad (i = 1, 2, \cdots, k) \tag{2.3}$$

and

$$G = 12N^2(N - 1)/r[(r^2 - 1) + 3N(N - r)]. \tag{2.4}$$

Define n_{ir} as the cumulative number of failures from the ith population among the first r failures and R_{ir} as the sum of the ranks of these n_{ir} failures. Clearly, then we have both

$$\textstyle\sum_{i=1}^k n_{ir} = r, \qquad \sum_{i=1}^k R_{ir} = r(r + 1)/2 \tag{2.5}$$

and

$$n_i = n_{iN} \qquad (i = 1, 2, \cdots, k).$$

In terms of the quantities

$$n_{ir} = \textstyle\sum_{\alpha=1}^r Z_\alpha^{(i)}, \qquad R_{ir} = \sum_{\alpha=1}^r \alpha Z_\alpha^{(i)} \tag{2.6}$$

we can rewrite (2.3) as

$$NS_i = R_{ir} - n_{ir}(N + r + 1)/2. \tag{2.7}$$

Substituting the value of S_i from (2.7) in (2.2) we obtain

$$B_r^{(N)} = GN^{-2} \sum_{i=1}^{k} n_i (R_{ir}/n_i - ((N + r + 1)/2)n_{ir}/n_i + r/2)^2. \tag{2.8}$$

Putting $r = N$ in (2.8) we get

$$B_N^{(N)} = (12/N(N + 1)) \sum_{i=1}^{k} n_i(\bar{R}_i - (N + 1)/2)^2, \tag{2.9}$$

which is the H-statistic proposed by Kruskal [7] and is related to Terpstra's k-sample statistic [10]; here $\bar{R}_i = R_i/n_i$ and $R_i = R_{iN}$ is the sum of ranks of all the n_i observations from the ith population. Thus $B_r^{(N)}$ may be considered as a generalization of the Kruskal H-statistic. It is also instructive to note that

$$R_{ir}/n_i - ((N + r + 1)/2)n_{ir}/n_i + r/2 = R'_{ir}/n_i - (N + 1)/2 \tag{2.10}$$

where

$$\begin{aligned} R'_{ir} &= R_{ir} + ((N + r + 1)/2)(n_i - n_{ir}) \\ &= R_{ir} + [(n_i - n_{ir}) / (N - r)] (N(N + 1)/2 - r(r + 1)/2) \end{aligned} \tag{2.11}$$

is the *expected* sum of ranks of observations from ith sample (of which n_{ir} have been observed only).

If we sum both sides of (2.7) we obtain

$$\sum_{i=1}^{k} S_i = -r/2. \tag{2.12}$$

Using (2.12), Equation (2.2) can be written in the form

$$B_r^{(N)} = G[\sum_{i=1}^{k} S_i^2/n_i - r^2/4N]. \tag{2.13}$$

For the special case $k = 2$ we obtain from (2.13)

$$B_r^{(N)} = (NG/n_1n_2)(S_1 + n_1r/2N)^2. \tag{2.14}$$

The statistic V_r proposed by Sobel [9] is defined by

$$V_r = \sum_{i=1}^{r} (n_2n_{1i} - n_1n_{2i}) + ((N - r - 1)/2)(n_2n_{1r} - n_1n_{2r}) \tag{2.15}$$

and it has been shown by Basu [3] that

$$\begin{aligned} G_r^{(N)} &= \sum_{\alpha=1}^{r} ((2\alpha - N - r - 1)/2N)Z_\alpha^{(1)} + n_1r/2N \\ &= S_1 + n_1r/2N = -V_r/N^2. \end{aligned} \tag{2.16}$$

Hence by (2.14) and (2.16) the two-tailed tests based on any one of the four statistics $B_r^{(N)}$, S_1, $G_r^{(N)}$ and V_r are all equivalent for $k = 2$.

3. Mean and variance of $B_r^{(N)}$ under the null hypothesis. In this section we shall find the mean and variance of $B_r^{(N)}$ under the null hypothesis.

Let $\{X_{ij}\}$ be a random permutation of N fixed numbers $\{b_{ij}, j = 1, 2, \cdots, n_i;$

$i = 1, 2, \cdots, k; \sum_{i=1}^{k} n_i = N\}$ such that each permutation is equally likely to occur. We also define k additional associated sequences $\{a_\alpha^{(i)}\}(i = 1, 2, \cdots, k)$ with N fixed numbers in each and to each of the $N!$ possible observations (i.e., permutations) on $\{X_{ij}\}$ we assign in a 1:1 manner a fixed permutation in each of the associated $\{a_\alpha^{(i)}\}$ sequences $(i = 1, 2, \cdots, k)$. Let the random sequences associated with $\{a_\alpha^{(i)}\}$ be denoted by $\{\xi_\alpha^{(i)}\}$, so that the one sequence of random variables $\{X_{ij}\}$ is associated with k sequences of random variables $\{\xi_\alpha^{(i)}\}$ $(i = 1, 2, \cdots, k)$. For each i we define k linear statistics

$$Y_i = \sum_{\alpha=1}^{N} c_\alpha \xi_\alpha^{(i)} \qquad (i = 1, 2, \cdots, k) \tag{3.1}$$

where the coefficients c_α's are arbitrary constants. Let $\bar{c} = N^{-1} \sum_{\alpha=1}^{N} c_\alpha$, $\bar{a}^{(i)} = N^{-1} \sum_{\alpha=1}^{N} a_\alpha^{(i)}$ $(i = 1, 2, \cdots, k)$ and let $E(Y_i)$, $\sigma^2(Y_i)$ and $\sigma(Y_i, Y_j)$ denote as usual the mean of Y_i, the variance of Y_i and the covariance between Y_i and Y_j respectively $(i, j = 1, 2, \cdots, k)$. We now prove the following useful theorem.

THEOREM 3.1. *For the above structure with Y_i defined in* (3.1)

$$E(Y_i) = N\bar{c}\bar{a}^{(i)}, \tag{3.2}$$

$$\sigma^2(Y_i) = (N-1)^{-1} \sum_{\alpha=1}^{N} (c_\alpha - \bar{c})^2 \sum_{\beta=1}^{N} (a_\beta^{(i)} - \bar{a}^{(i)})^2, \tag{3.3}$$

and

$$\sigma(Y_i, Y_j) = (N-1)^{-1} \sum_{\alpha=1}^{N} (c_\alpha - \bar{c})^2 \cdot \sum_{\beta=1}^{N} (a_\beta^{(i)} - \bar{a}^{(i)})(a_\beta^{(j)} - \bar{a}^{(j)}) \quad (i, j = 1, 2, \cdots, k). \tag{3.4}$$

PROOF. Result (3.2) is obvious. To prove (3.4) we have for any pair (i, j)

$$\begin{aligned}
\sigma(Y_i, Y_j) &= \sum_{\alpha=1}^{N} c_\alpha^2 \sigma(\xi_\alpha^{(i)}, \xi_\alpha^{(j)}) + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{N} c_\alpha c_\beta \sigma(\xi_\alpha^{(i)}, \xi_\beta^{(j)}) \\
&= \sum_{\alpha=1}^{N} c_\alpha^2 \{(\textstyle\sum_{\beta=1}^{N} (a_\beta^{(i)} a_\beta^{(j)}/N) - \bar{a}^{(i)}\bar{a}^{(j)}\} \\
&\quad + \{(\textstyle\sum_{\alpha=1}^{N} c_\alpha)^2 - \sum_{\alpha=1}^{N} c_\alpha^2\}\{\sum_{\beta\neq\gamma=1}^{N} a_\beta^{(i)} a_\gamma^{(j)}/N(N-1) - \bar{a}^{(i)}\bar{a}^{(j)}\} \\
&= \sum_{\alpha=1}^{N} c_\alpha^2 \{\textstyle\sum_{\beta=1}^{N} a_\beta^{(i)} a_\beta^{(j)}/N - \bar{a}^{(i)}\bar{a}^{(j)}\} \\
&\quad + \{(\textstyle\sum_{\alpha=1}^{N} c_\alpha)^2 - \sum_{\alpha=1}^{N} c_\alpha^2\}\{\bar{a}^{(i)}\bar{a}^{(j)}/(N-1) - \sum_{\beta=1}^{N} a_\beta^{(i)} a_\beta^{(j)}/N(N-1)\} \\
&= (N-1)^{-1} \sum_{\alpha=1}^{N} (c_\alpha - \bar{c})^2 \sum_{\beta=1}^{N} (a_\beta^{(i)} - \bar{a}^{(i)})(a_\beta^{(j)} - \bar{a}^{(j)}).
\end{aligned}$$

Proof of (3.3) follows from that of (3.4) by taking $i = j$.

In tying the above theorem up with our problem it may be noted that the null hypothesis corresponds to the case in which all permutations of $\{X_{ij}\}$ are equally likely. Hence letting $E_0(\cdot)$, $\sigma_0^2(\cdot)$, $\sigma_0(\cdot, \cdot)$ denote the mean, variance and covariance under H_0 we have the following corollaries.

COROLLARY 3.1. *Under the null hypothesis* H_0

$$E_0(S_i) = -rn_i/2N, \tag{3.5}$$

$$\sigma_0^2(S_i) = n_i(N - n_i)/GN, \tag{3.6}$$

and

$$\sigma_0(S_i, S_j) = -n_i n_j/GN \qquad (i, j = 1, 2, \cdots, k; i \neq j). \tag{3.7}$$

PROOF. The results directly follow from Theorem 3.1 by taking

$$\begin{aligned} c_\alpha &= (2\alpha - N - r - 1)/2N, && 1 \leqq \alpha \leqq r \\ &= 0, && r + 1 \leqq \alpha \leqq N \end{aligned} \tag{3.8}$$

and the definition of $Z_\alpha^{(i)}$ as given in (2.1).

COROLLARY 3.2. *Under the null hypothesis*

$$E_0(B_r^{(N)}) = k - 1. \tag{3.9}$$

PROOF.

$$E_0(B_r^{(N)}) = G \sum_{i=1}^k n_i^{-1} E_0(S_i - E_0 S_i)^2 = (k - 1).$$

It is important to notice that the expected value of $B_r^{(N)}$ is independent of r, the point of censoring.

To find the variance of $B_r^{(N)}$ under the null hypothesis we first compute $E(B_r^{(N)})^2$. The details of the computation are given in Appendix I.

4. Asymptotic distribution of $B_r^{(N)}$. In this section we shall find the asymptotic distribution of $B_r^{(N)}$ when N, n_i and r all become infinitely large in such a way that

$$\lim_{N\to\infty} r/N = p > 0, \qquad \lim_{N\to\infty} n_i/N = \lambda_i \qquad (i = 1, 2, \cdots, k) \tag{4.1}$$

where $0 < \lambda_0 \leqq \lambda_i \leqq 1 - \lambda_0 < 1$ $(i = 1, 2, \cdots, k)$ and λ_0 is a constant not greater than $1/k$. Asymptotic normality of $B_r^{(N)}$ will be shown with the help of the k-sample version of the Chernoff-Savage theorem [4] as given by Puri [8]. We accomplish this by showing that $B_r^{(N)}$ is an L-statistic as defined in [8].

We shall give Puri's theorem and show how the theorem applies to our case. Define

$$H(x) = \sum_{i=1}^k \lambda_i F_i(x) \tag{4.2}$$

and

$$H_N(x) = \sum_{i=1}^k \lambda_i F_i(x; n_i) \tag{4.3}$$

where $F_i(x; n_i)$ is the empirical cdf based on $x_{i1}, x_{i2}, \cdots, x_{in_i}$ and H_N then denotes a combined sample cdf with weights λ_i. Now define

$$T_{N,i} = n_i^{-1} \sum_{\alpha=1}^N E_{N,\alpha} Z_{N,\alpha}^{(i)} \tag{4.4}$$

where the $E_{N,\alpha}$'s are given numbers, and $Z_{N,\alpha}^{(i)}$ have been defined before as $Z_{N,\alpha}^{(i)} \equiv Z_\alpha^{(i)}$. (The additional subscript N in $Z_{N,\alpha}^{(i)}$ is needed to study the asymptotic properties as $N \to \infty$.) We can represent $T_{N,i}$ by

$$T_{N,i} = \int_{-\infty}^{\infty} J_N[H_N(x)]\, dF_i(x; n_i) \qquad (i = 1, 2, \cdots, k) \tag{4.5}$$

where $J_N(u)$ is an arbitrary weight function defined on the interval (0, 1]. We shall use $J^{(j)}(H)$ for $j = 0, 1, 2$ to denote, respectively $J(H)$ and the first two derivatives $J'(H)$ and $J''(H)$ of $J(H)$. Puri's theorem can now be stated.

THEOREM 4.1. *If*

(a) $J(H) = \lim_{N\to\infty} J_N(H)$ *exists for* $0 < H < 1$ *and is not a constant,*

(b) $\int_{I_N} [J_N(H_N) - J(H_N)]\, dF_i(x; n_i) = o_p(N^{-\frac{1}{2}})$, *where* $I_N = \{x: 0 < H_N(x) < 1\}$,

(c) $J_N(1) = o(N^{\frac{1}{2}})$

(d) $|J^{(j)}(H)| \leqq M[H(1 - H)]^{-j-\frac{1}{2}+\delta}$

for $j = 0, 1, 2$, *and some* $\delta > 0$ *and almost all* x; *here* M *is a generic constant.*

(e) *the quantity* $\sigma_{N,j}^2$ *defined in* (4.8) *below is positive, then the random vector*

$$N^{\frac{1}{2}}(T_{N,1} - \mu_{N,1}), \cdots, N^{\frac{1}{2}}(T_{N,k} - \mu_{N,k}) \tag{4.6}$$

where

$$\mu_{N,i} = \int_{-\infty}^{\infty} J[H(x)]\, dF_i(x) \tag{4.7}$$

has a limiting $(k - 1)$ *variate normal distribution with mean vector zero and covariance matrix given by*

$$\Sigma = (\sigma_{N,i,j}) \qquad i, j = 1, 2, \cdots, k$$

where

$$\begin{aligned} \sigma_{N,i,i} &= N\sigma_{N,i}^2 \\ &= \sum_{j=1, j\neq i}^{k} 2\lambda_j \int_{-\infty<x<y<\infty} F_j(x)[1 - F_j(y)]J'(H(x)) \\ &\qquad \cdot J'(H(y))\, dF_i(x)\, dF_i(y) \\ &\quad + (2/\lambda_i) \int_{-\infty<x<y<\infty} F_i(x)[1 - F_i(y)]J'(H(x)) \\ &\qquad \cdot J'(H(y))\, d[H(x) - \lambda_i F_i(x)]\, d[H(y) - \lambda_i F_i(y)]. \end{aligned} \tag{4.8}$$

$$\begin{aligned} \sigma_{N,i,j} &= E\{N(T_{N,i} - \mu_{N,i})(T_{N,j} - \mu_{N,j})\} \\ &= \sum_{t=1, t\neq i,j}^{k} [\textstyle\int\int_{-\infty<x<y<\infty} F_t(x)\{1 - F_t(y)\}J'(H(x)) \\ &\qquad \cdot J'(H(y))\, dF_j(x)\, dF_j(y) \\ &\quad + \textstyle\int\int_{-\infty<x<y<\infty} F_t(x)\{1 - F_t(y)\}J'(H(y)) \\ &\qquad \cdot J'(H(x))\, dF_i(y)\, dF_j(x)] \\ &\quad - \textstyle\int\int_{-\infty<x<y<\infty} F_i(x)\{1 - F_i(y)\}J'(H(y)) \\ &\qquad \cdot J'(H(x))\, dF_j(x)\, d\{H(y) - \lambda_i F_i(y)\} \\ &\quad - \textstyle\int\int_{-\infty<x<y<\infty} F_i(x)\{1 - F_i(y)\}J'(H(x)) \\ &\qquad \cdot J'(H(y))\, dF_j(y)\, d\{H(x) - \lambda_i F_i(x)\} \\ &\quad - \textstyle\int\int_{-\infty<x<y<\infty} F_j(x)\{1 - F_j(y)\}J'(H(x)) \\ &\qquad \cdot J'(H(y))\, dF_i(x)\, d\{H(y) - \lambda_j F_j(y)\} \\ &\quad - \textstyle\int\int_{-\infty<x<y<\infty} F_j(x)\{1 - F_j(y)\}J'(H(y)) \\ &\qquad \cdot J'(H(x))\, dF_i(y)\, d\{H(x) - \lambda_j F_j(x)\}. \end{aligned} \tag{4.9}$$

We now define

$$(4.10) \qquad A^2 = \int_0^1 J^2(u)\,du - \left(\int_0^1 J(u)\,du\right)^2,$$

$$(4.11) \qquad W_i = n_i^{\frac{1}{2}}(T_{N,i} - \mu_{N,i}(\theta))/A \qquad (i = 1, 2, \cdots, k)$$

where $\mu_{N,i}(\theta)$ is the mean of $T_{N,i}$ when $F_i(x) = F(x, \theta_i)$, and

$$(4.12) \qquad L = \sum_{i=1}^k W_i^2 = \sum_{i=1}^k \{n_i^{\frac{1}{2}}(T_{N,i} - \mu_{N,i}(\theta))/A\}^2.$$

It follows from the normality result above that the limiting distribution of L is a chi-square (χ_{k-1}^2) with $(k-1)$ degrees of freedom.

To show that the asymptotic distribution of $B_r^{(N)}$ is the χ^2-distribution all we need to show is that the conditions of Theorem 4.1 are met and that $B_r^{(N)}$ is an L-statistic as defined in (4.12). Define the statistic $T_{N,i}$ by

$$(4.13) \qquad \begin{aligned} n_i T_{N,i} &= \sum_{\alpha=1}^r ((2\alpha - N - 1)/2N) Z_\alpha^{(i)} + \sum_{\alpha=r+1}^N (r/2N) \cdot Z_\alpha^{(i)} \\ &= \sum_{\alpha=1}^r ((2\alpha - N - r - 1)/2N) Z_\alpha^{(i)} + (rn_i/2N) \end{aligned}$$

or equivalently

$$T_{N,i} = \int_{-\infty}^{\infty} J_N[H_N(x)]\,dF_i(x; n_i) \qquad (i = 1, 2, \cdots, k).$$

Clearly then, by (3.5)

$$(4.14) \qquad \mu_{N,i}(0) = E_0(S_i) + rn_i/2N = 0.$$

From (4.13) we easily see that

$$(4.15) \qquad \begin{aligned} \lim_{N\to\infty} J_N(u) = J(u) &= u - \tfrac{1}{2}, & 0 \leqq u \leqq p \\ &= p/2, & u > p, \end{aligned}$$

which can be seen to satisfy the first four conditions of Theorem 4.1. Also we note from (4.15) that

$$(4.16) \qquad \int_0^1 J(u)\,du = 0.$$

Now

$$(4.17) \qquad \begin{aligned} B_r^{(N)} &= G \sum_{i=1}^k n_i (S_i/n_i + r/2N)^2 \\ &= \sum_{i=1}^k \{n_i^{\frac{1}{2}}(T_{N,i} - \mu_{N,i}(0))/A_N\}^2 \end{aligned}$$

which is of the form of an L-statistic where $A_N^2 = G^{-1}$ and

$$\lim_{N\to\infty} A_N^2 = A^2 = \int_0^1 J^2(u)\,du = (p/12)(p^2 - 3p + 3).$$

Hence, under the H_0 the statistic $B_r^{(N)}$ is asymptotically distributed as a $\chi_{(k-1)}^2$ with $(k-1)$ degrees of freedom. It follows directly from Puri [8] that under the alternative hypothesis $H_1: F_i(x) = F(x, \theta_i)$ where the θ_i's are not all equal the limiting distribution of $B_r^{(N)}$ will be a noncentral χ_{k-1}^2 with $(k-1)$ degrees of freedom and with noncentrality parameter given by

$$\lambda(H; L) = (12/p(p^2 - 3p + 3)) \sum_{i=1}^{k} \lim_{N\to\infty} [n_i^{\frac{1}{2}} \int_{-\infty}^{F^{-1}(p)} \sum_{\alpha=1}^{k} \lambda_\alpha \cdot \{F(x + (\theta_\alpha - \theta_i)/N) - F(x)\}\, dF(x)]^2.$$

In the special case where $p = 1$, the above reduces to

$$(12/N^2) \sum_{i=1}^{k} n_i \sum_{\alpha=1}^{k} n_\alpha [\lim_{N\to\infty} \int_{-\infty}^{\infty} N\{F(x + (\theta_\alpha - \theta_i)/N^{\frac{1}{2}}) - F(x)\}\, dF(x)]^2$$

which is the corresponding expression for the noncentrality parameter associated with the Kruskal statistic H and was derived by Andrews [1].

5. An example. The following example will illustrate the computation of $B_r^{(N)}$. In a bio-assay problem a certain drug is administered simultaneously to 21 animals belonging to three groups A, B and C until all of them are dead. Table I

TABLE I

Group	Lethal dose
A	84, 47, 34, 41, 60, 45
B	40, 108, 117, 95, 86, 59, 98, 67, 61, 92
C	90, 93, 100, 46, 93

gives the lethal dose (in some suitable unit) of each animal at death. The data in Table I can be *naturally* ordered as:

34,	40,	41,	45,	46,	47,	59,	60,	61,	67,	84,	86,	90,	91,	92,	93,	95,	98,	100,	108,	117
A	B	A	A	C	A	B	A	B	B	A	B	C	C	B	C	B	B	C	B	B

Denoting the data from group A as first population, from group B as second population and data from group C as third population we have

$$n_1 = 6 \qquad R_1 = 33 \qquad N = 21$$
$$n_2 = 10 \qquad R_2 = 131$$
$$n_3 = 5 \qquad R_3 = 67.$$

Using (2.9), the Kruskal statistic $H(\equiv B_N^{(N)})$ can be computed as 6.61. By comparing the above value with that of χ^2 with 2 degrees of freedom we see that the H_0 of equality of the three populations will be rejected.

Now, let $r = 14$. We can then compute from above

$$n_{1r} = 6 \qquad R_{1r} = 33 \qquad NS_1 = -75$$
$$n_{2r} = 5 \qquad R_{2r} = 40 \qquad NS_2 = -50$$
$$n_{3r} = 3 \qquad R_{3r} = 32 \qquad NS_3 = -22$$

so that using (2.13) we find that $B_{14}^{(21)} = 6.88$ which also leads to the rejection of null hypothesis. Similarly computation with $r = 9$ gives $B_9^{(21)} = 6.11$ which also leads to rejection of H_0, as it should. It is interesting to note that the usual one-way analysis of variance test gives the value of the F-ratio with (2, 18) degrees of freedom as $F_{2,18} = 4.22$ which also leads to the rejection of H_0 at the 5% level of significance.

It must be pointed out that in the above example sample sizes were quite small and the conclusion based on the limiting distribution is only approximate. The exact distribution of $B_r^{(N)}$ and its rate of convergence to the limiting distribution will be investigated later.

Thus the $B_r^{(N)}$ statistic seems to be suitable for testing the equality of k populations against location or scalar alternatives. In the next few sections of the paper we shall study the $V(N, r)$ statistic which is suitable for ordered alternatives.

6. Definition of the statistic $V(N, r)$. In this section we define the $V(N, r)$ statistic which is useful for testing the null hypothesis (1.1) against the alternative hypothesis H_2 given by (1.3). As before, let $n_{i\alpha}$ be the cumulative number of observations from the ith population among the first α-ordered observations so that

$$\sum_{i=1}^{k} n_{i\alpha} = \alpha \qquad (\alpha = 1, 2, \cdots, r) \tag{6.1}$$

Define V_{ij} by

$$V_{ij} = \sum_{\alpha=1}^{r} (n_j n_{i\alpha} - n_i n_{j\alpha}) + \tfrac{1}{2}(N - r - 1)(n_j n_{ir} - n_i n_{jr}) \qquad (i, j = 1, 2, \cdots, k; i < j). \tag{6.2}$$

Then the statistic $V(N, r)$ is defined by

$$V(N, r) = \sum_{i<j} V_{ij} \tag{6.3}$$

where the summation in (2.3) is over all pairs (i, j) with $i < j$. For $r = N$ $V(N, r)$ reduces to the Jonckhere statistic [6] which is related to a statistic of Terpstra [11] and for $k = 2$ it is clear that $V(N, r)$ is the same as the V_r statistic proposed by Sobel [9] which is shown by Basu [3] to be asymptotically equivalent to modified Wilcoxon-Mann-Whitney statistic as modified by Gastwirth [5] for censored data.

7. Mean and variance of $V(N, r)$ under H_0. To find the mean and variance of $V(N, r)$ under H_0 we define for each pair (i, j) with $i < j$ a sequence of random variables $Z_{ij\alpha}$ ($\alpha = 1, 2, \cdots, r$) by

$$\begin{aligned} Z_{ij\alpha} &= +n_j \text{ if the } \alpha\text{th ordered observation is from the } i\text{th population} \\ &= -n_i \text{ if the } \alpha\text{th ordered observation is from the } j\text{th population} \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{7.1}$$

We have then the following:

LEMMA 3.1. *For each given pair (i, j) the statistic V_{ij} can be expressed as a linear combination of the $Z_{ij\alpha}$.*

PROOF. Letting $u_\beta = n_j n_{i\beta} - n_i n_{j\beta} = \sum_{\alpha=1}^{\beta} Z_{ij\alpha}$ from (6.2)

$$V_{ij} = \sum_{\beta=1}^{r} u_\beta + \tfrac{1}{2}(N - r - 1)u_r = \sum_{\alpha=1}^{r} \tfrac{1}{2}(N + r + 1 - 2\alpha)Z_{ij\alpha}. \tag{7.2}$$

Using (7.1) we obtain by routine computation the following results which we state as

THEOREM 7.1. *Under H_0*

$$E_0(Z_{ij\alpha}) = 0 \tag{7.3}$$

$$E_0(Z_{ij\alpha}^2) = n_i n_j(n_i + n_j)/N \tag{7.4}$$

$$E_0(Z_{ij\alpha}Z_{ij\alpha'}) = -n_i n_j(n_i + n_j)/\{N(N - 1)\} \qquad (\alpha \neq \alpha') \tag{7.5}$$

$$E_0(Z_{ij\alpha}Z_{i'j'\alpha'}) = 0 \qquad (i \neq j', i' \neq j, i \neq i', j \neq j') \tag{7.6}$$

$$E_0(Z_{ij\alpha}Z_{ij'\alpha}) = n_i n_j n_{j'}/N \qquad (j \neq j') \tag{7.7}$$

$$E_0(Z_{ij\alpha}Z_{ij'\alpha'}) = -n_i n_j n_{j'}/\{N(N - 1)\} \qquad (j \neq j', \alpha \neq \alpha') \tag{7.8}$$

$$E_0(Z_{ij\alpha}Z_{jt\alpha}) = -n_i n_j n_t/N \tag{7.9}$$

$$E_0(Z_{ij\alpha}Z_{jt\alpha'}) = n_i n_j n_t/\{N(N - 1)\} \qquad (\alpha \neq \alpha') \tag{7.10}$$

$$E_0(Z_{ij\alpha}Z_{i'j\alpha}) = n_i n_{i'} n_j/N \qquad (i \neq i') \tag{7.11}$$

$$E_0(Z_{ij\alpha}Z_{i'j\alpha'}) = -n_i n_{i'} n_j/\{N(N - 1)\} \qquad (i \neq i'). \tag{7.12}$$

Using the results of Theorem 3.1 we prove:

THEOREM 7.2. *Under H_0 the mean and variance of V_{ij} are given by*

$$E_0(V_{ij}) = 0, \tag{7.13}$$

and

$$\sigma_0^2(V_{ij}) = n_i n_j(n_i + n_j)N^2/G, \tag{7.14}$$

where G is given by (2.4).

PROOF. Using (7.2), (7.3), (7.4) and (7.5) we obtain

$$E_0(V_{ij}) = \sum_{\alpha=1}^{r} \tfrac{1}{2}(N + r + 1 - 2\alpha)E_0(Z_{ij\alpha}) = 0$$

and

$$\begin{aligned}\sigma_0^2(V_{ij}) &= \sum_{\alpha=1}^{r} [\tfrac{1}{2}(N + r + 1 - 2\alpha)]^2\sigma_0^2(Z_{ij\alpha}) \\ &\quad + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{r} \tfrac{1}{2}(N + r + 1 - 2\alpha)\tfrac{1}{2}(N + r + 1 - 2\beta) \\ &= [n_i n_j(n_i + n_j)/4N][\sum_{\alpha=1}^{r} (N + r + 1 - 2\alpha)^2 \\ &\quad - \sum_{\alpha,\beta=1,\alpha\neq\beta}^{r} (N + r + 1 - 2\alpha)(N + r + 1 - 2\beta)/(N - 1)] \\ &= n_i n_j(n_i + n_j)N^2/G.\end{aligned}$$

From Theorem 7.2 we have the obvious:

COROLLARY 7.1. $E_0(V(N, r)) = 0$.

To find the variance of $V(N, r)$ under H_0 we need a few more results. Let

$$n_{(12\cdots j)} = \sum_{i=1}^{j} n_i \quad \text{and} \quad n_{(12\cdots j)\alpha} = \sum_{i=1}^{j} n_{i\alpha}.$$

Define $V_{(12\cdots j)t}$ for $j < t$ by:

$$V_{(12\cdots j)t} = \sum_{\alpha=1}^{r} (n_t n_{(12\cdots j)\alpha} - n_{(12\cdots j)} n_{t\alpha}) + \tfrac{1}{2}(N - r - 1)(n_t n_{(12\cdots j)r} - n_{(12\cdots j)} n_{tr}). \tag{7.15}$$

That is, $V_{(12\cdots j)t}$ is the usual V_{ij} statistic as defined in (6.2) where the first j groups have been pooled together to form one sample. From the definition (7.15) it is clear that

$$V_{(12\cdots j)t} = \sum_{i=1}^{j} V_{it}. \tag{7.16}$$

We can now prove:

LEMMA 7.2.

$$V(N, r) = \sum_{j=2}^{k} V_{(12\cdots j-1)j}, \tag{7.17}$$

$$E_0\{V_{(12\cdots j-1)j}\} = 0, \tag{7.18}$$

$$\sigma_0^2\{V_{(12\cdots j-1)j}\} = n_{(12\cdots j-1)} n_j n_{(12\cdots j)} N^2/G, \tag{7.19}$$

$$\sigma_0\{V_{(12\cdots j-1)j},\quad V_{(12\cdots j'-1)j'}\} = 0 \qquad (j \neq j'). \tag{7.20}$$

PROOF. Using (6.3) we readily prove (7.17), since

$$V_{(N,r)} = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} V_{ij} = \sum_{j=2}^{k} \left(\sum_{i=1}^{j-1} V_{ij}\right) = \sum_{j=2}^{k} V_{(12\cdots j-1)j}.$$

(7.18) obviously follows from definition. Now

$$\sigma_0^2\{V_{(12\cdots j-1)j}\} = \sum_{i=1}^{j-1} \sigma_0^2(V_{ij}) + \sum_{i\neq i'=1}^{j-1} \sigma_0(V_{ij}, V_{i'j}). \tag{7.21}$$

From (7.11) and (7.12) we obtain

$$\begin{aligned} \sigma_0(V_{ij}, V_{i'j}) &= n_i n_{i'} n_j N^{-1} \Big[\sum_{\alpha=1}^{r} [\tfrac{1}{2}(N + r + 1 - 2\alpha)]^2 \\ &\quad - \sum_{\alpha\neq\beta=1}^{r} \tfrac{1}{2}(N + r + 1 - 2\alpha)\tfrac{1}{2}(N + r + 1 - 2\beta)/ \\ &\quad (N - 1)\Big] \\ &= n_i n_{i'} n_j N^2/G. \end{aligned} \tag{7.22}$$

Using (7.14) and (7.22) in (7.21) we obtain

$$\begin{aligned} \sigma_0^2\{V_{(12\cdots j-1)j}\} &= (n_j N^2/G)\Big[\sum_{i=1}^{j-1} n_i(n_i + n_j) + \sum_{i\neq i'=1}^{j-1} n_i n_{i'}\Big] \\ &= n_{(12\cdots j-1)} n_j n_{(12\cdots j)} N^2/G, \end{aligned}$$

which proves (7.19).

Finally to prove (7.20) we assume without any loss of generality that $j < j'$. From (7.2) and (7.6) we obtain

$$E_0(V_{ij} V_{i'j'}) = 0 \quad \text{for} \quad i \neq i',\ j \neq j',\ j \neq i' \quad \text{and} \quad i \neq j'. \tag{7.23}$$

Hence, we have, using (7.2), (7.7), (7.8), (7.9) and (7.10)

$$\begin{aligned}
&\sigma_0\{V_{(12\cdots j-1)j}V_{(12\cdots j'-1)j'}\} \\
&\quad = E_0 \textstyle\sum_{i=1}^{j-1} (V_{ij}V_{ij'} + V_{ij}V_{jj'}) \\
&\quad = \textstyle\sum_{i=1}^{j-1} [n_i n_j n_{j'} N^{-1}\{\sum_{\alpha=1}^{r} [\frac{1}{2}(N + r + 1 - 2\alpha)]^2 \\
&\qquad - \textstyle\sum_{\alpha\neq\alpha'=1}^{r} \frac{1}{2}(N + r + 1 - 2\alpha)\frac{1}{2}(N + r + 1 - 2\alpha')\} + n_i n_j n_{j'} N^{-1} \\
&\qquad\quad \cdot\{-\textstyle\sum_{\alpha=1}^{r} [\frac{1}{2}(N + r + 1 - 2\alpha)]^2 \\
&\qquad + \textstyle\sum_{\alpha\neq\alpha'=1}^{r} \frac{1}{2}(N + r + 1 - 2\alpha)\frac{1}{2}(N + r + 1 - 2\alpha')\}] \\
&\quad = 0.
\end{aligned}$$

The variance of $V(N, r)$ under H_0 now follows directly from Lemma 7.2 and is given as

THEOREM 7.3. *Under H_0 the variance of $V(N, r)$ is given by*

$$\sigma_0^2(V(N, r)) = (N^2/G) \textstyle\sum_{j=2}^{k} \{n_j n_{(12\cdots j-1)} n_{(12\cdots j)}\}. \tag{7.24}$$

8. Extreme values of $V(N, r)$. It is of interest to know the extreme values of $V(N, r)$. From the definition of $V_{(12\cdots j)t}$ it can be seen that the values of $V_{(12\cdots j)t}$ for $t \geqq j + 1$ remain unchanged for any permutation of the observations corresponding to different cdf's F_1 through F_j. Thus to find the maximum value of $V(N, r)$ we shall find the permutation of the observations belonging to the first j samples (keeping all observations from other samples fixed) for which $V_{(12\cdots j-1)j}$ is maximum. Afterwards, we shall permute between the $(j + 1)$st and the pooled set consisting of the first j samples so that $V_{(12\cdots j)j+1}$ is a maximum. Continuing this process we find the maximum values for each $V_{(12\cdots j-1)j}$ $(j = 2, 3, \cdots, k)$. From (7.17) the maximum of $V(N, r)$ follows. Similarly, we can also find the minimum of $V(N, r)$.

From (6.2) and (7.2) it is clear that V_{12} is maximum if the observations from the first sample precede the observations from the second sample (among the first r observations). Similarly, V_{12} and $V_{(12)3}$ are simultaneously maximum if the observations from the first sample precede those from the second sample and observations from the second sample precede the observations from the third sample (while observations from all other samples are kept fixed). Proceeding in this way we see that all the $V_{(12\cdots j-1)j}$ are simultaneously maximum (and therefore $V(N, r)$ is maximum) when the first r_1 observations are from the first sample, the next r_2 observations are from the second sample and so forth where

$$\begin{aligned}
r_i &= \min(n_i, r - n_1 - n_2 - \cdots - n_{i-1}) \quad \text{if} \quad r \geqq n_1 \\
&= 0 \text{ otherwise} \qquad\qquad (i = 1, 2, \cdots, k).
\end{aligned} \tag{8.1}$$

Below we compute for $k = 3$ the maximum values of $V(N, r)$ for different possible values of r. For $r \leqq n_1$, $V_{12} = \frac{1}{2}(Nrn_2)$, $V_{(12)3} = \frac{1}{2}(Nrn_3)$ and

$$V(N, r) = Nr(n_2 + n_3)/2. \tag{8.2}$$

For $n_1 < r \leqq n_1 + n_2$, $V_{12} = \frac{1}{2}n[n_2(N + r - n_1) - (r - n_1)(N - n_1)]$

$$V_{(12)3} = \tfrac{1}{2}(Nrn_3) \quad \text{and} \tag{8.3}$$

$$V(N, r) = \tfrac{1}{2}[n_1n_2(N + r - n_1) - n_1(r - n_1)(N - n_1) + N_rn_3] .$$

Finally, for $n_1 + n_2 < r \leqq n_1 + n_2 + n_3$

$$V_{12} = \tfrac{1}{2}(n_1n_2(n_1 + n_2)), \qquad V_{(12)3} = \tfrac{1}{2}(n_1 + n_2)[n_3(N + r - n_1 - n_2) - (r - n_1 - n_2)(N - n_1 - n_2)]$$

and

$$V(N, r) = \tfrac{1}{2}(n_1 + n_2)[n_1n_2 + n_3(N + r - n_1 - n_2) - (r - n_1 - n_2)(N - n_1 - n_2)]. \tag{8.4}$$

Similarly we can compute the minimum value of $V(N, r)$ where the order in which the observations occur is exactly the reverse of the case for which $V(N, r)$ is maximum. That is, observations from the kth population will precede all other observations, observations of the $(k - 1)$st population will precede all but the observations from the kth population and so forth. In the special case $k = 3$, we compute the minimum values of $V(N, r)$ below. For $r \leqq n_3$

$$V(N, r) = -(n_1 + n_2)rN/2. \tag{8.5}$$

For $n_3 < r \leqq n_2 + n_3$

$$V(N, r) = -\tfrac{1}{2}[n_2n_3(N + r - n_3) - n_3(r - n_3)(N - n_3) + Nrn_1] \tag{8.6}$$

and for $n_2 + n_3 < r \leqq n_1 + n_2 + n_3$

$$V(N, r) = -\tfrac{1}{2}(n_2 + n_3)[n_2n_3 + n_1(N + r - n_2 - n_3) - (r - n_2 - n_3)(N - n_2 - n_3)]. \tag{8.7}$$

It can be easily seen from the above equations that in each case if

$$(n_1, n_2, n_3) = (n_3, n_2, n_1) \quad \text{then} \tag{8.8}$$

$$\text{Maximum value of } V(N, r) + \text{Minimum value of } V(N, r) = 0.$$

Also if $n_1 = n_2 = n_3 = n$ the minimum difference between two successive values of $V(N, r)$ will be $2n$ since V_{12} changes by steps of $2n$ when all other $V_{(12\cdots j-1)j}$ are kept fixed.

9. Asymptotic normality of $V(N, r)$. In this section we shall prove the asymptotic normality of $V(N, r)$ as $N, n_i \to \infty$ with $n_i/N \to \lambda_i > 0$. From (7.2) and (4.13) we have

$$-V_{ij}/N = n_i(T_{N,i} - rn/2N) \tag{9.1}$$

and in Section 4 the $T_{N,i}$'s are shown to have limiting $(k - 1)$ dimensional normal distribution. It follows that the V_{ij}'s are asymptotically normally distributed under the null as well as the non-null hypothesis.

Since $V(N, r)$ is a linear combination of the V_{ij}'s the asymptotic normality of $V(N, r)$ both under the null and the alternative hypothesis follows.

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APPENDIX I

Variance of $B_r^{(N)}$ under H_0

In this appendix we compute the variance of $B_r^{(N)}$ under the null hypothesis. To this end we first compute the expected value of $(B_r^{(N)})^2$ under H_0. From (2.13) we have

$$(B_r^{(N)})^2 = G^2\{\textstyle\sum_{i=1}^k (S_i^4/n_i^2) + \sum_{i\neq j=1}^k (S_i^2 S_j^2/n_i n_j) - (r^2/2N)\sum_{i=1}^k (S_i^2/n_i) + (r^4/16N^2)\}. \tag{A.1}$$

For further computation we shall make use of the following elementary results where all the summations are from $\alpha = 1$ to $\alpha = r$ and within any term of a summation involving several indices (α, β, γ and δ) no two of which are equal.

(a)
$$\begin{aligned}(\textstyle\sum \alpha)^2 &= \textstyle\sum \alpha^2 + \sum \alpha\beta,\\ (\textstyle\sum \alpha)^3 &= \textstyle\sum \alpha^3 + 3\sum \alpha^2\beta + \sum \alpha\beta\gamma,\\ (\textstyle\sum \alpha)^4 &= \textstyle\sum\alpha^4 + 4\sum\alpha^3\beta + 3\sum\alpha^2\beta^2 + 6\sum\alpha^2\beta\gamma\\ &\quad + \textstyle\sum \alpha\beta\gamma\delta;\end{aligned}$$

(b)
$$\begin{aligned}\textstyle\sum \alpha^2\beta\gamma &= (\textstyle\sum \alpha^2)(\sum \alpha)^2 - 2\sum \alpha^3)(\sum \alpha) - (\sum \alpha^2)^2\\ &\quad + 2\textstyle\sum \alpha^4;\end{aligned}$$

(c)
$$\begin{aligned}\textstyle\sum \alpha\beta\gamma\delta &= (\textstyle\sum \alpha)^4 + 8(\sum \alpha^3)(\sum \alpha) + 3(\sum \alpha^2)^2\\ &\quad -6(\textstyle\sum \alpha^2)(\sum \alpha)^2 - 6\sum \alpha^4,\end{aligned}$$

(d)
$$\begin{aligned}\textstyle\sum (2\alpha - N - r - 1) &= -Nr,\\ \textstyle\sum (2\alpha - N - r - 1)^2 &= r(r^2 + 3N^2 - 1)/3 = c \quad \text{(say)},\\ \textstyle\sum (2\alpha - N - r - 1)^3 &= \tfrac{1}{2}(r(r+1)(2r^2 - 2Nr + r - N - 1))\\ &\quad + \tfrac{1}{2}(r(N + r + 1)^2(r + 1 - 2N))\\ &= d \quad \text{(say)},\\ \textstyle\sum (2\alpha - N - r - 1)^4 &= 8r(r+1)(2r+1)(3r^2 + 3r - 1)/15\\ &\quad + r(N + r + 1)^4 - 4r(r+1)(N + r + 1)\\ &\quad \cdot(r^2 + r + N^2 + N) = a \quad \text{(say)}.\end{aligned}$$

Using above results we get

$$
\begin{aligned}
(2N)^4E_0(S_i^4) &= E_0[\textstyle\sum_{\alpha=1}^{r}(2\alpha - N - r - 1)Z_\alpha^{(i)}]^4 \\
&= an_i/N - [4(Nrd + a) - 3(c^2 - a)]n_i^{(2)}/N^{(2)} \\
&\quad + 6(N^2r^2c + 2Nrd - c^2 + 2a)n_i^{(3)}/N^{(3)} \\
&\quad + (N^4r^4 - 8Nrd + 3c^2 - 6N^2r^2c - 6a)n_i^{(4)}/N^{(4)},
\end{aligned} \tag{A.2}
$$

where $x^{(t)} = x(x-1)\cdots(x-t+1)$. Hence

$$
\begin{aligned}
&(2N)^4\textstyle\sum_{i=1}^{k}(E_0(S_i^4)/n_i^2) \\
&\quad = aN^{-1}\textstyle\sum_1^k n_i^{-1} + ((3c^2 - 4Nrd - 7a)/N^{(2)})(k - \sum_1^k n_i^{-1}) \\
&\qquad + [6(N^2r^2c + 2Nrd - c^2 + 2a)/N^{(3)}](N - 3k + 2\textstyle\sum n^{-1}) \\
&\qquad + [(N^2r^4 - 8Nrd + 3c^2 - 6N^2r^2c - 6a)/N^{(4)}] \\
&\qquad\quad \cdot(\textstyle\sum_1^k n_i^2 - 6N + 11k - 6\sum n_i^{-1}).
\end{aligned} \tag{A.3}
$$

Similarly, denoting $2\alpha - N - r - 1$ by g_α, we have

$$
\begin{aligned}
(2N)^4E_0(S_i^2S_j^2) &= \textstyle\sum g_\alpha g_{\alpha'} g_\beta g_{\beta'} E_0\{Z_\alpha^{(i)}Z_{\alpha'}^{(i)}Z_\beta^{(j)}Z_{\beta'}^{(j)}\} \\
&\quad + [\textstyle\sum g_\alpha g_{\alpha'} g_\beta^2 E_0\{Z_\alpha^{(i)}Z_{\alpha'}^{(i)}(Z_\beta^{(j)})^2\} \\
&\quad + \textstyle\sum g_\alpha^2 g_\beta g_{\beta'} E_0\{(Z_\alpha^{(i)})^2 Z_\beta^{(j)}Z_{\beta'}^{(j)}\}] \\
&\quad + \textstyle\sum g_\alpha^2 g_\beta^2 E_0\{(Z_\alpha^{(i)})^2(Z_\beta^{(j)})^2\} \\
&= [(N^4r^4 - 8Nrd + 3c^2 - 6N^2r^2c - 6a)n_i^{(2)}n_j^{(2)}]/N^{(4)} \\
&\quad + (N^2r^2c + 2Nrd - c^2 + 2a) \\
&\qquad \cdot\{n_in_j(n_i + n_j - 2)\}/N^{(3)} + (c^2 - a)n_in_j/N^{(2)}.
\end{aligned} \tag{A.4}
$$

Hence

$$
\begin{aligned}
&(2N)^4\textstyle\sum_{i\neq j=1}^{k}E_0(S_i^2S_j^2)/n_in_j \\
&\quad = ((c^2 - a)/N^{(2)})k(k-1) \\
&\qquad + 2(N^2r^2c + 2Nrd - c^2 + 2a)(N - k)(k - 1)/N^{(3)} \\
&\qquad + [(N^4r^4 - 8Nrd + 3c^2 - 6N^2r^2c - 6a) \\
&\qquad\quad \cdot\{N^2 - \textstyle\sum_1^k n_i^2 - 2N(k-1) + k(k-1)\}]/N^{(4)}.
\end{aligned} \tag{A.5}
$$

Finally,

$$
(2N)^2\textstyle\sum_{i=1}^{k}E_0(S_i^2)/n_i = ck/N + (N^2r^2 - c)(N - k)/N^{(2)}. \tag{A.6}
$$

From (A.1), (A.3), (A.5) and (A.6) we obtain

$$
\begin{aligned}
&(2N)^4E_0(B_r^{(N)})^2/G^2 \\
&\quad = aN^{-1}\textstyle\sum_1^k 1/n_i + (3c^2 - 4Nrd - 7a)(k - \sum 1/n_i)/N^{(2)} \\
&\quad + (c^2 - a)k(k-1)/N^{(2)} - 2Nr^2\{ck/N + (N^2r^2 - c)/(N - k)/N^{(2)}\}
\end{aligned}
$$

$$+ N^2r^4 + 2(N^2r^2c + 2Nr\,d - c^2 + 2a)(2N - 8k + 6\textstyle\sum_1^k 1/n_i + Nk - k^2)/N^{(3)} + (N^4r^4 - 8Nr\,d + 3c^2 - 6N^2r^2c - 6a) \cdot(10k - 4N - 6\textstyle\sum 1/n_i - 2Nk + N^2 + k^2)/N^{(4)}. \tag{A.7}$$

From (A.7) the variance of $B_r^{(N)}$ follows.

As a check on our computation let us compute the variance of $B_r^{(N)}$ when $r = N$, that is when all the observations are available. In this case

$$c = N(4N^2 - 1)/3, \qquad d = -N^2(2N^2 - 1)$$

$$a = N\{48N^4 - 40N^2 + 7\}/15 \quad \text{and} \quad G = 12N/(N + 1).$$

Substituting the above values in (A.7), and after some simplification we obtain

$$E_0(B_N^{(N)})^2 = (5N + 6)(N - 1)k^2/5N(N + 1) + 12k/5N - (5N^2 + 12N + 7/5(N + 1)^2) - 6\textstyle\sum_1^k n_i^{-1}/5. \tag{A.8}$$

Hence, from (3.9) and (A.8) we get

$$\sigma_0^2(B_N^{(N)}) = 2(k - 1) - (2/5)N(N + 1) \cdot[3k^2 - 6k + N(2k^2 - 6k + 1)] - 6\textstyle\sum_1^k n_i^{-1}/5, \tag{A.9}$$

which is the result obtained by Kruskal [7].

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