INADMISSIBILITY OF THE USUAL CONFIDENCE SETS FOR THE MEAN OF A MULTIVARIATE NORMAL POPULATION

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1. Introduction. We consider an m-dimensional random vector $\mathbf{X}$, distributed normally with unknown mean vector $\boldsymbol{\theta}$, and known covariance matrix equal to the identity matrix. The usual confidence sets for estimating $\boldsymbol{\theta}$, are spheres of fixed volume centered at the sample mean. We consider the admissibility of these confidence sets in the general class of confidence procedures. Let x denote the set of observed values of $\mathbf{X}$. Then a confidence procedure C is defined as a procedure which assigns to each possible point x, a Lebesgue measurable subset $C(x, \cdot)$ of the parameter space within which $\boldsymbol{\theta}$ is estimated to lie. Let $vC(x, \cdot)$ denote the Lebesgue measure of the set $C(\mathrm{x}, \cdot)$. Then the confidence procedure C_0 consisting of spheres of fixed volume, centered at the observed sample mean, has the property, that amongst the class of confidence procedures C, with a given lower confidence level $(1 - \alpha)$, C_0 minimizes the maximum expected measure of the confidence set, viz.

$$\sup_\theta E_\theta vC(x, \cdot).$$

Stein (1962) raises the question whether the usual procedure is unique in having this property and conjectures that it is probably unique for $m = 1$, and probably not unique for $m \geqq 3$, the case $m = 2$ being doubtful. In this paper we investigate the case $m \geqq 3$, and for this case Stein's conjecture is proved to be true, by showing that the usual procedure is not unique. It is in fact shown that the usual procedure is inadmissible as there exists a uniformly superior procedure.

2. Notation. $\mathbf{X}$ denotes an m-dimensional random vector distributed normally with unknown mean vector $\boldsymbol{\theta}$ and covariance matrix equal to the $m \times m$ identity matrix; X_i, $i = 1, 2, \cdots, m$, denote the components of $\mathbf{X}$; $\mathbf{x}_r$, $r = 1, 2, \cdots, n$, denote the observed values of $\mathbf{X}$, the observed values of the components X_i being x_{ir}; $x = \{x_{ir}\}$, $i = 1, 2 \cdots, m$; $r = 1, 2, \cdots n$, denotes a point in the mn dimensional sample space R and $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_m)$ denotes a point in the m-dimensional parameter sapce Ω; $\bar{\mathbf{x}}$ denotes the observed sample mean value of $\mathbf{X}$ and $\bar{x}_i$ the observed sample mean value of the component X_i, so that

$$\bar{\mathbf{x}} = n^{-1} \textstyle\sum_{r=1}^{n} \bar{\mathbf{x}}_r; \qquad \bar{x}_i = n^{-1} \sum_{r=1}^{n} x_{ir}, \quad i = 1, 2, \cdots, m. \tag{1}$$

We denote by $\bar{R}$ the m-dimensional space of the points $\bar{\mathbf{x}}$.

Next following Wallace (1959), we define a confidence procedure C as a Lebesgue measurable subset of the Cartesian product space $\Omega \times R$; $C(x, \cdot)$ and

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$C(\cdot, \boldsymbol{\theta})$ respectively denote the crossections of C for given x, and $\boldsymbol{\theta}$, $C(x, \cdot)$ being the confidence sets. Then we denote by C_0 the procedure in which the confidence sets are m-dimensional spheres of fixed volume centered at the sample mean, i.e.

$$(2) \qquad C_0(x,\cdot): \left[\theta \epsilon \Omega; \sum_{i=1}^{m} (\theta_i - \bar{x}_i)^2 \leqq h^2\right];$$

$$C_0(\cdot,\theta): \left[x \epsilon R; \sum_{i=1}^{m} (\bar{x}_i - \theta_i)^2 \leqq g^2\right].$$

Let v_0 denote the volume of the m-dimensional spheres $C_0(x, \cdot)$ and $(1 - \alpha)$ the fixed confidence level of C_0, so that

$$(3) \qquad vC_0(x, \cdot) = v_0 \quad \text{for all } x \varepsilon R;$$

$$P_\theta[C_0(\cdot, \boldsymbol{\theta})] = 1 - \alpha \quad \text{for all } \boldsymbol{\theta} \varepsilon \Omega.$$

3. Main result. We define on the sample space R a vector $\mathbf{y}$ by

$$(4) \qquad \mathbf{y} = (1 + b/(a + \bar{x}^2))^{-1} \cdot \bar{\mathbf{x}}$$

where $\bar{x}$ denotes the magnitude of $\bar{\mathbf{x}}$, i.e.

$$(5) \qquad \bar{x}^2 = \textstyle\sum_{i=1}^{m} \bar{x}_i^2,$$

and a and b are positive constants. We suppose that b is small and a is large; how small and how large will be determined during the process of the following computation. We now consider the confidence procedure C_1 in which the confidence sets are m-dimensional spheres of the fixed radius h in (2), and centered at the point $\mathbf{y}$ in (4), so that the confidence sets $C_1(x, \cdot)$ are given by

$$(6) \qquad C_1(x, \cdot) = [\theta \varepsilon \Omega; |\boldsymbol{\theta} - \mathbf{y}| \leqq h],$$

where $|\boldsymbol{\theta} - \mathbf{y}|$ denotes the magnitude of the vector $(\boldsymbol{\theta} - \mathbf{y})$. Then the cross section $C_1(\cdot, \boldsymbol{\theta})$ is given by

$$(7) \qquad C_1(\cdot, \boldsymbol{\theta}) = [x \varepsilon R; |\mathbf{y} - \boldsymbol{\theta}| \leqq h].$$

Clearly,

$$vC_1(x, \cdot) = vC_0(x, \cdot) \quad \text{for all } x \varepsilon R.$$

We shall prove that, if $m \geqq 3$, then for sufficiently large a and sufficiently small b,

$$P_\theta[C_1(\cdot, \boldsymbol{\theta})] > P_\theta[C_0(\cdot, \boldsymbol{\theta})] \quad \text{for all } \boldsymbol{\theta} \varepsilon \Omega,$$

so that the procedure C_1 is uniformly superior to C_0.

[Explanatory note: as the following argument involves a rather lengthy computation, we shall indicate its main steps. The co-ordinate axes are transformed taking one of the axes in the direction of $\boldsymbol{\theta}$. Next by using the transformed form of the inequality defining $C_1(\cdot, \boldsymbol{\theta})$, we determine a subset $C_1'(\cdot, \boldsymbol{\theta})$ which is wholly contained within $C_1(\cdot, \boldsymbol{\theta})$. The argument is then completed by showing that the inclusion probability for the subsets $C_1'(\cdot, \boldsymbol{\theta})$ always exceeds $(1 - \alpha)$.]

PROOF. Substituting (4) in (7), the defining equation of $C_1(\cdot, \boldsymbol{\theta})$ reduces to

$$|\bar{\mathbf{x}} - \boldsymbol{\theta} - (b/(a + \bar{x}^2))\boldsymbol{\theta}|^2 \leqq h^2[1 + b/(a + \bar{x}^2)]^2$$

i.e.

$$(8) \quad |\bar{\mathbf{x}} - \boldsymbol{\theta}|^2 - (2b/(a + \bar{x}^2))\boldsymbol{\theta}\cdot(\bar{\mathbf{x}} - \boldsymbol{\theta}) + b^2\theta^2/(a + \bar{x}^2)^2 \leqq h^2[1 + b/(a + \bar{x}^2)]^2$$

where in the 3rd term in the left hand side of (8), θ denotes the absolute magnitude of $\boldsymbol{\theta}$, i.e. $\theta^2 = \sum_{i=1}^m \theta_i^2$. Now in the space $\bar{R}$ of the points $\bar{\mathbf{x}}$, we can by an orthogonal transformation of co-ordinates take one axis in the direction of $\boldsymbol{\theta}$ and the remaining axes perpendicular to $\boldsymbol{\theta}$. In this transformed system of co-ordinates, let

$$(9) \quad \boldsymbol{\theta} = \{\theta, 0, 0, \cdots, 0\},$$
$$\bar{\mathbf{x}} = \{u + \theta, z_1, z_2, \cdots, z_{m-1}\},$$

with $z^2 = \sum_{i=1}^{m-1} z_i^2$. Then using (9), (8) reduces to

$$(10) \quad \begin{aligned} u^2 + z^2 - 2bu\theta/(a + \theta^2 + 2\theta u + u^2 + z^2) \\ + b^2\theta^2/(a + \theta^2 + 2\theta u + u^2 + z^2)^2 \\ \leqq h^2[1 + b/(a + \theta^2 + 2\theta u + u^2 + z^2)].^2 \end{aligned}$$

It is seen from (10), that $(u^2 + z^2)$ is bounded above on $C_1(\cdot, \boldsymbol{\theta})$. For by transposing terms in (10), we get,

$$\begin{aligned} u^2 + z^2 &< h^2[1 + b/(a + (\theta + u)^2 + z^2)]^2 + 2b|u|\theta/(a + z^2 + (u + \theta)^2) \\ &< h^2(1 + b/a)^2 + b. \end{aligned}$$

Since a is to be large and b small, we may assume that $a \geqq 1$ and $b \leqq 1$, so that $u^2 + z^2 < 4h^2 + 1 < (1 + 2h)^2$. Let h_1 be a fixed number independent of a, b and θ, and such that $h_1 \geqq 1 + 2h$. Then for all points $x \in C_1(\cdot, \boldsymbol{\theta})$

$$(11) \quad u^2 + z^2 \leqq h_1^2, \quad \text{and} \quad |u| \leqq h_1.$$

Then in the third term in the left hand side of (10), noting that $\theta/(a + \theta^2)$ assumes its maximum value at $\theta = a^{\frac{1}{2}}$,

$$(12) \quad \begin{aligned} (|2u\theta| + u^2 + z^2)/(a + \theta^2) &\leqq 2h_1\theta/(a + \theta^2) + h_1^2/(a + \theta^2) \\ &< h_1/a^{\frac{1}{2}} + h_1^2/a. \end{aligned}$$

Hence by making a sufficiently large but independently of θ we can make

$$(13) \quad \frac{|2u\theta + u^2 + z^2|}{(a + \theta^2)} < \epsilon$$

where ϵ may be any arbitrarily small number > 0. For the present we assume only only that $0 < \epsilon < 1$.

Next in the third term in the left hand side of (10), put

$$(14)\quad \theta u/((a+\theta^2)+2u\theta+u^2+z^2) = \theta u/(a+\theta^2) - 2\theta^2u^2/(a+\theta^2)^2 + \beta.$$

We shall show that for large a, $\beta = o((a+\theta^2)^{-1})$. From (14), we get,

$$\beta = [(a+\theta^2)+2\theta u+u^2+z^2]^{-1}$$
$$\cdot\{4\theta^3u^3/(a+\theta^2)^2 + 2u^2\theta^2(u^2+z^2)/(a+\theta^2)^2 - (u^2+z^2)u\theta/(a+\theta^2)\}.$$

Now using (13) and (11)

$$|\beta| \leqq (a+\theta^2)^{-1}(1-\epsilon)^{-1}[4h_1^3\theta^3/(a+\theta^2)^2 + 2h_1^4/a + h_1^3\theta/(a+\theta^2)]$$

and noting that $\theta^3/(a+\theta)^2$ is maximized for $\theta = (3a)^{\frac{1}{2}}$ and $\theta/(a+\theta^2)$ for $\theta = a^{\frac{1}{2}}$, we get

$$(15)\quad |\beta| \leqq (a+\theta^2)^{-1}(1-\epsilon)^{-1}[\tfrac{1}{4}h_1^3(27)^{\frac{1}{2}}\cdot a^{-\frac{1}{2}} + 2h_1^4/a + \tfrac{1}{2}h_1^3/a^{\frac{1}{2}}]$$
$$\leqq (a+\theta^2)^{-1}\delta$$

where δ which is independent of θ can be made arbitrarily small by increasing a sufficiently. Now using (14), (15) and (13), we see that

(16) left hand side of (10)

$$\leqq u^2+z^2-2bu\theta/(a+\theta^2)+4b\theta^2u^2/(a+\theta^2)^2+2b\cdot\delta/(a+\theta^2)$$
$$+\, b^2\theta^2/(a+\theta^2)^2(1-\epsilon)^2.$$

In the 4th and 6th terms in the right-hand side of (16),

$$\theta^2/(a+\theta^2)^2 = (a+\theta^2)^{-1}\cdot\theta^2/(a+\theta^2) < (a+\theta^2)^{-1}.$$

Hence we obtain from (16),

(17) left hand side of (10)

$$< u^2+z^2-2bu\theta/(a+\theta^2)+4bu^2/(a+\theta^2)+2b\delta/(a+\theta^2)$$
$$+\, b^2/(a+\theta^2)(1-\epsilon)^2$$
$$< (1+2b/(a+\theta^2))^2(u-b_1\theta/(a+\theta^2))^2+z^2+2b\delta/(a+\theta^2)$$
$$+\, b^2/(a+\theta^2)(1-\epsilon)^2$$

where

$$(18)\quad b_1 = b/(1+2b/(a+\theta^2)).$$

Similarly by using (13), and noting that $(1+\epsilon)^{-1} > 1-\epsilon$ for $\epsilon > 0$

(19) right hand side of (10)

$$\geqq h^2[1+b/(a+\theta^2)(1+\epsilon)]^2$$
$$\geqq h^2[1+b(1-\epsilon)/(a+\theta^2)]^2.$$

Next let $C_1'(\cdot, \boldsymbol{\theta})$ be the subset of R defined by

$$(20) \quad (1 + 2b/(a + \theta^2))^2(u - b_1\theta/(a + \theta^2))^2 + z^2$$
$$\leqq h^2[1 + b(1 - \epsilon)/(a + \theta^2)]^2 - 2b\delta/(a + \theta^2) - b^2/(a + \theta^2)(1 - \epsilon)^2.$$

We shall show that for sufficiently large a and sufficiently small b, (the limits for a, and b being independent of θ),

$$C_1(\cdot, \boldsymbol{\theta}) \supset C_1'(\cdot, \boldsymbol{\theta}),$$

so that,

$$(21) \qquad P_\theta[C_1(\cdot, \boldsymbol{\theta}] \geqq P_\theta[C_1'(\cdot, \boldsymbol{\theta})], \quad \text{for all } \boldsymbol{\theta} \,\varepsilon\, \Omega.$$

To show this, put

$$(22) \qquad \text{right hand side of (20)} = h^2[1 + (b/(a + \theta^2))(1 - g)]^2.$$

It is seen that given any arbitrarily small positive number δ' we can determine a_0 and b_0, independent of θ, such that $g \leqq \delta'$ if $a \geqq a_0$ and $b \leqq b_0$.

By (22), the defining relation of the set $C_1'(\cdot, \boldsymbol{\theta})$ becomes,

$$(23) \quad (1 + 2b/(a + \theta^2))^2(u - b_1\theta/(a + \theta^2))^2 + z^2$$
$$\leqq h^2[1 + (b/(a + \theta^2))(1 - g)]^2.$$

It is again seen from (23), that $u^2 + z^2$ is bounded above on $C_1'(\cdot, \boldsymbol{\theta})$. For we have from (23),

$$|u - b_1\theta/(a + \theta^2)| < h(1 + (b/(a + \theta^2))(1 - g)](1 + 2b/(a + \theta^2))^{-1}$$
$$< h(1 + b/a).$$

Hence,

$$|u| < h(1 + b/a) + b_1\theta/(a + \theta^2).$$

Using the value of b_1 in (18), $b_1\theta/(a + \theta^2)$ is maximized for $\theta = (a + 2b)^{\frac{1}{2}}$, so that

$$b_1\theta/(a + \theta^2) \leqq b/2(a + 2b)^{\frac{1}{2}} < b/2a^{\frac{1}{2}}.$$

Hence, $|u| < h(1 + b/a) + b/2a^{\frac{1}{2}} < 2h + \frac{1}{2}$, since by assumption in (11), $a \geqq 1, b \leqq 1$. (23) similarly gives, $|z| \leqq h(1 + b/a) \leqq 2h$. We thus get, for all points $x \,\varepsilon\, C_1'(\cdot, \boldsymbol{\theta})$

$$u^2 + z^2 < (2h + \tfrac{1}{2})^2 + 4h^2$$
$$< (4h + \tfrac{1}{2})^2 < (4h + 1)^2.$$

We assume in (11) that $h_1 \geqq 1 + 2h$. Consistently with this we take $h_1 = 1 + 4h$. Then for all points $\mathbf{x} \,\varepsilon\, C_1'(\cdot, \boldsymbol{\theta})$, $u^2 + z^2 \leqq h_1^2$, so that (11) holds; consequently (17) and (19) also hold and in the result (10) holds for every point $\mathbf{x}$ of $C_1'(\cdot, \boldsymbol{\theta})$. We have thus proved (21).

Now the probability density on the sample space R can be expressed in the form

$$f(x \mid \boldsymbol{\theta}) = k \cdot \exp\left[-\tfrac{1}{2}n \textstyle\sum_{i=1}^{m} (\bar{x}_i - \theta_i)^2\right] \cdot L(x \mid \bar{\mathbf{x}}) \tag{24}$$

where, $k = (n/2\pi)^{m/2}$.

By an orthogonal transformation of co-ordinates in R, we can take $\bar{x}_i$, $i = 1, 2, \cdots, m$, as independent co-ordinates. Let x denote the group of the remaining $(mn - n)$ transformed co-ordinates. Then for each fixed $\bar{\mathbf{x}}$,

$$\int L(x \mid \bar{\mathbf{x}})\, dx' = 1. \tag{25}$$

In the space $\bar{R}$ of the points $\bar{\mathbf{x}}$, let E denote the subset defined by (23). Then because of (25), the probability $P_\theta[C_1(\cdot, \boldsymbol{\theta})]$ reduces to the integral over E of the probability density $k \cdot \exp[-\frac{1}{2}n \sum_{i=1}^{m} (\bar{\mathbf{x}}_i - \theta_i)^2]$, which using (9) becomes,

$$f(\bar{\mathbf{x}} \mid \boldsymbol{\theta}) = k \cdot \exp[-\tfrac{1}{2}n(u^2 + z^2)]. \tag{26}$$

Hence

$$k^{-1} P_\theta[C_1'(\cdot, \boldsymbol{\theta})] = \int_E \exp[-\tfrac{1}{2}n(u^2 + z^2)]\, du\, dz \tag{27}$$

where dz is written for short for $\prod_{i=1}^{m-1} dz_i$.

Now make a transformation of variables by putting

$$w = u - b_1\theta/(a + \theta^2). \tag{28}$$

The Equation (23) defining the set E then reduces to

$$(1 + 2b/(a + \theta^2))^2 w^2 + z^2 \leqq h^2[1 + b(1 - g)/(a + \theta^2)]^2. \tag{29}$$

Now, $u = w + b_1\theta/(a + \theta^2)$ by (28) and

$$\exp(t) \geqq 1 + t, \qquad -\infty < t < \infty. \tag{30}$$

Hence in (26), using (30),

$$\begin{aligned} \exp(-\tfrac{1}{2}nu^2) &= \exp[-\tfrac{1}{2}n(w^2 + 2b_1\theta w/(a + \theta^2) + b_1^2\theta^2/(a + \theta^2)^2)] \\ &\geqq [1 - nb_1^2/2(a + \theta^2)][1 - nb_1\theta w/(a + \theta^2)] \exp(-\tfrac{1}{2}nw^2) \end{aligned} \tag{31}$$

by using in the 2nd term in the first square bracket in the right hand side the relation $\theta^2/(a + \theta^2) < 1$.

Hence combining (31) and (27) we get,

$$\begin{aligned} k^{-1}P_\theta[C_1'(\cdot, \boldsymbol{\theta})] &\geqq [1 - nb_1^2/2(a + \theta^2)] \int_E (1 - nb_1\theta w/(a + \theta^2)) \\ &\qquad \cdot \exp[-\tfrac{1}{2}n(w^2 + z^2)]\, dw\, dz \\ &\geqq [1 - nb_1^2/2(a + \theta^2)] \int_E \exp[-\tfrac{1}{2}n(w^2 + z^2)]\, dw\, dz \end{aligned} \tag{32}$$

as, $\int_E w \exp[(-\frac{1}{2}n(w^2 + z^2)]\, dw\, dz = 0$ because of the symmetry of (29) which defines E.

Proceeding further from (32), denoting the volume vE of the set E by v_1 we have from (32)

$$(33)\quad k^{-1}P_\theta[C_1'(\cdot\,,\boldsymbol{\theta})] \geqq \int_E \exp\,[-\tfrac{1}{2}n(w^2 + z^2)]\,dw\,dz - [nb_1^2/2(a + \theta^2)]v_1$$
$$\geqq \int_E \exp\,[-\tfrac{1}{2}n(w^2 + z^2)]\,dw\,dz - [nb^2/2(a + \theta^2)]v_1$$

since by (18), $b > b_1$.

We next evaluate the first term in the right hand side of (33). Since $(b/(a + \theta^2))$ can be made arbitrarily small uniformly in θ, by making a sufficiently large, we shall make the calculation up to the 1st power of $(b/(a + \theta^2))$ only. Let D denote the subset of $\bar{R}$ defined by

$$(34)\qquad w^2 + z^2 \leqq h^2.$$

Then (3) implies that

$$(35)\qquad \int_D \exp\,[-\tfrac{1}{2}(w^2 + z^2)]\,dw\,dz = (1 - \alpha)/k.$$

Define sets A and B by

$$(36)\qquad A = D - D\cdot E \quad\text{and}\quad B = E - D\cdot E.$$

Then in the right hand side of (33) using (35)

$$(37)\quad \int_E \exp\,[-\tfrac{1}{2}n(w^2 + z^2)]\,dw\,dz = (1 - \alpha)/k + \int_B \exp\,[-\tfrac{1}{2}n(w^2 + z^2)]\,dw\,dz - \int_A \exp\,[-\tfrac{1}{2}n(w^2 + z^2)]\,dw\,dz.$$

Comparing (34) and (29), it is seen that the volumes vA and vB of the sets A and B are of the order of $b/(a + \theta^2)$. Also when integrating on the sets A and B, the term $(w^2 + z^2)$ differs from h^2 by a term of the order of $b/(a + \theta^2)$. Hence neglecting terms $O(b/(a + \theta^2))^2$, we have

(38) right hand side of (37)

$$= (1 - \alpha)/k + \exp\,(-\tfrac{1}{2}nh^2)(vB - vA) + O\,(b/(a + \theta^2))^2$$
$$= (1 - \alpha)/k + \exp\,(-\tfrac{1}{2}nh)^2(v_1 - v_0) + O(b/(a + \theta^2))^2.$$

Now let the volume of an m-dimensional sphere of radius h be Kh^m, so that

$$(39)\qquad v_0 = Kh^m.$$

It is then easily seen that the volume v_1 of the set E defined by (29) which is an ellipsoid is given by

$$v_1 = Kh^m[1 + (b/(a + \theta^2))(1 - g)]^m[1 + 2b/(a + \theta^2)]^{-1}$$

so that using (39), and retaining terms of the 1st degree in $b/(a + \theta^2)$,

$$(40)\quad v_1 = v_0[1 + (m - 2)b/(a + \theta^2) - mgb/(a + \theta^2)] + O(b/(a + \theta^2))^2.$$

Combining (40), (38), (37) and (33), we get

$$P_\theta[C_1'(\cdot\,,\theta)]$$
$$\geqq 1 - \alpha + kv_0 \exp\,(-\tfrac{1}{2}nh^2)\cdot[(m - 2)b/(a + \theta^2) - mgb/(a + \theta^2)]$$
$$(41)\qquad - [nb^2/2(a + \theta^2)]\cdot kv_0 + O(b/(a + \theta^2))^2$$
$$\geqq (1 - \alpha) + kv_0[b/[a + \theta^2)][(m - 2 - mg)\exp\,(-\tfrac{1}{2}nh^2) - \tfrac{1}{2}nb] + O(b/(a + \theta^2))^2.$$

We now take b to be sufficiently small, so that in the 2nd term in the right hand side of (41),

(42) $\frac{1}{2}nb + mg \exp(-\frac{1}{2}nh^2) \leqq f\cdot(m-2)\exp(-\frac{1}{2}nh^2)$ where $0 < f < 1$.

We note that b can be found so as to satisfy (42) only if $(m-2) > 0$, i.e. $m \geqq 3$. The present proof thus does not hold for $m = 1$ or 2.

The comparison of (22) and (20) shows that g cannot be made arbitrarily small merely by increasing a, as g contains a part which depends on b alone. (42) thus provides an upper bound for b. We then take a suffifficiently large so as to make the residual term in the right hand side of (41) viz.,

(43) $$O(b/(a+\theta^2))^2 \leqq \tfrac{1}{2}(1-f)kv_0b/(a+\theta^2).$$

Now combining (41), (42) and (43), and noting that $C_1'(\cdot, \boldsymbol{\theta}) \subset C_1(\cdot, \boldsymbol{\theta})$ by (21), we get

(44) $$P_\theta[C_1(\cdot, \boldsymbol{\theta})] \geqq 1 - \alpha + \tfrac{1}{2}(1-f)kv_0b/(a+\theta^2)$$
$$> 1 - \alpha \quad \text{for all } \boldsymbol{\theta} \,\varepsilon\, \Omega.$$

The procedure C_1 is thus uniformly superior to C_0, which was to be proved.

REMARK. By an argument similar to that in this paper it can be shown that the usual rectangular confidence sets defined by

$$|\bar{x}_i - \theta_i| \leqq h, \qquad i = 1, 2, \cdots, m,$$

are uniformly inferior to the confidence sets $|y_i - \theta_i| \leqq h$, $i = 1, 2, \cdots, m$, $\mathbf{y}$, being the vector defined in (4).

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