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A MULTI-DIMENSIONAL LINEAR GROWTH BIRTH AND DEATH PROCESS[1]

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1. Introduction and summary. In [5] Karlin and McGregor analyzed a class of linear growth birth and death processes admitting a representation formula of the transition probability function in terms of classical orthogonal polynomials. Birth and death processes belong to the category of reversible stochastic processes (see [12] and [13]). In this case by invoking the spectral resolution of the identity for Hermitian operators one achieves the representation formula of Karlin and McGregor described in [3] and [4]. This theory is mostly restricted to the case of one-dimensional birth and death processes and diffusion processes. Some special higher dimensional birth and death processes motivated by certain applications in studies of population growth are reversible. For these processes there are representation formulae for the transition probability function that can be explicitly determined. In [9] and [10] Karlin and McGregor develop a discretized version of the classical technique of solving Laplace's equation in terms of spherical harmonics. This paper applies their method to determine the representation formula for the transition probability function of two two-dimensional (Sections 2 and 3) and the corresponding higher dimensional (Sections 4 and 5) linear growth birth and death processes. In each case the representation formula is expressed explicitly in terms of semi-direct products of classical orthogonal polynomials somewhat reminiscent of such products of spherical harmonics. In the Appendix the properties of the orthogonal polynomials are summarized and the method of derivation of the explicit form of the representation formula is outlined.

2. A two-dimensional linear growth process. In this section we shall examine a two-dimensional linear growth birth and death process. We shall assume that a population of two distinct genotypes exists and its growth is governed by stochastic fluctuations to be described here in detail. The size of the population at any time t is adequately described by the random vector $\mathbf{X}(t) = (U(t), V(t))$. The conditional probability that the population size changes during the time interval $(s, s + t)$ from (m, n) to (m', n') will be denoted by

$$(2.1)\quad P(t; (m, n), (m', n')) = P\{X(s + t) = (m', n') \mid \mathbf{X}(s) = (m, n)\}$$

independent of s. Thus the transition probability function is assumed to be stationary in time. In order to make $\mathbf{X}(t)$ a birth and death process we assume that during an interval $(t, t + h)$ of infinitesimal length $h > 0$ only four types of

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changes are possible in the population, namely, an increase or decrease by at most one in either genotype. More precisely, we assume that

$$
\begin{aligned}
P(h;\ (m, n),\ (m', n')) &= \lambda_1(m, n)h + o(h) && \text{if } m' = m + 1 \text{ and } n' = n \\
&= \lambda_2(m, n)h + o(h) && \text{if } m' = m \text{ and } n' = n + 1 \\
&= \mu_1(m, n)h + o(h) && \text{if } m' = m - 1 \text{ and } n' = n \\
&= \mu_2(m, n)h + o(h) && \text{if } m' = m \text{ and } n' = n - 1 \\
&= 1 - [\lambda_1(m, n) + \lambda_2(m, n) + \mu_1(m, n) \\
&\quad + \mu_2(m, n)]h + o(h) \\
& && \text{if } m' = m \text{ and } n' = n \\
&= o(h) && \text{otherwise}
\end{aligned} \tag{2.2}
$$

where $\lambda_1(m, n)$, $\lambda_2(m, n)$, $\mu_1(m, n)$ and $\mu_2(m, n)$, usually called the infinitesimal birth and death rates of the process, are constant in time. When these rates are linear functions in m and n (the present sizes of the population of the two genotypes), we call $\mathbf{X}(t)$ a (two-dimensional) linear growth process.

An example of such a two-dimensional linear growth process is a population of type A and type a genes. In each time interval $(t, t + h)$ of infinitesimal length h a birth or a death of a gene of either type may occur with probability proportional to the number of genes of that type alive in the population. Each individual gene born, however, may mutate to the other type with probability α for a type A gene to type a and with probability β for a type a gene to a type A. In this case, we have

$$\lambda_1(m, n) = (1 - \alpha)m + \beta n, \qquad \lambda_2(m, n) = \alpha m + (1 - \beta)n$$

and

$$\mu_1(m, n) = m, \qquad \mu_2(m, n) = n.$$

Another example of a two-dimensional linear growth process is the one discussed in detail in this section. The probability of a birth of an individual of either genotype is assumed to be proportional to the *total* number of individuals in the population. The probability of a death of an individual of either genotype, however, is assumed to be proportional to the number of individuals of that genotype alone. More precisely, the infinitesimal birth and death rates of this process are given by

$$
\begin{aligned}
&\lambda_1(m, n) = (m + n + 1)p, \qquad \lambda_2(m, n) = (m + n + 1)q \\
&\text{and} \\
&\mu_1(m, n) = m, \qquad\qquad\qquad \mu_2(m, n) = n
\end{aligned} \tag{2.3}
$$

where $p > 0$ and $q > 0$ are constants.

The main result of this section is a representation theorem for the transition probability function defined by formula (2.1).

THEOREM 2.1 (Representation Theorem). *If* $\mathbf{X}(t)$ *is a two-dimensional linear growth birth and death process whose infinitesimal birth and death rates are defined by* (2.3), *then the transition probability function of* $\mathbf{X}(t)$ *is given by the formula*

$$(2.4)\quad P(t;\,(m,n),\,(m',n')) = \Pi(m',n')\sum_{(x,y)} e^{-yt} f(x,y;m,n) f(x,y;m',n')\rho(x,y).$$

Here the functions $f(x,y;\,m,\,n)$ *are so-called semi-direct products of various orthogonal polynomials; namely,*

$$(2.5)\quad \begin{aligned} f(x,y;m,n) &= \tbinom{M}{x}\phi_{M-x}((y-x)(1-(p+q))^{-1};\,x+1;\,p+q) \\ &\quad\cdot K_m(x;\,p(p+q)^{-1},\,M) \quad \text{if} \quad p+q<1 \\ &= \tbinom{M}{x}(p+q)^{-(M-x)}\phi_{M-x}((y-x)(p+q-1)^{-1} \\ &\quad -(x+1);\,x+1;\,(p+q)^{-1})K_m(x;\,p(p+q)^{-1},\,M) \\ &\quad \text{if} \quad p+q>1 \\ &= L^{x}_{M-x}(y-x)K_m(x;\,p,\,M) \end{aligned}$$

where $M = m + n$.

Here $\phi_i(z)$ *is the Meixner polynomial of the indicated parameters,* $L_i^x(z)$ *is the associated Laguerre polynomial of parameter* x, *and* $K_i(z)$ *is the Krawtchouk polynomial. All three are defined in detail in Appendix* 1.

The weight function $\Pi(m',n')$ *and the spectral measure* $\rho(x,y)$ *are given by*

$$(2.6)\quad \Pi(m,n) = \tbinom{M}{n}p^m q^n, \quad \text{with} \quad M = m+n$$

and

$$(2.7)\quad \begin{aligned} \rho(x,y) &= \tbinom{x+k}{k}(p+q)^k[1-(p+q)]^{x+1}[p(q(p+q))^{-1}]^x && \text{if } p+q<1 \\ &= \tbinom{x+k}{k}(p+q)^{-k}[1-(p+q)^{-1}]^{x+1} \\ &\quad\cdot[p(q(p+q))^{-1}]^x && \text{if } p+q>1 \\ &= e^{-(y-x)}(y-x)^x(x!)^{-1}(p/q)^x && \text{if } p+q=1 \end{aligned}$$

where y *and* k *are closely associated through*

$$(2.8)\quad \begin{aligned} y &= x+[1-(p+q)]k && \text{if } p+q<1 \\ &= x(p+q)+(p+q-1)(k+1) && \text{if } p+q>1. \end{aligned}$$

The summation $\sum_{(x,y)}$ *in formula* (2.4) *stands for a double summation over all non-negative integers* $x \leqq M$ *and all possible values of* y *as* k *runs over all non-negative integers when* $p + q \neq 1$. *In the case* $p + q = 1$, *the summation over* y *is replaced by the integral* $\int_x^\infty dy$.

PROOF. Define the matrix, A, of the infinitesimal birth and death rates, with elements

$$(2.9)\quad \begin{aligned} a((m, n), (m', n')) &= (M+1)p && \text{if } m' = m+1 \text{ and } n' = n \\ &= (M+1)q && \text{if } m' = m \text{ and } n' = n+1 \\ &= m && \text{if } m' = m-1 \text{ and } n' = n \\ &= n && \text{if } m' = m \text{ and } n' = n-1 \\ &= -[(M+1)(p+q) + M] && \\ & && \text{if } m' = m \text{ and } n' = n \\ &= 0 && \text{otherwise} \end{aligned}$$

where $M = m + n$.

Then the transition probability function is the unique solution (see [3] and [4]) of the differential equation

$$(2.10)\quad \begin{aligned} &(d/dt)P(t; (m, n), (m', n')) \\ &\qquad = \textstyle\sum_{m'',n''=0}^{\infty} a((m, n), (m'', n''))P(t; (m'', n''), (m, n)) \end{aligned}$$

with initial conditions

$$(2.11)\quad P(0; (m, n), (m', n')) = \delta_{m,m'}\delta_{n,n'} .$$

Direct substitution of the representation formula (2.4) in (2.10) and (2.11) proves Theorem 2.1.

Although the above proves the representation formula (2.4), it leaves the actual derivation of the functions $f(x, y; m, n)$ obscure. This, in fact, is accomplished with the aid of the spectral resolution of the identity of Hermitian operators. The procedure is outlined in Lemma 1 of Appendix 3.

In order to derive a generating function formula for the transition probability function, we first present two generating function formulae for the functions $f(x, y; m, n)$. This is done in

LEMMA 2.1. *If we define the generating functions*

$$G_1(u, w) = \textstyle\sum_{(x,y)} f(x, y; m, n)\binom{x+k}{k}u^x w^k \quad \textit{for} \quad |u| < 1, \quad |w| < 1$$

and

$$G_2(v_1, v_2) = \textstyle\sum_{m,n=0}^{\infty} f(x, y; m, n)\binom{m+n}{n}v_1^m v_2^n \quad \textit{for} \quad |v_1| < 1, \quad |v_2| < 1$$

where $f(x, y; m, n)$ and k are as defined in (2.5) *and* (2.8), *respectively, then the following generating function formulae are valid:*

$$(2.12)\quad \begin{aligned} G_1(u, w) &= (1 - w)^{-(M+1)}(1 + u - w(p+q)^{-1})^n \\ &\qquad \cdot (1 - qp^{-1}u - w(p+q)^{-1})^m && \textit{if } \ p + q < 1 \\ &= (p+q)^{-M}(1 - w)^{-(M+1)}(1 + (p+q)(u - w))^n \\ &\qquad \cdot (1 - (p+q)(qp^{-1}u + w))^m && \textit{if } \ p + q > 1 \\ &= \textit{no closed form} && \textit{if } \ p + q = 1 \end{aligned}$$

where $M = m + n$, *and*

$$
\begin{aligned}
G_2(v_1, v_2) &= (1 - (v_1 + v_2))^{-(k+x+1)}(1 - (v_1 + v_2)(p + q)^{-1})^{k} \\
&\qquad \cdot (v_2 - qp^{-1}v_1)^x \qquad \text{if } p + q < 1 \\
&= (1 - (v_1 + v_2)(p + q)^{-1})^{-(k+x+1)}(1 - (v_1 + v_2))^{k} \\
&\qquad \cdot (v_2 - qp^{-1}v_1)^x \qquad \text{if } p + q > 1 \\
&= \exp(-(y - x)(v_1 + v_2)[1 - (v_1 + v_2)]^{-1}) \\
&\qquad \cdot (1 - (v_1 + v_2))^{-(x+1)}(v_2 - qp^{-1}v_1)^x \qquad \text{if } p + q = 1.
\end{aligned} \tag{2.13}
$$

PROOF. These generating function formulae are derived by direct computation from the generating function formulae of the Krawtchouk, Meixner and Laguerre polynomials (see (9), (10), (18), (19), and (24) of Appendix 1).

Next we present formulae for the generating function of the transition probability function when $p + q \neq 1$.

THEOREM 2.2. *If we define the generating function*

$$G(t; v_1, v_2) = \sum_{m',n'=0}^{\infty} P(t; m, n), (m', n'))v_1^{m'}v_2^{n'} \quad \textit{for} \quad |v_1| < 1, \quad |v_2| < 1,$$

then

$$G(t; v_1, v_2) = AB^{-(M+1)}C^mD^n \tag{2.14}$$

where A, B, C *and* D *are functions of* t, v_1 *and* v_2. *Namely, with*

$$
\begin{aligned}
&P = p + q, \qquad Q = |1 - P|, \\
&E = e^{-Qt}, \qquad F = 1 - E, \\
&U = pv_1 + qv_2 \quad \textit{and} \quad V = v_1 - v_2,
\end{aligned} \tag{2.15}
$$

we have

$$
\begin{aligned}
A &= P^{-M}Q && \text{if } P < 1 \\
&= P^{-M}QE && \text{if } P > 1; \\
B &= 1 - PE - FU && \text{if } P < 1 \\
&= P - E - FU && \text{if } P > 1; \\
C &= PF + qQe^{-t}V - (P - E)U && \text{if } P < 1 \\
&= PF + qQe^{-Pt}V - (1 - PE)U && \text{if } P > 1; \\
D &= PF - pQe^{-t}V - (P - E)U && \text{if } P < 1 \\
&= PF - pQe^{-Pt}V - (1 - PE)U && \text{if } P > 1.
\end{aligned} \tag{2.16}
$$

PROOF. The theorem follows by direct computation from Theorem 2.1 and Lemma 2.1.

The generating function formula (2.14) may be used to compute the means and higher moments of the process. As an example, we compute

$$E[U(t) \mid \mathbf{X}(0) = (m, n)] = [\partial G(t; v_1, 1)/\partial v_1]\mid_{v_1=1}.$$

Note first that for $v_1 = v_2 = 1$ we have

$$\begin{aligned} B &= Q \quad \text{if } P < 1 \qquad \text{and} \qquad & C = D &= PQ \quad \text{if } P < 1 \\ &= QE \quad \text{if } P > 1 & &= PQE \quad \text{if } P > 1. \end{aligned}$$

Also, if prime denotes derivative with respect to v_1, we have:

$$\begin{aligned} B' = -pF, \qquad C' &= qQe^{-t} - p(P - E) \qquad \text{if } P < 1 \\ &= qQe^{-Pt} - p(1 - PE) \qquad \text{if } P > 1 \end{aligned}$$

and

$$\begin{aligned} D' &= -pQe^{-t} - p(P - E) \qquad \text{if } P < 1 \\ &= -pQe^{-Pt} - p(1 - PE) \qquad \text{if } P > 1. \end{aligned}$$

Since

$$G' = G[-(M + 1)B'/B + mC'/C + nD'/D],$$

we obtain

$$\begin{aligned} E[U(t) \mid \mathbf{X}(0) &= (m, n)] \\ &= (PQ)^{-1}[(mq - np)Qe^{-t} + MpQE + pPF] \qquad \text{if } P < 1 \\ &= (PQE)^{-1}[(mq - np)Qe^{-Pt} + MpQ + pPF] \qquad \text{if } P > 1. \end{aligned}$$

Similarly, we may compute the mean of $V(t)$. Thus, the mean of the total population size is:

$$\begin{aligned} E[U(t) + V(t) \mid \mathbf{X}(0) = (m, n)] &= ME + PQ^{-1}F \qquad \text{if } P < 1 \\ &= ME^{-1} + PQ^{-1}E^{-1}F \qquad \text{if } P > 1. \end{aligned}$$

To examine recurrence we select state $(0, 0)$ and compute $P(t; (0, 0), (0, 0))$ to find

$$\begin{aligned} &P(t; (0, 0), (0, 0)) \\ &\quad = [1 - (p + q)][1 - (p + q)\exp(-[1 - (p + q)]t)]^{-1} \qquad \text{if } p + q < 1 \\ &\quad = (t + 1)^{-1} \qquad \text{if } p + q = 1 \\ &\quad = (p + q - 1)[(p + q)\exp((p + q - 1)t) - 1]^{-1} \qquad \text{if } p + q > 1. \end{aligned}$$

From here we have

$$\begin{aligned} \int_0^\infty P(t; (0, 0), (0, 0))\, dt &= \infty \qquad \text{if } p + q < 1 \\ &= \infty \qquad \text{if } p + q = 1 \\ &= \ln[(p + q)(p + q - 1)^{-1}] < \infty \qquad \text{if } p + q > 1 \end{aligned}$$

and

$$\lim_{t\to\infty} P(t;\ (0,0),\ (0,0)) = 1 - (p+q) \quad \text{if } p+q<1$$
$$= 0 \quad \text{if } p+q=1$$
$$= 0 \quad \text{if } p+q>1.$$

Thus we may state

THEOREM 2.3. *If* $\mathbf{X}(t)$ *is a two-dimensional linear growth birth and death process whose infinitesimal birth and death rates are defined by* (2.3), *then the process is positive recurrent, null-recurrent or transient according as* $p + q < 1$, $p + q = 1$, *or* $p + q > 1$.

3. The associated two-dimensional absorption process. If the process studied in the previous section is "reversed," we obtain a process with infinitesimal birth rates

$$\lambda_1(m,n) = m+1, \qquad \lambda_2(m,n) = n+1 \tag{3.1}$$

and infinitesimal death rates

$$\mu_1(m,n) = (M+1)p, \qquad \mu_2(m,n) = (M+1)q, \tag{3.2}$$

where $M = m + n$. This is obviously an absorption process, since from any integer point $(m, 0)$ or $(0, n)$ along one of the non-negative coordinate axes absorption may occur with probability $q(m + 1)$ or $p(n + 1)$, respectively.

The results obtained for this absorption process are analogous to the ones presented for the original two-dimensional linear growth process and are sum marized in the following theorems and lemmas.

THEOREM 3.1 (Representation Theorem). *If* $\mathbf{X}(t)$ *is a two-dimensional linear growth birth and death process whose infinitesimal birth and death rates are defined by* (3.1) *and* (3.2), *respectively, then the transition probability function of* $\mathbf{X}(t)$ *is given by formula* (2.4). *The functions* $f(x, y; m, n)$ *are again semi-direct products of orthogonal polynomials:*

$$\begin{aligned} f(x,y;m,n) &= \tbinom{M}{m} p^m q^n \tbinom{M+1}{x+1} \phi_{M-x}((y-(x+1))(1-(p+q))^{-1}-1; \\ &\qquad x+2;\, p+q) K_m(x;\, p(p+q)^{-1}, M) \qquad \textit{if } p+q<1 \\ &= \tbinom{M}{m} p^m q^n \tbinom{M+1}{x+1} (p+q)^{-(M-x)} \cdot \phi_{M-x}((y-(x+1)) \\ &\quad \cdot (p+q-1)^{-1} - (x+1);\, x+2;\, (p+q)^{-1}) K_m(x; p(p+q)^{-1}, M) \\ &\qquad\qquad \textit{if } p+q>1 \\ &= \tbinom{M}{m} p^m q^n L_{M-x}^{x+1}(y-x-1) K_m(x;\, p, M) \qquad \textit{if } p+q=1. \end{aligned} \tag{3.3}$$

where $M = m + n$.

The weight function $\Pi(m, n)$ *and the spectral measure* $\rho(x, y)$ *are given respectively by*

$$\Pi(m,n) = [(M+1)\tbinom{M}{m} p^m q^n]^{-1} \tag{3.4}$$

and

$$\begin{aligned}(3.5)\qquad \rho(x, y) &= (x+1)\binom{x+k+1}{k}(p+q)^k[1-(p+q)]^{x+2} \\ &\qquad \cdot [p(q(p+q))^{-1}]^x \quad \text{if } p+q<1 \\ &= (x+1)\binom{x+k+1}{k}(p+q)^{-k}[1-(p+q)^{-1}]^{x+2} \\ &\qquad \cdot [p(q(p+q))^{-1}]^x \quad \text{if } p+q>1 \\ &= \exp(-[y-(x+1)])[y-(x+1)]^{x+1}(x!)^{-1}(p/q)^x \\ &\qquad \text{if } p+q=1\end{aligned}$$

where y and k are associated through

$$\begin{aligned}(3.6)\qquad y &= (x+1)+[1-(p+q)](k+1) \quad \text{if } p+q<1 \\ &= (p+q)(x+1)+(p+q-1)k \quad \text{if } p+q>1.\end{aligned}$$

PROOF. Define the matrix A of the infinitesimal birth and death rates with elements

$$\begin{aligned}(3.7)\qquad & a((m, n), (m', n')) \\ &= m+1 && \text{if } m' = m+1 \text{ and } n' = n \\ &= n+1 && \text{if } m' = m \text{ and } n' = n+1 \\ &= (M+1)p && \text{if } m' = m-1 \text{ and } n' = n \\ &= (M+1)q && \text{if } m' = m \text{ and } n' = n-1 \\ &= -[(M+2) \\ &\qquad + (M+1)(p+q)] && \text{if } m' = m \text{ and } n' = n \\ &= 0 && \text{otherwise}\end{aligned}$$

where $M = m + n$. Then the proof of Theorem 3.1 is accomplished by direct substitution of the representation formula in the differential equation (2.10) with initial conditions (2.11).

The actual derivation of the functions $f(x, y; m, n)$ is outlined in the proof of Lemma 1 of Appendix 3.

LEMMA 3.1. *If we define the generating functions*

$$G_1(u, w) = \sum_{(x,y)} f(x, y; m, n)(x+1)\binom{x+k+1}{k}u^x w^k \quad \textit{for} \quad |u| < 1 \quad \textit{and} \quad |w| < 1$$

and

$$G_2(v_1, v_2) = \sum_{m,n=0}^{\infty} f(x, y; m, n) v_1^m v_2^n \quad \textit{for} \quad |v_1| < 1 \quad \textit{and} \quad |v_2| < 1$$

where $f(x, y; m, n)$ *and* k *are given by formulae* (3.3) *and* (3.6), *respectively, then* *the following generating function formulae are valid:*

$$
\begin{aligned}
&G_1(u, w) \\
&= \Pi^{-1}(m, n)(1 - w)^{-(M+2)} \\
(3.8)\quad &\cdot[1 + u - w(p+q)^{-1}]^n[1 - qp^{-1}u - w(p+q)^{-1}]^m \quad \text{if } p + q < 1 \\
&= \Pi^{-1}(m, n)(p+q)^{-M}(1 - w)^{-(M+2)} \\
&\cdot[1 + (p+q)(u - w)]^n[1 - (p+q)(qp^{-1}u + w)]^m \quad \text{if } p + q > 1 \\
&= \text{no closed form} \quad \text{if } p + q = 1
\end{aligned}
$$

where $M = m + n$ *and*

$$
\begin{aligned}
&G_2(v_1, v_2) \\
&= [1 - (pv_1 + qv_2)]^{-(k+x+2)} \\
&\qquad \cdot[1 - (pv_1 + qv_2)(p+q)^{-1}]^k[q(v_2 - v_1)]^x \quad \text{if } p + q < 1 \\
(3.9)\quad &= [1 - (pv_1 + qv_2)(p+q)^{-1}]^{-(k+x+2)} \\
&\qquad \cdot[1 - (pv_1 + qv_2)]^k[q(v_2 - v_1)]^x \quad \text{if } p + q > 1 \\
&= \exp\left(-[y - (x+1)](pv_1 + qv_2)(1 - (pv_1 + qv_2))^{-1}\right) \\
&\qquad \cdot[1 - (pv_1 + qv_2)]^{-(x+2)}[q(v_2 - v_1)]^x \quad \text{if } p + q = 1.
\end{aligned}
$$

PROOF. The derivation of these formulae is analogous to that of (2.12) and (2.13).

THEOREM 3.2. *If (ignoring absorbing states) we define the generating function*

$$G(t; v_1, v_2) = \sum_{m', n'=0}^{\infty} P(t; (m, n), (m', n'))\Pi^{-1}(m', n')v_1^{m'}v_2^{n'}$$

for $|v_1| < 1$, $|v_2| < 1$, *then*

$$G(t; v_1, v_2) = AB^{-(M+2)}C^mD^n \tag{3.10}$$

where

$$A = \Pi^{-1}(m, n)P^{-M}Q^2e^{-(1+Q)t} \tag{3.11}$$

and B, C *and* D *as well as* P, Q, E *and* F *are the same as defined by* (2.16) *and* (2.15), *respectively.*

PROOF. The derivation is analogous to that of formula (2.14).

Formula (3.10) may be used to compute absorption probabilities of the process. For example, the probability of absorption

$$
\begin{aligned}
&P\{U(t) = -1 \mid \mathbf{X}(0) = m, n\} \\
&\quad = \sum_{n'=0}^{\infty} p(n' + 1)P(t; (m, n), (0, n')) \\
&\quad = pG(t; 0, q^{-1}) \\
&\quad = \Pi^{-1}(m, n)P^{-M}pe^{-Pt}(1 - e^{-Pt})^m(1 + pq^{-1}e^{-Pt})^n \quad \text{provided} \quad P \neq 1.
\end{aligned}
$$

Finally, we present a theorem that connects the eigenvectors $f(x, y; m, n)$ of the original two-dimensional linear growth process discussed in Section 2 with those of the associated absorption process of this section. For this purpose we change the notation of the latter functions from $f(x, y; m, n)$ to $f^*(x, y; m, n)$. Similarly, an asterisk will be added to the notation of the weight function as well as the spectral measure of the absorption process.

THEOREM 3.3. *Let* $f(x, y; m, n)$, $\Pi(m, n)$ *and* $\rho(x, y)$ *denote the functions defined in Section* 2 *by formulae* (2.5), (2.6) *and* (2.7), *respectively. Further, let* $f^*(x, y; m, n)$, $\Pi^*(m, n)$ *and* $\rho^*(x, y)$ *denote the functions defined in this section by formulae* (3.3), (3.4) *and* (3.5), *respectively. Then we have the following relationships:*

$$\begin{aligned} f^*(x, y; m, n) = {} & (M + 1)\Pi(m, n)(x - (y - 1))^{-1} \\ & \cdot[pf(x, y - 1; m + 1, n) + qf(x, y - 1; m, n + 1) \\ & - (p + q)f(x, y - 1; m, n)], \end{aligned} \tag{3.12}$$

$$\Pi^*(m, n) = [(M + 1)\Pi(m, n)]^{-1} \tag{3.13}$$

and

$$\rho^*(x, y) = qp^{-1}(p + q)(x + 1)\rho(x + 1, y + p + q - 1). \tag{3.14}$$

PROOF. Formulae (3.13) and (3.14) are obvious. Formula (3.12) follows from the consecutive application of formula (2) of Appendix 1 for the Krawtchouk polynomials and then either formula (14) or (15) of Appendix 1 for the Meixner polynomials or formula (22) of Appendix 1 for the Laguerre polynomials, depending on whether $p + q < 1$, $p + q > 1$, or $p + q = 1$.

4. An $(N + 1)$-dimensional linear growth process. In this section the two-dimensional linear growth process is generalized to $(N + 1)$-dimensions. We consider a population of $N + 1$ genotypes. The number of individuals of each genotype is a random variable dependent on time and will be represented as a component of the $(N + 1)$-dimensional random vector $\mathbf{X}(t)$. The range space of $\mathbf{X}(t)$ is then the set of all $(N + 1)$-dimensional vectors $\mathbf{m}$ with non-negative integer components $m_0, m_1, \cdots, m_N$. The transition probability function

$$P(t; \mathbf{m}, \mathbf{m}') = P\{\mathbf{X}(s + t) = \mathbf{m}' \mid \mathbf{X}(s) = \mathbf{m}\} \tag{4.1}$$

is assumed to be stationary in time. We further assume that $\mathbf{X}(t)$ is a generalization of the linear growth birth and death process of Section 2; i.e., we assume that for small $h > 0$

$$\begin{aligned} & P(h; \mathbf{m}, \mathbf{m}') \\ & = (M + 1)p_j h + o(h) \qquad \text{if } \mathbf{m}' = \mathbf{m} + \mathbf{e}_j \text{ for } j = 0, \cdots, N \\ & = m_j h + o(h) \qquad \text{if } \mathbf{m}' = \mathbf{m} - \mathbf{e}_j \text{ for } j = 0, \cdots, N \\ & = 1 - [(M + 1)P + M]h + o(h) \\ & \qquad \text{if } \mathbf{m}' = \mathbf{m} \end{aligned} \tag{4.2}$$

$$= o(h) \qquad \text{otherwise.}$$

Here $\mathbf{e}_j$ denotes an $(N + 1)$-dimensional vector whose jth component is one and all others vanish, thus, e.g., $\mathbf{e}_0 = (1, 0, \cdots, 0)$ and $\mathbf{e}_1 = (0, 1, 0, \cdots, 0)$, etc. Further

$$M = \sum_{i=0}^{N} m_i, \qquad P = \sum_{i=0}^{N} p_i \tag{4.3}$$

and the p_j's are positive constants.

The generalization of Theorem 2.1 is the following:

THEOREM 4.1 (Representation Theorem). *If $\mathbf{X}(t)$ is the $(N + 1)$-dimensional linear growth birth and death process defined above, with infinitesimal birth and death rates as given in (4.2), then the transition probability function has the representation*

$$P(t; \mathbf{m}, \mathbf{m}') = \Pi(\mathbf{m}') \sum_{(\mathbf{x},y)} e^{-yt} f(\mathbf{x}, y; \mathbf{m}) f(\mathbf{x}, y; \mathbf{m}') \rho(\mathbf{x}, y) \tag{4.4}$$

where $f(\mathbf{x}, y; \mathbf{m})$ is the semi-direct product of orthogonal polynomials. Namely,

$$\begin{aligned} f(\mathbf{x}, y; \mathbf{m}) &= \tbinom{M}{x_N} \phi_{M-x_N}((y - x_N)(1 - P)^{-1}; x_N + 1; P) \\ &\qquad \cdot \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}) \qquad \text{if } P < 1 \\ &= \tbinom{M}{x_N} P^{-(M-x_N)} \phi_{M-x_N}((y - x_N)(P - 1)^{-1} \\ &\qquad - (x_N + 1); x_N + 1; P^{-1}) \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}) \qquad \text{if } P > 1 \\ &= L_{M-x_N}^{x_N}(y - x_N) \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}) \qquad \text{if } P = 1\cdot \end{aligned} \tag{4.5}$$

Here, $\phi_i(z)$ and $L_i^{\alpha}(z)$ are the Meixner and Laguerre polynomials, respectively, defined in Appendix 1; $\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m})$ is the Krawtchouk polynomial of order N introduced in Appendix 2 and so called because it has many of the properties of ordinary Krawtchouk polynomials. The weight function $\Pi(\mathbf{m})$ and the spectral measure $\rho(\mathbf{x}, y)$ are given by

$$\Pi(\mathbf{m}) = \tbinom{M}{m_0, \cdots m_N} p_0^{m_0} \cdots p_N^{m_N} \tag{4.6}$$

and

$$\rho(\mathbf{x}, y) = \tbinom{x_N}{x_1 - x_0, \cdots, x_N - x_{N-1}} r(x_N, y; P) \prod_{j=1}^{N} (P_{j-1}/p_j P_j)^{x_j - x_{j-1}} \tag{4.7}$$

where

$$\begin{aligned} r(x_N, y; P) &= \tbinom{x_N + k}{k} (1 - P)^{x_N + 1} P^k \qquad &&\text{if } P < 1 \\ &= \tbinom{x_N + k}{k} (1 - P^{-1})^{x_N + 1} P^{-k} &&\text{if } P > 1 \\ &= e^{-(y - x_N)} (y - x_N)^{x_N} (x_N!)^{-1} &&\text{if } P = 1. \end{aligned} \tag{4.7'}$$

Here, $x_0 = 0$, $\mathbf{x} = (x_0, x_1, \cdots, x_N)$ and y and k are closely associated through

$$\begin{aligned} y &= x_N + (1 - P)k \qquad &&\text{if } P < 1 \\ &= P x_N + (P - 1)(k + 1) &&\text{if } P > 1. \end{aligned} \tag{4.8}$$

The summatiom in formula (4.4) *stands for an* $(N+1)$*-tuple summation over all possible values of* y *as* k *runs over all non-negative integers and all integer values of* $x_1, \cdots, x_N$ *such that* $x_{j-1} \leqq x_j \leqq M_j$, *for* $j = 1, \cdots, N$ *where* $M_j = \sum_{i=0}^{j} m_i$, *for* $j = 0, \cdots, N$ (*with* $M_N = M$) *when* $P \neq 1$. *In the case* $P = 1$ *the summation over* y *is replaced by an integral* $\int_{x_N}^{\infty} dy$.

Proof. Define the matrix A of the infinitesimal birth and death rates, with elements

$$
\begin{aligned}
a(\mathbf{m}, \mathbf{m}') &= (M+1)p_j && \text{if } \mathbf{m}' = \mathbf{m} + \mathbf{e}_j \text{ for } j = 0, \cdots, N \\
&= m_j && \text{if } \mathbf{m}' = \mathbf{m} - \mathbf{e}_j \text{ for } j = 0, \cdots, N \\
&= -[(M+1)P + M] && \text{if } \mathbf{m}' = \mathbf{m} \\
&= 0 && \text{otherwise.}
\end{aligned}
\tag{4.9}
$$

Then the transition probability function is the unique solution of the differential equations

$$(d/dt)P(t; \mathbf{m}, \mathbf{m}') = \sum_{\mathbf{m}''} a(\mathbf{m}, \mathbf{m}'')P(t; \mathbf{m}'', \mathbf{m}) \tag{4.10}$$

with initial conditions

$$P(0; \mathbf{m}, \mathbf{m}') = \delta(\mathbf{m}, \mathbf{m}') = \delta_{m_0, m_0'} \cdots \delta_{m_N, m_N'} . \tag{4.11}$$

Direct substitution of the representation formula in (4.10) and (4.11) proves the theorem.

The actual derivation of the functions $f(\mathbf{x}, y; \mathbf{m})$ is the generalization of the procedure applied in the two-dimensional case. In the course of the proof of Lemma 1 of Appendix 4 we give an outline of this derivation.

The $(N+1)$-dimensional generalizations of the generating function formulae of Section 2 are given below in Lemma 4.1 and Theorem 4.2.

Lemma 4.1. *If we define the generating functions*

$$
\begin{aligned}
G_1(\mathbf{u}, w) = \sum_{(\mathbf{x}, y)} f(\mathbf{x}, y; \mathbf{m}) &\binom{x_N}{x_1 - x_0, \cdots, x_N - x_{N-1}} \\
&\cdot \binom{x_N + k}{k} u_1^{x_1 - x_0} \cdots u_N^{x_N - x_{N-1}} w^k, \quad \text{for} \quad |\mathbf{u}| < 1, \quad |w| < 1
\end{aligned}
$$

and

$$G_2(\mathbf{v}) = \sum_{\mathbf{m}} f(\mathbf{x}, y; \mathbf{m}) \binom{M}{m_0, \cdots, m_N} v_0^{m_0} \cdots v_N^{m_N}, \quad \text{for} \quad |\mathbf{v}| < 1,$$

where $\mathbf{u} = (u_0, u_1, \cdots, u_N)$ *with* $u_0 \equiv 0$, $\mathbf{v} = (v_0, v_1, \cdots, v_N)$, *and* $f(\mathbf{x}, y; \mathbf{m})$ *and* k *are as given by* (4.5) *and* (4.8), *respectively, then the following formulae hold:*

$$
\begin{aligned}
&G_1(\mathbf{u}, w) \\
&\quad = (1 - w)^{-(M+1)}(1 + u_N - P^{-1}w)^{m_N} \\
&\qquad \cdot \prod_{j=1}^{N} [1 + u_{j-1} - P^{-1}w - \textstyle\sum_{i=j}^{N} P_i P_{i-1}^{-1} u_i]^{m_{j-1}} \qquad \text{if} \quad P < 1 \\
&\quad = P^{-M}(1 - w)^{-(M+1)}[1 + P(u_N - w)]^{m_N}
\end{aligned}
\tag{4.12}
$$

$$\cdot \prod_{j=1}^{N} [1 + P(u_{j-1} - w - \sum_{i=j}^{N} P_i P_{i-1}^{-1} u_i)]^{m_{j-1}} \qquad \text{if } P > 1$$
$$= \text{no closed form} \qquad \text{if } P = 1$$

and

$$\begin{aligned} G_2(\mathbf{v}) &= (1 - \textstyle\sum_{i=0}^{N} v_i)^{-(k+x_N+1)} (1 - P^{-1} \sum_{i=0}^{N} v_i)^k \\ &\quad \cdot \textstyle\prod_{j=1}^{N} (v_j - p_j P_{j-1}^{-1} \sum_{i=0}^{j-1} v_i)^{x_j - x_{j-1}} \qquad \text{if } P < 1 \\ &= (1 - P^{-1} \textstyle\sum_{i=0}^{N} v_i)^{-(k+x_N+1)} (1 - \sum_{i=0}^{N} v_i)^k \\ &\quad \cdot \textstyle\prod_{j=1}^{N} (v_j - p_j P_{j-1}^{-1} \sum_{i=0}^{j-1} v_i)^{x_j - x_{j-1}} \qquad \text{if } P > 1 \\ &= (1 - \textstyle\sum_{i=0}^{N} v_i)^{-(x_N+1)} \exp\left(-(y - x_N)(\sum_{i=0}^{N} v_i)(1 - \sum_{i=0}^{N} v_i)^{-1}\right) \\ &\quad \cdot \textstyle\prod_{j=0}^{N} (v_j - p_j P_{j-1}^{-1} \sum_{i=0}^{j-i} v_i)^{x_j - x_{j-1}} \qquad \text{if } P = 1 \end{aligned} \tag{4.13}$$

where $P_i = \sum_{i=0}^{j} p_i$ *for* $j = 0, 1, \cdots, N$ (*with* $P_N = P$).

PROOF. These generating function formulae follow from those of the Krawtchouk polynomials of order N (see Theorem 4 of Appendix 2) and similar relations for the Meixner and Laguerre polynomials (see formulae (18), (19), and (24) of Appendix 1).

Next we derive generating function formulae for the transition probability function. Thus we have the following:

THEOREM 4.2. *If we define the generating function of the transition probability function*

$$G(t; \mathbf{v}) = \sum_{\mathbf{m}'} P(t; \mathbf{m}, \mathbf{m}') v_0^{m_0'} \cdots v_N^{m_N'}, \quad \text{for} \quad |\mathbf{v}| < 1,$$

where $\mathbf{v} = (v_0, \cdots, v_N)$, *then*

$$G(t; \mathbf{v}) = AB^{-(M+1)} \prod_{j=0}^{N} C_j^{m_j} \tag{4.14}$$

where with the notation

$$P = \sum_{i=0}^{N} p_i, \qquad Q = |1 - P|, \qquad E = e^{-Qt},$$
$$F = 1 - E \qquad \text{and} \qquad U = \sum_{i=0}^{N} p_i v_i \tag{4.15}$$

we have

$$\begin{aligned} A &= P^{-M} Q && \text{if } P < 1 \\ &= P^{-M} QE && \text{if } P > 1; \\ B &= 1 - PE - FU && \text{if } P < 1 \\ &= P - E - FU && \text{if } P > 1; \\ C_j &= PF - (P + Qe^{-t} - E)U + PQe^{-t} v_j && \text{if } P < 1 \\ &= PF - (1 + Qe^{-Pt} - PE)U + PQe^{-Pt} v_j && \text{if } P > 1. \end{aligned} \tag{4.16}$$

PROOF. Combining the two generating function formulae of Lemma 4.1 results in formulae (4.14) and (4.16) through the use of Theorem 4.1.

As in the two-dimensional case, the generating function formula (4.14) may be used to compute the means and higher moments of the process.

To examine recurrence we compute that

$$\begin{aligned} P(t; \mathbf{0}, \mathbf{0}) &= (1 - P)(1 - Pe^{-(1-P)t})^{-1} && \text{if } P < 1 \\ &= (t + 1)^{-1} && \text{if } P = 1 \\ &= (P - 1)(Pe^{(P-1)t} - 1)^{-1} && \text{if } P > 1. \end{aligned}$$

Since this is the same result as that obtained for the two-dimensional linear growth process, we have the equivalent of Therem 2.3:

THEOREM 4.3. *If* $\mathbf{X}(t)$ *is an* $(N + 1)$*-dimensional linear growth birth and death process whose infinitesimal birth and death rates are defined by* (4.2), *then the process is positive recurrent, null-recurrent or transient according as* $P < 1$, $P = 1$ *or* $P > 1$.

5. The associated $(N + 1)$**-dimensional absorption process.** The absorption process discussed in Section 3 may be generalized to $(N + 1)$-dimensions. The infinitesimal birth and death rates are given by

$$(5.1)\quad \begin{aligned} P(h; \mathbf{m}, \mathbf{m}') &= (m_j + 1)h + o(h) \quad \text{if } \mathbf{m}' = \mathbf{m} + \mathbf{e}_j \text{ for } j = 0, \cdots, N \\ &= (M + 1)p_j h + o(h) \\ &\qquad \text{if } \mathbf{m}' = \mathbf{m} - \mathbf{e}_j \text{ for } j = 0, \cdots, N \\ &= 1 - [(M + N) + (M + 1)P]h + o(h) \\ &\qquad \text{if } \mathbf{m}' = \mathbf{m} \\ &= o(h) \qquad \text{otherwise} \end{aligned}$$

where the same notation is used as in (4.2). This is obviously an absorption process, where integer points along the coordinate axes form "lines" of absorption. The results for this process are summarized in the following theorems and lemmas.

THEOREM 5.1 (Representation Theorem). *If* $\mathbf{X}(t)$ *is an* $(N + 1)$*-dimensional linear growth birth and death process whose infinitesimal birth and death rates are given by* (5.1), *then the transition probability function has the representation* (4.4). *The functions* $f(\mathbf{x}, y; \mathbf{m})$ *are now given by*

$$(5.2)\quad \begin{aligned} f(\mathbf{x}, y; \mathbf{m}) &= [(x_N + 1)\Pi(\mathbf{m})]^{-1} \\ &\quad \cdot \phi_{M-x_N}((y - N - x_N)(1 - P)^{-1} - 1; x_N + 2; P) \\ &\quad \cdot \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}) && \textit{if } P < 1 \\ &= [(x_N + 1)\Pi(\mathbf{m})]^{-1} P^{-(M-x_N)} \\ &\quad \cdot \phi_{M-x_N}((y - N - x_N)(P - 1)^{-1} - (x_N + 1); x_N + 2; P^{-1}) \\ &\quad \cdot \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}) && \textit{if } P > 1 \\ &= [(M + 1)\Pi(\mathbf{m})]^{-1} L_{M-x_N}^{x_N+1}(y - N - x_N)\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}) \\ &&& \textit{if } P = 1 \end{aligned}$$

where $\phi_i(z)$ *and* $L_i^{\alpha}(z)$ *are the Meixner and Laguerre polynomials, respectively, defined in Appendix* 1, *and* $\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m})$ *is the Krawtchouk polynomial of order* N *introduced in Appendix* 2. *The weight function* $\Pi(\mathbf{m})$ *and the spectral measure* $\rho(\mathbf{x}, y)$ *are now given by*

$$\Pi(\mathbf{m}) = [(M+1)\binom{M}{m_0, \cdots, m_N} p_0^{m_0} \cdots p_N^{m_N}]^{-1} \tag{5.3}$$

and

$$\rho(\mathbf{x}, y) = \binom{x_N+1}{x_1-x_0, \cdots, x_N - x_{N-1}} r(x_N, y; P) \prod_{j=1}^{N} (P_{j-1}/p_j P_j)^{x_j - x_{j-1}} \tag{5.4}$$

where

$$\begin{aligned} r(x_N, y; P) &= \binom{x_N+k+1}{k}(1-P)^{x_N+2}P^{k} && \text{if } P < 1 \\ &= \binom{x_N+k+1}{k}(1-P^{-1})^{x_N+2}P^{-k} && \text{if } P > 1 \\ &= e^{-(y-N-x_N)}(y-N-x_N)^{x_N+1}((x_N+1)!)^{-1} && \text{if } P = 1. \end{aligned} \tag{5.4'}$$

Here, $x_0 = 0$, $\mathbf{x} = (x_0, \cdots, x_N)$ *and* y *and* k *are closely associated through*

$$\begin{aligned} y &= N + x_N + (1-P)(k+1) && \text{if } P < 1 \\ &= N - 1 + P(x_N+1) + (P-1)k && \text{if } P > 1. \end{aligned} \tag{5.5}$$

PROOF. Define the matrix A of the infinitesimal birth and death rates with elements

$$\begin{aligned} a(\mathbf{m}, \mathbf{m}') &= m_j + 1 && \text{if } \mathbf{m}' = \mathbf{m} + \mathbf{e}_j \text{ for } j = 0, \cdots, N \\ &= (M+1)p_j && \text{if } \mathbf{m}' = \mathbf{m} - \mathbf{e}_j \text{ for } j = 0, \cdots, N \\ &= -[(M+N) + (M+1)P] && \\ & && \text{if } \mathbf{m}' = \mathbf{m} \\ &= 0 && \text{otherwise.} \end{aligned} \tag{5.6}$$

Then the transition probability function is the unique solution of the differential equations (4.10) with initial conditions (4.11). Direct substitution of the representation formula proves the theorem.

The actual derivation of the function $f(\mathbf{x}, y; \mathbf{m})$ is outlined in the proof of Lemma 1 of Appendix 4.

LEMMA 5.1. *If we define the generating functions*

$$G_1(\mathbf{u}, w) = \sum_{(\mathbf{x},y)} f(\mathbf{x}, y; \mathbf{m}) \binom{x_N+1}{x_1-x_0, \cdots, x_N-x_{N-1}} \binom{x_N+k+1}{k} u_1^{x_1-x_0} \cdots u_N^{x_N-x_{N-1}} w^k$$

$$\text{for} \quad |\mathbf{u}| < 1, |w| < 1,$$

and

$$G_2(\mathbf{v}) = \sum_{\mathbf{m}} f(\mathbf{x}, y; \mathbf{m}) v_0^{m_0} \cdots v_N^{m_N} \quad \text{for} \quad |\mathbf{v}| < 1$$

where $\mathbf{u} = (u_0, u_1, \cdots, u_N)$ *with* $u_0 \equiv 0$, $\mathbf{v} = (v_0, v_1, \cdots, v_N)$ *and* $f(\mathbf{x}, y; \mathbf{m})$ *and* k *are as given by* (5.2) *and* (5.5), *respectively, then the following formulae* *hold:*

$$
\begin{aligned}
G_1(\mathbf{u}, w) &= \Pi^{-1}(\mathbf{m})(1 - w)^{-(M+2)}(1 + u_N - wP^{-1})^{m_N} \\
&\quad \cdot \textstyle\prod_{j=1}^{N} (1 + u_{j-1} - wP^{-1} - \sum_{i=j}^{N} p_i P_{i-1}^{-1} u_i)^{m_j - 1} && \text{if } P < 1 \\
&= \Pi^{-1}(\mathbf{m})(1 - w)^{-(M+2)}(P^{-1} + u_N - w)^{m_N} \\
&\quad \cdot \textstyle\prod_{j=1}^{N} (P^{-1} + u_{j-1} - w - \sum_{i=j}^{N} p_i P_{i-1}^{-1} u_i)^{m_j - 1} && \text{if } P > 1 \\
&= \text{no closed form} && \text{if } P = 1
\end{aligned}
\tag{5.7}
$$

and

$$
\begin{aligned}
G_2(\mathbf{v}) &= (1 - \textstyle\sum_{i=0}^{N} p_i v_i)^{-(k+x_N+2)}(1 - P^{-1}\sum_{i=0}^{N} p_i v_i)^{k} \\
&\quad \cdot \textstyle\prod_{j=1}^{N} (p_j v_j - p_j P_{i-1}^{-1} \sum_{i=0}^{j-1} p_i v_i)^{x_j - x_{j-1}} && \text{if } P < 1 \\
&= (1 - P^{-1}\textstyle\sum_{i=0}^{N} p_i v_i)^{-(k+x_N+2)}(1 - \sum_{i=0}^{N} p_j v_j)^{k} \\
&\quad \cdot \textstyle\prod_{j=1}^{N} (p_j v_j - p_j P_{j-1}^{-1} \sum_{i=0}^{j-1} p_i v_i)^{x_j - x_{j-1}} && \text{if } P > 1 \\
&= (1 - \textstyle\sum_{i=0}^{N} p_i v_i)^{-(x_N+2)} \\
&\quad \cdot \exp\left(-(y - N - x_N)(\textstyle\sum_{i=0}^{N} p_i v_i)(1 - \sum_{i=0}^{N} p_i v_i)^{-1}\right) \\
&\quad \cdot \textstyle\prod_{j=1}^{N} (p_j v_j - p_j P_{j-1}^{-1} \sum_{i=0}^{j-1} p_i v_i)^{x_j - x_{j-1}} && \text{if } P = 1
\end{aligned}
\tag{5.8}
$$

where $P_j = \sum_{i=0}^{j} p_i$ *for* $j = 0, 1, \cdots, N$ (*with* $P_N = P$).

PROOF. These generating function formulae follow from those of the Krawtchouk polynomials of order N (see Theorem 4 of Appendix 2) and similar relations for the Meixner and Laguerre polynomials (see Appendix 1).

THEOREM 5.2. *If we define the generating function of the transition probability function*

$$G(t; \mathbf{v}) = \textstyle\sum_{\mathbf{m}} P(t; \mathbf{m}, \mathbf{m}')\Pi^{-1}(\mathbf{m}')v_0^{m_0'} \cdots v_N^{m_N'} \quad \text{for} \quad |\mathbf{v}| < 1$$

where $\mathbf{v} = (v_0, \cdots, v_N)$, *then*

$$G(t; \mathbf{v}) = AB^{-(M+2)} \textstyle\prod_{j=0}^{N} C_j^{m_j} \tag{5.9}$$

where

$$A = \Pi^{-1}(\mathbf{m})P^{-M}Q^2 e^{-(N+Q)t} \tag{5.10}$$

and B *and* C_j *as well as* P, Q, E, F *and* U *are as defined by* (4.16) *and* (4.15), *respectively.*

PROOF. The proof is analogous to that of Theorem 4.2.

As in the two-dimensional case the generating function formula (5.9) may be used to compute the absorption probabilities of the process.

Finally we have the following generalization of Theorem 3.3:

THEOREM 5.3. *Let* $f(\mathbf{x}, y; \mathbf{m})$, $\Pi(\mathbf{m})$ *and* $\rho(\mathbf{x}, y)$ *denote the functions defined in Section* 4 *by formulae* (4.5), (4.6) *and* (4.7), *respectively. Further, let* $f^*(\mathbf{x}, y; \mathbf{m})$, $\Pi^*(\mathbf{m})$ *and* $\rho^*(\mathbf{x}, y)$ *denote the functions defined in this section by formulae* (5.2), (5.3) *and* (5.4), *respectively. Then we have the following relationships:*

$$(5.11)\quad f^*(\mathbf{x}, y; \mathbf{m}) = (M+1)\Pi(\mathbf{m})(x_N - (y-N))^{-1} \cdot [\textstyle\sum_{i=0}^{N} p_i f(\mathbf{x}, y-N; \mathbf{m}+\mathbf{e}_i) - Pf(\mathbf{x}, y-N; \mathbf{m})],$$

$$(5.12)\quad \Pi^*(\mathbf{m}) = [(M+1)\Pi(\mathbf{m})]^{-1},$$

$$(5.13)\quad \textit{and} \quad \rho^*(\mathbf{x}, y) = \rho(\mathbf{x}+\mathbf{E}, y+P-N)$$

where $\mathbf{E}$ *is an* $(N+1)$*-dimensional vector with all its components equal to one.*

PROOF. For all cases of P we may write

$$f(\mathbf{x}, y-N; \mathbf{m}) = \psi(M)\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m})$$

where $\psi(M)$ is a function of N, x_N, y and P as well as M. Then, by Theorem 1 of Appendix 2, we have

$$\begin{aligned}\textstyle\sum_{i=0}^{N} p_i f(\mathbf{x}, y-N; \mathbf{m}+\mathbf{e}_i) &= \psi(M+1)\textstyle\sum_{i=0}^{N} p_i\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}+\mathbf{e}_i)\\ &= \psi(M+1)P(M-x_N+1)(M+1)^{-1}\\ &\quad\cdot\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}).\end{aligned}$$

Thus, to prove formula (5.11) we must show that

$$(5.14)\quad P\Pi_{\mathbf{m}}(x_N-(y-N))^{-1}[(M-x_N+1)\psi(M+1)-(M+1)\psi(M)] = \psi^*(M)$$

where $\psi^*(M)$ is defined through

$$f^*(\mathbf{x}, y; \mathbf{m}) = \psi^*(M)\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m})$$

for all cases of P. But equation (5.14) is equivalent to formula (14) or (15) of Appendix 1 for Meixner polynomials according as $P < 1$ or $P > 1$. When $P = 1$, equation (5.14) is equivalent to formula (22) of Appendix 1 for Laguerre polynomials. Formula (5.12) is obvious and (5.13) may be proved directly by comparing formulae (4.7) and (5.4).

Appendix 1. *The Krawtchouk, Meixner and Laguerre Polynomials.* In this section we summarize the definitions and some basic relationships of the classical Krawtchouk, Meixner and Laguerre polynomials.

The hypergeometric function is defined by (see [1])

$${}_2F_1(a, b; c; z) = \textstyle\sum_{j=0}^{\infty}(a)_j(b)_j((c)_j j!)^{-1}z^j$$

where $(a)_j = a(a+1)\cdots(a+j-1)$. Then

$$(1)\quad K_m(x) = K_m(x; p(p+q)^{-1}, M) = {}_2F_1(-m, -x; -M; (p+q)q^{-1})$$

defines the Krawtchouk polynomials, where $p > 0$, $q > 0$ and $m \leqq M$ are non-negative integers ([2] and [14]).

The following two relations may be obtained for the Krawtchouk polynomials from two of the well-known relations of Gauss between contiguous hypergeometric functions (see formulae (38) and (35) on page 103 of [1]):

$$(2)\quad pK_{m+1}(x; p(p+q)^{-1}, M+1) + qK_m(x; p(p+q)^{-1}, M+1) = (p+q)(M-x+1)(M+1)^{-1}K_m(x; p(p+q)^{-1}M)$$

and

$$(3)\quad mK_{m-1}(x; p(p+q)^{-1}, M-1) + nK_m(x; p(p+q)^{-1}, M-1) = MK_m(x; p(p+q)^{-1}, M)$$

where $M = m + n$.

The Krawtchouk polynomials satisfy ([5], [7]) the recursive relations

$$(4)\quad -xK_m(x) = mq(p+q)^{-1}K_{m-1}(x) + nq(p+q)^{-1}K_{m+1}(x) - [mq(p+q)^{-1} + np(p+q)^{-1}]K_m(x)$$

where $M = m + n$.

The Krawtchouk polynomials are also known to satisfy orthogonal and dual orthogonal relations ([7]):

$$(5)\qquad \textstyle\sum_{x=0}^{M} K_m(x)K_{m'}(x)\rho(x) = \delta_{m,m'}\Pi_m^{-1}$$

and

$$(6)\qquad \textstyle\sum_{m=0}^{M} K_m(x)K_m(x')\Pi_m = \delta_{x,x'}(\rho(x))^{-1}$$

where

$$(7)\qquad \Pi_m = \binom{M}{m}p^m q^n,$$

$$(8)\qquad \rho(x) = \binom{M}{x}(pq^{-1})^x(p+q)^{-M}$$

and $M = m + n$. There are two generating functions associated with the Krawtchouk polynomials ([2] and [14]):

$$(9)\qquad \textstyle\sum_{x=0}^{M} K_m(x)\binom{M}{x}u^x = (1+u)^n(1-qp^{-1}u)^m$$

and

$$(10)\qquad \textstyle\sum_{m=0}^{M} K_m(x)\binom{M}{m}v^m = (1+v)^{M-x}(1-qp^{-1}v)^x$$

where $M = m + n$.

Next we define the Meixner polynomials as

$$(11)\qquad \phi_i(z) = \phi_i(z; \beta, \gamma) = {}_1F_2(-i, -z; \beta; 1-\gamma^{-1})$$

for $\beta > 0$ and $0 < \gamma < 1$, although it is more customary (see [2]) to take $(\beta)_i$ times the above function as the definition of Meixner polynomials. In [5], Karlin and McGregor find the Meixner polynomials (11) as the solutions of recursive equations and thereby entering the representation formula for the one-dimensional linear growth process. In particular, the recursive equations

$$(12)\quad -zQ_i(z) = (i+\beta-1)\lambda Q_{i-1}(z) + (i+1)\mu Q_{i+1}(z) - [(i+1)\mu + (i+\beta-1)\lambda]Q_i(z)$$

are satisfied by

$$(13)\quad Q_i(z) = \binom{i+\beta-1}{i}\phi_i(z(\mu-\lambda)^{-1} - (\beta-1); \beta; \lambda\mu^{-1}) \qquad \text{if } \lambda < \mu$$
$$= \binom{i+\beta-1}{i}(\mu\lambda^{-1})^i\phi_i(z(\lambda-\mu)^{-1} - 1; \beta; \mu\lambda^{-1}) \qquad \text{if } \mu < \lambda.$$

We further have the following relations for the function $\phi_i(z; \beta, \gamma)$ (see page 85 of [11] and formula (38) on page 103 of [1]):

$$(14)\quad \phi_{i+1}(z;\beta;\gamma) - \phi_i(z;\beta;\gamma) = z\beta^{-1}(1-\gamma^{-1})\phi_i(z-1;\beta+1;\gamma);$$

$$(15)\quad \phi_{i+1}(z;\beta;\gamma) - \gamma^{-1}\phi_i(z;\beta;\gamma) = (\beta+z)\beta^{-1}(1-\gamma^{-1})\phi_i(z;\beta+1;\gamma).$$

The orthogonality relations and generating functions of the function $\phi_i(z;\beta;\gamma)$ as given in [2] and [5] are as follows:

$$(16)\quad \sum_{z=0}^{\infty}\phi_i(z;\beta;\gamma)\phi_{i'}(z;\beta;\gamma)\binom{z+\beta-1}{z}\gamma^z(1-\gamma)^\beta = \delta_{i,i'}/\binom{i+\beta-1}{i}\gamma^i;$$

$$(17)\quad \sum_{i=0}^{\infty}\phi_i(z;\beta;\gamma)\phi_i(z';\beta;\gamma)\binom{i+\beta-1}{z}\gamma^i = \delta_{z,z'}/\binom{z+\beta-1}{z}\gamma^z(1-\gamma)^\beta.$$

Further

$$(18)\quad \sum_{z=0}^{\infty}\phi_i(z;\beta;\gamma)\binom{z+\beta-1}{z}s^z = (1-s\gamma^{-1})^i(1-s)^{-(i+\beta)} \quad \text{for } |s| < 1$$

and

$$(19)\quad \sum_{i=0}^{\infty}\phi_i(z;\beta;\gamma)\binom{i+\beta-1}{i}s^i = (1-s\gamma^{-1})^z(1-s)^{-(z+\beta)} \quad \text{for } |s| < 1.$$

Finally, we summarize the formulae to be used in this paper concerning the the Laguerre polynomials. These are defined in [2] and [14] as

$$L_i^\alpha(z) = \sum_{v=0}^{i}\binom{i+\alpha}{i-\nu}(-z)^\nu(v!)^{-1}.$$

Then the function

$$(20)\quad Q_i(z) = L_i^\alpha(z)$$

turns out to be the solution to the recurrence relations (see [5])

$$(21)\quad -zQ_i(z) = (i+1)Q_{i+1}(z) + (i+\alpha)Q_{i-1}(z) - (2i+\alpha+1)Q_i(z).$$

We also have the relations (see [2])

$$(22)\quad (i+1)L_{i+1}^\alpha(z) - (i+\alpha+1)L_i^\alpha(z) = (-z)L_i^{\alpha+1}(z).$$

The orthogonality relations and generating functions are (see [2], [5], and [14]):

$$(23)\quad \int_0^\infty L_i^\alpha(z)L_{i'}^\alpha(z)e^{-z}z^\alpha(\Gamma(\alpha+1))^{-1}\,dz = \delta_{i,i'}\binom{i+\alpha}{i}$$

and

(24) $\sum_{i=0}^{\infty} L_i^{\alpha}(z)s^i = (1 - s)^{-(\alpha+1)} \exp(-zs(1 - s)^{-1})$ for $|s| < 1$.

There are no simple "dual" orthogonality or generating function relations available for the Laguerre polynomials.

Appendix 2. *The Krawtchouk Polynomials of Order N.* In this section we shall generalize the Krawtchouk polynomials to higher dimensions. We give the following:

DEFINITION. Let $\mathbf{x} = (x_0, \cdots, x_N)$ and $\mathbf{m} = (m_0, \cdots, m_N)$ be $(N + 1)$-dimensional vectors of non-negative integer components such that $x_0 = 0$ and $x_{j-1} \leqq x_j \leqq M_j$ for $j = 1, \cdots, N$, where $M_j = \sum_{i=0}^{j} m_i$ for $j = 0, \cdots, N$. Let $\mathbf{p} = (p_0, \cdots, p_N)$ be a vector of positive real numbers and $P_j = \sum_{i=0}^{j} p_i$ for $j = 0, \cdots, N$. Then the function

(1) $$\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}) = \prod_{j=1}^{N} \binom{M_{j-1}}{x_{j-1}} \binom{M_j}{x_{j-1}}^{-1} \cdot K_{M_{j-1}-x_{j-1}}(x_j - x_{j-1}; P_{j-1}P_j^{-1}, M_j - x_{j-1})$$

is called the Krawtchouk polynomial of order N.

Note that this function is actually a semi-direct product of N ordinary Krawtchouk polynomials and for $N = 1$ reduces to an ordinary Krawtchouk polynomial.

This definition will be justified by showing that all the properties listed in Appendix 1 of ordinary Krawtchouk polynomials have natural extensions valid for Krawtchouk polynomials of order N. As a generalization for formulae (2) and (3) of Appendix 1, we first prove the following:

THEOREM 1. *The Krawtchouk polynomials of order N satisfy the relations*

(2) $$\sum_{i=0}^{N} p_i \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m} + \mathbf{e}_i) = P_N(M_N - x_N + 1)(M_N + 1)^{-1}\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m})$$

and

(3) $$\sum_{j=0}^{N} m_j \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m} - \mathbf{e}_j) = M_N \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m})$$

where $\mathbf{e}_i$ *denotes the* $(N + 1)$*-dimensional vector whose* ith *component is one and all others vanish.*

PROOF BY INDUCTION ON N. For $N = 1$ these equations reduce to formulae (2) and (3), respectively, of Appendix 1. To facilitate our induction proof it will be helpful to indicate the dimensionality of vectors through the use of superscripts. Thus, let $\mathbf{x}^N = (x_0, \cdots, x_N)$, $\mathbf{m}^N = (m_0, \cdots, m_N)$, and $\mathbf{p}^N = (p_0, \cdots, p_N)$. Further, we shall use the abbreviated notations

$$\mathcal{K}^N = \mathcal{K}(\mathbf{x}^N; \mathbf{p}^N, \mathbf{m}^N);$$

$$\mathcal{K}^N(\mathbf{e}_i) = \mathcal{K}(\mathbf{x}^N; \mathbf{p}^N, \mathbf{m}^N + \mathbf{e}_i^N);$$

$$\mathcal{K}^N(-\mathbf{e}_j) = \mathcal{K}(\mathbf{x}^N; \mathbf{p}^N, \mathbf{m}^N - \mathbf{e}_j^N).$$

From the definition directly we have the recursion formula

$$(1') \quad \mathcal{K}^{N+1} = \binom{M_N}{x_N}\binom{M_{N+1}}{x_N}^{-1} K_{M_N - x_N}(x_{N+1} - x_N;\, P_N P_{N+1}^{-1},\, M_{N+1} - x_N)\mathcal{K}^N.$$

Next we assume that formula (2) holds for N and proceed to prove it for $N + 1$. Using the recursion formula (1′) and the induction assumption, we have

$$\begin{aligned}
\sum_{i=0}^{N+1} p_i \mathcal{K}^{N+1}(\mathbf{e}_i) &= \binom{M_N+1}{x_N}\binom{M_{N+1}+1}{x_N}^{-1} K_{m+1}(x;\, p(p+q)^{-1},\, m+n+1) \sum_{i=0}^{N} p_i \mathcal{K}^N(\mathbf{e}_i) \\
&\quad + p_{N+1}\mathcal{K}^{N+1}(\mathbf{e}_{N+1}) \\
&= \binom{M_N}{x_N}\binom{M_{N+1}+1}{x_N}^{-1}\{pK_{m+1}(x;\, p(p+q)^{-1},\, m+n+1) \\
&\qquad + qK_m(x;\, p(p+q)^{-1},\, m+n+1)\}\mathcal{K}^N
\end{aligned}$$

where we have temporarily put

$$x = x_{N+1} - x_N, \quad m = M_N - x_N, \quad n = m_{N+1}, \quad p = P_N \text{ and } q = p_{N+1}.$$

Then by formula (2) of Appendix 1 and the recursion formula (1′) again we obtain

$$\sum_{i=0}^{N+1} p_i \mathcal{K}^{N+1}(\mathbf{e}_i) = P_{N+1}(M_{N+1} - x_{N+1} + 1)(M_{N+1} + 1)^{-1}\mathcal{K}^{N+1}.$$

This completes the proof of formula (2). The proof of formula (3) proceeds along similar lines. We omit the details.

Next we generalize the recurrence relations (4) of Appendix 1 by stating

THEOREM 2. *The Krawtchouk polynomials of order N satisfy the recursion relations*

$$(4) \quad P(M_N - x_N)\mathcal{K}(\mathbf{x};\, \mathbf{p},\, \mathbf{m}) = \sum_{i,j=0}^{N} p_i m_j \mathcal{K}(\mathbf{x};\, \mathbf{p},\, \mathbf{m} + \mathbf{e}_i - \mathbf{e}_j).$$

PROOF BY INDUCTION ON N. For $N = 1$, equation (4) reduces to formula (4) of Appendix 1. In proving formula (4) for $N + 1$, assuming it valid for N we shall make use of the notation introduced in the proof of Theorem 1. First, through the use of recursion formula (1′) we have

$$\begin{aligned}
S &= \sum_{i,j=0}^{N+1} p_i m_j \mathcal{K}^{N+1}(\mathbf{e}_i - \mathbf{e}_j) \\
&= \binom{M_N}{x_N}\binom{M_{N+1}}{x_N}^{-1}\{K_m(x)\sum_{i,j=0}^{N} p_i m_j \mathcal{K}^N(\mathbf{e}_i - \mathbf{e}_j) \\
&\quad + m_{N+1}(M_N + 1)(M_N - x_N + 1)^{-1}K_{m+1}(x)\sum_{i=0}^{N} p_i \mathcal{K}^N(\mathbf{e}_i) \\
&\quad + p_{N+1}(M_N - x_N)M_N^{-1}K_{m-1}(x)\sum_{j=0}^{N} m_j \mathcal{K}^N(-\mathbf{e}_j) + m_{N+1}p_{N+1}K_m(x)\mathcal{K}^N\}
\end{aligned}$$

where

$$\begin{aligned}
K_m(x) &= K_m(x;\, p(p+q)^{-1},\, m+n) \\
&= K_{M_N - x_N}(x_{N+1} - x_N;\, P_N P_{N+1}^{-1},\, M_{N+1} - x_N).
\end{aligned}$$

Next we apply the induction assumption as well as both equations of Theorem 1. Then

$$S = \binom{M_N}{x_N}\binom{M_{N+1}}{x_N}^{-1}\mathcal{K}^N\{pmK_m(x) + pnK_{m+1}(x) + qmK_{m-1}(x) + qnK_m(x)\}.$$

Finally, we use the recursion formula (4) of Appendix 1 of ordinary Krawtchouk polynomials. Then

$$S = \binom{M_N}{x_N}\binom{M_{N+1}}{x_N}^{-1}\mathcal{K}^N(p+q)(m+n-x)K_m(x) = P(M_{N+1} - x_{N+1})\mathcal{K}^{N+1}.$$

This completes the proof of the theorem.

As generalizations of the orthogonality relations (see (5) and (6) of Appendix 1) and the generating function formulae (see (9) and (10) of Appendix 1) for ordinary Krawtchouk polynomials, we have the following theorems.

THEOREM 3. *The Krawtchouk polynomials of order N satisfy the orthogonality relations*

$$\sum_{\mathbf{x}} \mathcal{K}(\mathbf{x}, \mathbf{p}, \mathbf{m})\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}')\rho(\mathbf{x}) = \delta(\hat{\mathbf{m}}, \hat{\mathbf{m}}')\Pi_{\mathbf{m}}^{-1} \tag{5}$$

and dual orthogonality relations

$$\sum_{\hat{\mathbf{m}}} \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m})\mathcal{K}(\mathbf{x}'; \mathbf{p}, \mathbf{m})\Pi_{\mathbf{m}} = \delta(\mathbf{x}, \mathbf{x}')(\rho(\mathbf{x}))^{-1} \tag{6}$$

where $\hat{\mathbf{m}}$ *denotes the N-dimensional vector* $\hat{\mathbf{m}} = (m_0, \cdots, m_{N-1})$ *and*

$$\Pi(\mathbf{m}) = \binom{M}{m_0, \ldots, m_N} p_0^{m_0} \cdots p_N^{m_N}, \tag{7}$$

$$\rho(\mathbf{x}) = \binom{M}{x_1 - x_0, \ldots, x_N - x_{N-1}} P^{x_N - M} \prod_{j=1}^{N} (P_{j-1}/p_j P_j)^{x_j - x_{j-1}} \tag{8}$$

with $\binom{M}{m_0, \ldots, m_N} = M!/m_0! \cdots m_N!$ *being the multinomial coefficient.*

THEOREM 4. *The Krawtchouk polynomials of order N possess the following generating function formulae:*

$$\sum_{\mathbf{x}} \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}) \binom{M}{x_1 - x_0, \ldots, x_N - x_{N-1}} u_1^{x_1 - x_0} \cdots u_N^{x_N - x_{N-1}} = (1 + u_N)^{m_N} \prod_{j=1}^{N} \Big(1 + u_{j-1} - \sum_{i=j}^{N} p_i P_{i-1}^{-1} u_i\Big)^{m_{j-1}} \quad \text{for} \quad |\mathbf{u}| < 1$$

and

$$\sum_{\hat{\mathbf{m}}} \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m}) \binom{M}{m_0, \ldots, m_N} v_0^{m_0} \cdots v_{N-1}^{m_{N-1}} = \Big(\sum_{i=0}^{N} v_i\Big)^{M - x_N} \prod_{j=1}^{N} \Big(v_j - p_j P_{j-1}^{-1} \sum_{i=0}^{j-1} v_i\Big)^{x_j - x_{j-1}} \quad \textit{for} \quad |\mathbf{v}| < 1$$

where $\mathbf{u} = (u_0, \cdots, u_N)$ *with* $u_0 = 0$ *and* $\mathbf{v} = (v_0, \cdots, v_N)$ *with* $v_N = 1$.

The proofs of Theorems 3 and 4 may be carried out by induction in a routine manner and are omitted here.

Appendix 3. *Derivation of the Functions* $f(x, y; m, n)$.

LEMMA 1. *Let A be the matrix of infinitesimal birth and death rates defined by either formula* (2.9) *or* (3.7). *Then as* $x = 0, \cdots, M$, *y is given by formula* (2.8) *or* (3.6), *respectively, and k runs over the non-negative integers, the functions* $f(x, y; m, n)$, *given by formulae* (2.5) *or* (3.3), *respectively, form a complete set of orthonormal eigenvectors with eigenvalues* $-y$ *for the matrix A when* $p + q \neq 1$. *For the case* $p + q = 1$ *the same statement holds with the difference that y must* *take values* $\geqq x$ *continuously. Thus, we have the orthogonality relations*

$$\sum_{(x,y)} f(x, y; m, n) f(x, y; m', n') \rho(x, y) = \delta_{m,m'}\delta_{n,n'}/\Pi(m, n).$$

When $p + q \neq 1$, *we also have the dual orthogonality relations*

$$\sum_{m,n=0}^{\infty} f(x, y; m, n) f(x', y'; m, n) \Pi(m, n) = \delta_{x,x'}\delta_{y,y'}/\rho(x, y)$$

where the weight function $\Pi(m, n)$ *is given by formula* (2.6) *or* (3.4), *respectively, and the spectral measure* $\rho(x, y)$ *is given by formula* (2.7) *or* (3.5), *respectively.*

PROOF. Let $\Pi(m, n)$ be defined by either formula (2.6) or (3.4). Then the elements of A satisfy the symmetry relations

$$a((m, n), (m', n'))\Pi(m, n) = a((m', n'), (m, n))\Pi(m', n'). \tag{1}$$

Next we consider the space $\mathfrak{F}$ of all complex valued functions $f(m, n)$ of pairs of non-negative integers (m, n) such that $\sum_{m,n=0}^{\infty} |f(m, n)|^2 \Pi(m, n) < \infty$ and define the inner product

$$(f, g) = \sum_{(m,n)} f(m, n)\bar{g}(m, n)\Pi(m, n)$$

for all $f, g \in \mathfrak{F}$. Further, we introduce the linear transformation $g = Af$ in $\mathfrak{F}$ as

$$g(m, n) = \sum_{(m',n')} a((m, n), (m', n'))f(m', n')$$

for all $f \in \mathfrak{F}$. Due to the symmetry relations (1) this transformation is Hermitian with respect to weights $\Pi(m, n)$, i.e., $(Af, g) = (f, Ag)$. Because of the Hermitian property matrix A possesses a complete set of orthonormal eigenvectors. The method of finding the eigenvectors (and eigenvalues) of A is based on the classical technique of separation of variables. The technique is demonstrated here for the matrix A as defined by formula (2.9). We find we must solve the equation

$$\begin{aligned} -yf(m, n) = {} & (M + 1)p\, f(m + 1, n) + (M + 1)q\, f(m, n + 1) \\ & + m\, f(m - 1, n) + n\, f(m, n - 1) \\ & - [(M + 1)(p + q) + M]\, f(m, n). \end{aligned} \tag{2}$$

We use the technique of separation of variables as suggested by [9]. Let

$$f(m, n) = Q_M(y)g(m, n), \quad \text{with} \quad M = m + n \tag{3}$$

where the functions on the right hand side depend only on the parameters indicated through their notation. Then with a_M and b_M defined such that

$$\begin{aligned} p\, g(m + 1, n) + q\, g(m, n + 1) &= a_M\, g(m, n), \\ m\, g(m - 1, n) + n\, g(m, n - 1) &= b_M\, g(m, n), \end{aligned} \tag{4}$$

we have, from equation (2),

$$\begin{aligned} -yQ_M(y) = {} & (M + 1)a_M Q_{M+1}(y) + b_M Q_{M-1}(y) \\ & - [(M + 1)(p + q) + M]Q_M(y). \end{aligned} \tag{5}$$

From equations (4) we have

$$a_M b_{M+1}\, g(m, n) = p[(m + 1)\, g(m, n) + n\, g(m + 1, n - 1)] + q[m\, g(m - 1, n + 1) + (n + 1)\, g(m, n)]$$

and

$$(6)\quad a_{M-1} b_M\, g(m, n) = m[p\, g(m, n) + q\, g(m - 1, n + 1)] + n[p\, g(m + 1, n - 1) + q\, g(m, n)].$$

From the last two equations we obtain $a_M b_{M+1} - a_{M-1} b_M = p + q$ or solving recursively

$$(7)\qquad a_{M-1} b_M = (p + q)(m - x)$$

where x is an arbitrary constant. Substituting (7) in equation (6), we obtain after rearrangement

$$-x\, g(m, n) = mq(p + q)^{-1} g(m - 1, n + 1) + nq(p + q)^{-1} g(m + 1, n - 1) - [mq(p + q)^{-1} + np(p + q)^{-1}]\, g(m, n).$$

Comparing this with the recursive relations (4) of Appendix 1 valid for the Krawtchouk polynomials, we find that

$$(8)\qquad g(m, n) = C(x; M) K_m(x; p(p + q)^{-1}, M)$$

with $C(x; M)$ an arbitrary function of x and M and $m = 0, 1, \cdots, M$. With this solution a_M and b_M can be evaluated. In fact, if we substitute solution (8) in in equations (4) and then compare them with formulae (2) and (3) of Appendix 1, we find that

$$(9)\qquad a_M = (p + q)(M - x + 1)/(M + 1) \quad \text{and} \quad b_M = M$$

provided we choose

$$C(x; M + 1)/C(x; M) = C(x; M - 1)/C(x; M) = 1$$

The function$C(x; M)$ may indeed be chosen such if, for example, $C(x;M) = C(x)$ independent of M. For simplicity we choose

$$(10)\qquad C(x; M) \equiv C(x) \equiv 1.$$

Next we put (9) in equation (5). Then after rearrangement we have

$$(11)\quad -[y - (p + q)x]Q_M(y) = (p + q)(M - x + 1)Q_{M+1}(y) + MQ_{M-1}(y) - [(M - x + 1)(p + q) + M]Q_M(y).$$

Depending on whether $p + q \neq 1$ or $p + q = 1$, we compare this equation with the recursive equations (12) or (21), respectively, of Appendix 1 satisfied by the Meixner or Laguerre polynomials, respectively. Thus we find the solution of

(11) as follows:

$$Q_M(y) = \binom{M}{x}\phi_{M-x}((y - x)(1 - (p + q))^{-1}; x + 1; p + q) \quad \text{if } p + q < 1$$

$$(12) \qquad = \binom{M}{x}(p + q)^{x-M}\, \phi_{M-x}((y - x)(p + q - 1)^{-1} - (x + 1); x + 1; (p + q)^{-1}) \quad \text{if } p + q > 1$$

$$= L^x_{M-x}(y - x) \quad \text{if } p + q = 1$$

where $\phi_i(z)$ and $L_i^\alpha(z)$ are the Meixner and Laguerre polynomials, respectively, defined in Appendix 1.

Finally, we put formulae (8), (10), and (12) in equation (3), and thereby the expressions for the eigenvectors $f(m, n) = f(x, y; m, n)$, given by formula (2.5), are verified. The procedure to verify formula (3.3) by the same technique is only slightly different from the above development. The formula for the spectral measure $\rho(x, y)$ (see 2.7) or (3.5)) may now be verified by direct computation based on the orthogonality of the Krawtchouk, Meixner and Laguerre polynomials as given by formulae (5), (16), and (23) of Appendix 1. Similar statement is valid for the proof of the dual orthogonality of the eigenvectors when $p + q \neq 1$.

Appendix 4. *Derivation of the Functions* $f(\mathbf{x}, y; \mathbf{m})$.

Lemma 1. *Let A be the matrix of infinitesimal birth and death rates defined by either* (4.9) *or* (5.6). *Then as $x_1, \cdots, x_N$ take integer values such that $x_{j-1} \leqq x_j \leqq M_j$ for $j = 1, \cdots, N$ (with $x_0 = 0$), y is given by either* (4.8) *or* (5.5), *respectively, and k runs over the non-negative integers, the functions $f(\mathbf{x}, y; \mathbf{m})$, given by either formula* (4.5) *or* (5.2), *respectively, form a complete set of orthonormal eigenvectors with eigenvalues $-y$ for the matrix A when $P \neq 1$. For the case $P = 1$ the same statement holds with the difference that y must take values $\geqq x_N$ continuously. Thus we have the orthogonality relations*

$$\sum\nolimits_{(\mathbf{x},y)} f(\mathbf{x}, y; \mathbf{m})f(\mathbf{x}, y; \mathbf{m}')\rho(\mathbf{x}, y) = \delta(\mathbf{m}, \mathbf{m}')/\Pi(\mathbf{m}).$$

When $P \neq 1$, we also have the dual orthogonality relations

$$\sum\nolimits_{\mathbf{m}} f(\mathbf{x}, y; \mathbf{m})f(\mathbf{x}', y'; \mathbf{m})\Pi(\mathbf{m}) = \delta(\mathbf{x}, \mathbf{x}')\delta_{y,y'}/\rho(\mathbf{x}, y),$$

where the weight function $\Pi(\mathbf{m})$ is given by formula (4.6) *or* (5.3), *respectively, and the spectral function $\rho(\mathbf{x}, y)$ is given by formula* (4.7) *or* (5.4), *respectively.*

Proof. Let $\Pi(\mathbf{m})$ be defined as in formula (4.6) or (5.3), respectively. Then the elements of matrix A as defined in (4.9) or (5.6), respectively, satisfy the symmetry relations

$$(1) \qquad a(\mathbf{m}, \mathbf{m}')\Pi(\mathbf{m}) = a(\mathbf{m}', \mathbf{m})\Pi(\mathbf{m}').$$

Next we consider the space $\mathfrak{F}$ of all complex valued functions $f(\mathbf{m})$ such that

$\sum_{\mathbf{m}} |f(\mathbf{m})|^2 \Pi(\mathbf{m}) < \infty$ and define the inner product

$$(f, g) = \sum_{\mathbf{m}} f(\mathbf{m}) \bar{g}(\mathbf{m}) \Pi(\mathbf{m})$$

for all $f, g \in \mathfrak{F}$. Further, we introduce the linear transformation $g = Af$ in $\mathfrak{F}$ as $g(\mathbf{m}) = \sum_{\mathbf{m}'} a(\mathbf{m}, \mathbf{m}') f(\mathbf{m}')$ for all $f \in \mathfrak{F}$. Due to the symmetry relations (1) this transformation is Hermitian with respect to weights $\Pi(\mathbf{m})$, i.e., $(Af, g) = (f, Ag)$. Because of the Hermitian property, matrix A possesses a complete set of orthonormal eigenvectors. The method of finding the eigenvalues and eigenvectors of A is sketched here for the matrix defined by formula (4.9). We must solve the equation

$$(2) \quad -yf(\mathbf{m}) = (M+1) \sum_{i=0}^{N} p_i f(\mathbf{m} + \mathbf{e}_i) + \sum_{j=0}^{N} m_j f(\mathbf{m} - \mathbf{e}_j) - [P(M+1) + M] f(\mathbf{m}).$$

As in the two-dimensional case, we employ the technique of separation of variables. Let

$$(3) \qquad f(\mathbf{m}) = Q_M(y) g(\mathbf{m})$$

where $Q_M(y)$ depends on $\mathbf{m}$ only through M and $g(\mathbf{m})$ is independent of y. Substituting (3) in equation (2), the procedure is completely analogous to that of the solution of the equation for the two-dimensional case as given in Appendix 3. Thus, by choosing a_M and b_M such that

$$(4) \quad \sum_{i=0}^{N} p_i\, g(\mathbf{m} + \mathbf{e}_i) = a_M\, g(\mathbf{m}) \quad \text{and} \quad \sum_{j=0}^{N} m_j\, g(\mathbf{m} - \mathbf{e}_j) = b_M\, g(\mathbf{m})$$

we have that $Q_M(y)$ must satisfy

$$(5) \quad -yQ_M(y) = (M+1) a_M Q_{M+1}(y) + b_M Q_{M-1}(y) - [P(M+1) + M] Q_M(y).$$

From equations (4) we obtain the equation

$$(6) \qquad a_M b_{M+1}\, g(\mathbf{m}) = \sum_{i,j=0}^{N} p_i m_j\, g(\mathbf{m} + \mathbf{e}_i - \mathbf{e}_j) + P g(\mathbf{m})$$

and, then, solving recursively, we find that

$$(7) \qquad a_M b_{M+1} = P(M - x_N + 1)$$

where x_N is an arbitrary constant. Substituting (7) in equation (6), we have

$$P(M - x_N) g(\mathbf{m}) = \sum_{i,j=0}^{N} p_i m_j\, g(\mathbf{m} + \mathbf{e}_i - \mathbf{e}_j).$$

According to Theorem 2 of Appendix 2, the solution is

$$(8) \qquad g(\mathbf{m}) = C(x_N; M) \mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m})$$

where $C(x_N; M)$ is an arbitrary function of x_N and M only, and $\mathcal{K}(\mathbf{x}; \mathbf{p}, \mathbf{m})$ is the Krawtchouk polynomial of order N introduced in Appendix 2. Then a_M and b_M are evaluated as

$$(9) \qquad a_M = P(M - x_N + 1)(M+1)^{-1} \quad \text{and} \quad b_M = M$$

after arguing as in Appendix 3 that we may choose

$$C(x_N\,;M) \equiv C(x_N) \equiv 1. \tag{10}$$

Next we put (9) in equation (5). Then, after rearrangement, we have

$$-(y - Px_N)Q_M(y) = P(M - x_N + 1)Q_{M+1}(y) + MQ_{M-1}(y) - [P(M - x_N + 1) + M]Q_M(y).$$

This equation is identical to equation (11) of Appendix 3, and thus its solutions are given by (12) of Appendix 3 with the obvious notational changes that $x = x_N$ and $p + q = P$. Using these solutions as well as (3), (8), and (10), the expressions for the eigenvectors $f(\mathbf{m}) = f(\mathbf{x}, y; \mathbf{m})$, as given by formula (4.5) of Section 4, are verified. This procedure is only slightly varied when solving for the eigenvectors of A as defined by formula (5.6). The correct expressions for the spectral measure $\rho(\mathbf{x}, y)$ (see (4.7) and (5.4)) may now be verified by direct computation based on the orthogonality and dual orthogonality relations of the Krawtchouk polynomials of order N as presented in Theorem 3 of Appendix 2 and similar formulae for the Meixner and Laguerre polynomials given in Appendix 1.

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REFERENCES

[1] ERDÉLYI, A. (editor) (1953). *Higher Transcendental Functions*, **1**. McGraw-Hill, New York.

[2] ERDÉLYI, A. (editor) (1953). *Higher Transcendental Functions*, **2**. McGraw-Hill, New York.

[3] KARLIN, S. and McGREGOR, J. L. (1957). The differential equations of birth-and-death processes and the Stieltjes moment problem. *Trans. Amer. Math. Soc.* **85** 489–546.

[4] KARLIN, S. and McGREGOR, J. L. (1957). The classification of birth-and-death processes. *Trans. Amer. Math. Soc.* **86** 366–400.

[5] KARLIN, S. and McGREGOR, J. L. (1958). Linear growth, birth-and-death processes. *J. Math. Mech.* **7** 643–662.

[6] KARLIN, S. and McGREGOR, J. L. (1960). Classical diffusion processes and total positivity. *J. Math. Analysis Appl.* **1** 163–183.

[7] KARLIN, S. and McGREGOR, J. L. (1961). The Hahn polynomials, formulas and an application. *Scripta Mathematica*, **26** 33–46.

[8] KARLIN, S. and McGREGOR, J. L. (1962). On a genetics model of Moran. *Proc. Cambridge Philos. Soc.* **58** 299–311.

[9] KARLIN, S. and McGREGOR, J. L. (1963). On some stochastic models in genetics. Technical Report No. 1, National Institute of Health, GM 10452-01A1, Stanford Univ.

[10] KARLIN, S. and McGREGOR, J. L. (1964). Ehrenfest urn models. Technical Report No. 40, Office of Naval Research, NR-047-019, Stanford Univ.

[11] KARLIN, S. and SZEGÖ, G. (1960). On certain determinants whose elements are orthogonal polynomials. *J. d'Analyze Math. Jerusalem* **8** 1–157.

[12] KENDALL, D. G. (1959). Unitary dilations of Markov transition operators and the corresponding integral representation for transition probability matrices. *Surveys in Probability and Statistics, The Herald Cramér Volume,* Stockholm, 139–161.

[13] KENDALL, D. G. (1959). Unitary dilations of one-parameter semigroups of Markov transition operators, and the corresponding integral representations for Markov processes with a countable infinity of states. *Proc. London Math. Soc.* **9** 417–431.

[14] SZEGÖ, G. (1939). *Orthogonal Polynomials.* Amer. Math. Soc. Colloquium Publications 23, New York.