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ADMISSIBILITY AND DISTRIBUTION OF SOME PROBABILISTIC FUNCTIONS OF DISCRETE FINITE STATE MARKOV CHAINS[1]

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1. Introduction and summary. In [1], Anderson and Goodman investigated statistical inference about Markov chains for random samples of independently, identically distributed chains, in [6] Goodman obtained results for single chains and in [3] Baum and Petrie studied inferences for probabilistic functions of Markov chains. Most of their results are either for a large number of identical chains or for very long chains. In this paper, we shall be mainly concerned with the distributions of certain probabilistic functions of a finite chain and the admissibility of these functions as test statistics. Some similar and related results in this respect, especially those for a two state Markov chain may be found in the works of many authors, such as Fisz [5], chapter 11, Goodman [6], Lehmann [8], pp. 115–6, and Mood [9]. The results in Goodman [6] dealt more generally with s state Markov chain with $s \geqq 2$ and certain extensions and supplementary results can be found in [2] and [7].

Throughout this paper, we shall assume that our Markov chain is defined on a finite state space and has a discrete time parameter which takes on non-negative integers. Our derivations make use of a vector representation of an arbitrary Markov chain as given in Section 2 below. In Section 3, a special sequence of probabilistic functions of a Markov chain is defined and shown to be independently, identically distributed multinomial random vectors under certain assumptions about the transition probabilities. The admissibility of test statistics based on some of these functions is shown in Section 4 and a few examples are given in Section 5.

2. Probability distribution function of a discrete finite state Markov chain. Let $s \geqq 2$ be a fixed positive integer and $\mathfrak{X}_s = \{z_1, z_2, \cdots, z_s\}$ be a finite state space consisting of the s states (vectors) $z_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{is}), i = 1, 2, \cdots, s$, where δ_{ij} is the Kronecker delta. Let $Z_0, Z_1, \cdots, Z_T$ be a Markov chain defined on $\mathfrak{X}_s$, that is

$$Z_t = (Z_{t1}, Z_{t2}, \cdots, Z_{ts}), \qquad t = 0, 1, \cdots, T,$$

where, for each t ($t = 0, 1, \cdots, T$), one and only one of the random components $Z_{t1}, Z_{t2}, \cdots, Z_{ts}$ assumes the value 1 and each of the other $(s - 1)$ components assumes the value 0. Let $Q = (q_1, q_2, \cdots, q_s)$ and $P = \|p_{ij}\|$ be the initial probability distribution and the stationary transition probability matrix of the Markov chain $Z_0, Z_1, \cdots, Z_T$. Then we may express the elements of Q and P as

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exponential functions of $z_1, z_2, \cdots, z_s$. That is, we have

$$q_i = \Pr[Z_0 = z_i] = \exp[L(Q)z_i'], \qquad i = 1, \cdots, s, \tag{2.1}$$

where $L(Q) = (\ln q_1, \cdots, \ln q_s)$ and z_i' is just z_i written as a column vector (transpose of z_i) and

$$p_{ij} = \Pr[Z_t = z_j \mid Z_{t-1} = z_i] = \exp[z_i L(P) z_j'], \qquad i, j = 1, \cdots, s, \tag{2.2}$$

where $L(P) = \|\ln p_{ij}\|$. Here, for notational convenience, we have defined $(\ln 0)0 = 0$ and $x(\ln 0)y = 0$ if either $x = 0$ or $y = 0$.

With this representation for a Markov chain, we may state

THEOREM 2.1. *Let $Q = (q_1, \cdots, q_s)$ and $P = \|p_{ij}\|$ be the initial distribution and the $s \times s$ stationary transition probability matrix of a finite state Markov chain $Z_0, \cdots, Z_T$. Then the probability density function of the chain may be written as*

$$f(z_{i_0}, \cdots, z_{i_T}) = \exp[L(Q)z_{i_0}' + \textstyle\sum_{t=1}^{T} z_{i_{t-1}} L(P) z_{i_t}'], \tag{2.3}$$

$$z_{i_t} \varepsilon \mathfrak{X}_s, \qquad t = 0, 1, \cdots, T.$$

3. Distribution of probabilistic functions of a Markov chain. We know that if the rows of the transition probability matrix $P = \|p_{ij}\|$ are identical, then $Z_1, Z_2, \cdots, Z_T$ are independently, identically distributed multinomial random vectors. This property is also shared by other functions of some Markov chains. The purpose of this section is to define a class of more general functions of $Z_0, Z_1, \cdots, Z_T$ and give a sufficient condition for these probabilistic functions to be stochastically independent.

Let $A = \|a(u, v)\|$ be an $s \times s$ matrix of positive integers such that the rows of A are s permutations of the positive integers $(1, 2, \cdots, s)$. Let h be a positive integer with the property $2 \leqq h \leqq s$ and $M = \|m_{ij}\|$ be an $s \times h$ matrix with the properties:

(3.1)
(a) each m_{ij} is a positive integer,
(b) $m_{i1} + \cdots + m_{ih} = s, \qquad i = 1, \cdots, s.$

Let $X_1, X_2, \cdots, X_T$ be T vector-valued probabilistic functions of $Z_0, Z_1, \cdots, Z_T$ defined by

$$X_t = (X_{t1}, \cdots, X_{th}), \qquad t = 1, \cdots, T, \tag{3.2}$$

where

$$X_{tj} = \textstyle\sum_{u=1}^{s} \sum_{v=M_{u(j-1)}+1}^{M_{uj}} Z_{(t-1)u} Z_{ta(u,v)}, \qquad j = 1, \cdots, h, \tag{3.3}$$

and where

$$M_{u0} = 0, \qquad u = 1, \cdots, s,$$

$$M_{uj} = m_{u1} + \cdots + m_{uj}, \qquad j = 1, \cdots, h.$$

It is easily seen that $X_1, \cdots, X_T$ is a sequence of random vectors, each defined on the state space $\mathfrak{X}_h$. These random vectors are not generally stochastically independent. However, if the transition probabilities satisfy certain mild conditions, they may become independent random vectors.

THEOREM 3.1. *Suppose the transition probability matrix* $P = \|p_{ij}\|$ *of the Markov chain* $Z_0, \cdots, Z_T$ *satisfy the property that for each* u ($u = 1, \cdots, s$) *and each* j ($j = 1, \cdots, h$), *the set of transition probabilities* $p_{ua(u,M_{u(j-1)}+1)}, \cdots, p_{ua(u,M_{uj})}$ *is a scalar* (p_j) *multiple of a* m_{uj}*-variate multinomial distribution* $\alpha_{uj} = (\alpha_{uj1}, \cdots, \alpha_{ujm_{uj}})$. *In other words, we have*

$$(3.4)\quad (p_{ua(u,M_{u(j-1)}+1)}, \cdots, p_{ua(u,M_{uj})}) = p_j(\alpha_{uj1}, \cdots, \alpha_{ujm_{uj}}) = p_j\alpha_{uj}, \quad j = 1, \cdots, h, \quad u = 1, \cdots, s,$$

where $(p_1, \cdots, p_h)$ *is an* h*-variate multinomial distribution with non-zero members. Then* $X_1, \cdots, X_T$ *are independently, identically distributed multinomial random vectors with the common distribution* $(p_1, \cdots, p_h)$.

PROOF. It is sufficient to prove that for any positive integer $\alpha \geqq 2$ and any subset $(t_1, \cdots, t_\alpha)$ of $(1, \cdots, T)$, the following identity is satisfied:

$$(3.5)\quad \Pr[X_{t_1} = z_{i_1}, \cdots, X_{t_\alpha} = z_{i_\alpha}] = \textstyle\prod_{j=1}^{\alpha} \Pr[X_{t_j} = z_{i_j}],$$

where each i_k is one of the numbers $1, \cdots, h$. We shall prove this identity by mathematical induction.

First of all, by the definition of X_t and assumption, we have, for any t ($t = 1, \cdots, T$) and any i ($i = 1, \cdots, h$),

$$(3.6)\quad \begin{aligned} \Pr[X_t = z_i] = \Pr[X_{ti} = 1] &= \textstyle\sum_{u=1}^{s} \sum_{v=M_{u(i-1)}+1}^{M_{ui}} \Pr[Z_{t-1u}Z_{ta(u,v)} = 1] \\ &= \textstyle\sum_u \Pr[Z_{t-1u} = 1] \sum_v p_{ua(u,v)} = p_i . \end{aligned}$$

Next, we shall show that for any $1 \leqq t_1 < t_2 \leqq T$ and any pair (i, k), $i, k = 1, \cdots, h$, the following is true:

$$(3.7)\quad \Pr[X_{t_1} = z_i, X_{t_2} = z_k] = \Pr[X_{t_1} = z_i] \Pr[X_{t_2} = z_k].$$

Now, if we denote by $p_{ij}^{(0)} = \delta_{ij}$ the Kronecker delta and $p_{ij}^{(n)}$ the n-step transition probability from state z_i to state z_j, $i, j = 1, 2, \cdots, s$; $n = 1, 2, \cdots$, then we have

$$(3.8)\quad \begin{aligned} &\Pr[X_{t_1} = z_i, X_{t_2} = z_k] \\ &\quad = \Pr[X_{t_1i}X_{t_2k} = 1] \\ &\quad = \textstyle\sum_{u=1}^{s} \sum_{v=M_{u(i-1)}+1}^{M_{ui}} \sum_{c=1}^{s} \sum_{d=M_{c(k-1)}+1}^{M_{ck}} \\ &\qquad\quad \Pr[Z_{t_1-1u}Z_{t_1a(u,v)}Z_{t_2-1c}Z_{t_2a(c,d)} = 1] \\ &\quad = \textstyle\sum_u \Pr[Z_{t_1-1u} = 1] \sum_v p_{ua(u,v)} \sum_c p_{a(u,v)c}^{(t_2-t_1-1)} \sum_d p_{ca(c,d)} \\ &\quad = p_ip_k = \Pr[X_{t_1} = z_i] \Pr[X_{t_2} = z_k]. \end{aligned}$$

This completes the proof of (3.7).

Finally, assuming that (3.5) is true for $\alpha = 2, \cdots, n$, we need to prove that it is also true for $\alpha = n + 1$. However, this can be easily proved by using the same argument as that in proving (3.7).

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1, *the probability distribution of the sum*

$$s_{hT} = X_1 + X_2 + \cdots + X_T$$

is given by the following multinomial density

$$\begin{aligned}(3.9)\quad \Pr[S_{hT} &= (T_1, T_2, \cdots, T_{h-1}, T - T_1 - \cdots - T_{h-1})] \\ &= T!\,[T_1! \cdots (T - T_1 - \cdots - T_{h-1})!]^{-1} p_1^{T_1} \cdots p_h^{T-T_1-\cdots-T_{h-1}}.\end{aligned}$$

In the case $h = 2$, we have the binomial distribution

$$(3.10)\qquad \Pr[S_{2T} = (x, T - x)] = \binom{T}{x} p^x q^{T-x}, \quad x = 0, 1, 2, \cdots, T,$$

where $p = p_1$ *and* $q = p_2$.

4. Some admissible tests based on S_{2T}. We shall now consider the problem of testing statistical hypotheses about the transition probability matrix. We are concerned only with the case where $h = 2$ and assumptions (3.4) in Theorem 3.1 hold.

Let r_0 and r_1 be two given numbers such that $0 < r_0, r_1 < 1$ and let

$$(4.1)\qquad \overset{0}{\alpha}_{ij} = (\overset{0}{\alpha}_{ij1}, \cdots, \overset{0}{\alpha}_{ijm_{ij}}), \qquad i = 1, \cdots, s; \qquad j = 1, 2,$$

be $2s$ distributions (known or unknown). Let H_k, $k = 0, 1, 2, 3, 4$, denote the following hypotheses:

$$(4.2)\qquad \begin{aligned} &H_0 : p = r_0, && \alpha_{ij} = \overset{0}{\alpha}_{ij}, \quad i = 1, 2, \cdots, s; j = 1, 2,\\ &H_1 : p = r_1, && \alpha_{ij} = \overset{0}{\alpha}_{ij}, \quad i = 1, 2, \cdots, s; j = 1, 2,\\ &H_2 : p > r_0, && \alpha_{ij} = \overset{0}{\alpha}_{ij}, \quad i = 1, 2, \cdots, s; j = 1, 2,\\ &H_3 : p < r_0, && \alpha_{ij} = \overset{0}{\alpha}_{ij}, \quad i = 1, 2, \cdots, s; j = 1, 2,\\ &H_4 : p \neq r_0, && \alpha_{ij} = \overset{0}{\alpha}_{ij}, \quad i = 1, 2, \cdots, s; j = 1, 2.\end{aligned}$$

We shall derive four optimum tests for testing H_0 against the four alternative hypotheses H_1, H_2, H_3 and H_4 respectively. In the following derivations, the transition probability matrices under the five hypotheses H_0, H_1, H_2, H_3, H_4 will be denoted by P_0, P_1, P_2, P_3, P_4 respectively.

THEOREM 4.1. *Suppose the initial distribution Q remains fixed (known or unknown). Then the most powerful test for testing H_0 against H_1 is given by*

$$(4.3)\qquad \begin{aligned}\varphi_1(x) &= 1 \quad \text{for } x > c_1\\ &= d_1 \quad \text{for } x = c_1\\ &= 0 \quad \text{for } x < c_1\end{aligned}$$

if $r_0 < r_1$ or

$$(4.4)\qquad \begin{aligned}\varphi_2(x) &= 1 \quad \text{for } x < c_2\\ &= d_2 \quad \text{for } x = c_2\\ &= 0 \quad \text{for } x > c_2\end{aligned}$$

if $r_0 > r_1$, *where* (c_1, d_1) *and* (c_2, d_2) *are determined by a preassigned level of significance and the value* r_0 *and where* x *is given by* $x = x_{11} + x_{21} + \cdots + x_{T1}$ *with*

$$x_{t1} = \sum_{u=1}^{s} \sum_{v=1}^{m_{u1}} \delta_{i_{t-1}u}\delta_{i_t a(u,v)}, \qquad t = 1, \cdots, T.$$

PROOF. We shall prove the theorem only for the case $r_0 < r_1$. The proof for the case $r_0 > r_1$ is similar.

By Theorem 2.1 and Neyman-Pearson's fundamental lemma, for any given level of significance, say α, the most powerful test is easily seen to be given by

$$\begin{aligned} \varphi(z_{i_0}, \cdots, z_{i_T}) &= 1 \quad \text{for} \quad R(z_{i_0}, \cdots, z_{i_T}) > c \\ &= d \quad \text{for} \quad R(z_{i_0}, \cdots, z_{i_T}) = c \\ &= 0 \quad \text{for} \quad R(z_{i_0}, \cdots, z_{i_T}) < c \end{aligned} \tag{4.5}$$

where (c, d) is determined by (α, r_0) and

$$\begin{aligned} R(z_{i_0}, \cdots, z_{i_T}) &= \sum_{t=1}^{T} z_{i_{t-1}}[L(P_1) - L(P_0)]z'_{i_t} \\ &= (\ln r_1 - \ln r_0)x + [\ln (1 - r_1) - \ln (1 - r_0)](T - x) \\ &= [\ln r_1(1 - r_0)/r_0(1 - r_1)]x + T[\ln (1 - r_1)/(1 - r_0)]. \end{aligned}$$

Thus, the most powerful test given by (4.5) is equivalent to the test given by (4.3).

This completes the proof of Theorem 4.1.

COROLLARY 4.1. *If the initial distribution* Q *remains fixed, then* (i) *the test* φ_1 *is uniformly most powerful for testing* H_0 *against* H_2 *and* (ii) *the test* φ_2 *is uniformly most powerful for testing* H_0 *against* H_3.

COROLLARY 4.2. *If* Q *remains fixed, then the test*

$$\begin{aligned} \varphi_3(x) &= 1 \quad \text{for} \quad x < c_1' \text{ or } x > c_2' \\ &= d_3 \quad \text{for} \quad x = c_1' \text{ or } c_2' \\ &= 0 \quad \text{for} \quad c_1' < x < c_2' \end{aligned} \tag{4.6}$$

where (c_1', c_2', d_3) *is determined by* (α, r_0), *is uniformly most powerful unbiased test for testing* H_0 *against the alternative hypothesis* H_4.

We note here that if we regard $\{\alpha_{ij}^0\}$ as nuisance parameters, then the three tests φ_1, φ_2, φ_3 are actually UMP unbiased similar tests. Secondly, since the power functions are all binomial with parameter (T, p), all tests are uniformly consistent. Thirdly, we may use the same tests for testing the null hypothesis H_0 against more general alternative hypotheses by relaxing the conditions on the distributions $\{\alpha_{ij}\}$ and/or on the initial distribution $Q = (q_1, \cdots, q_s)$, as we shall show below.

COROLLARY 4.3. *Let* λ *be a positive number such that* $0 < \lambda \leqq 1/s$. *Let the initial distribution* Q *of the Markov chain be such that* $Q \geqq \boldsymbol{\lambda} = (\lambda, \cdots, \lambda)$, *that is,* $q_i \geqq \lambda$, $i = 1, 2, \cdots, s$. *Then* φ_1, φ_2, φ_3 *are asymptotically* UMP *unbiased tests for testing the null hypothesis* H_0 *against the hypotheses* H_2, H_3 *and* H_4, *respectively.*

Let us now consider the problem of testing the hypothesis H_0 against one of the following extended alternative hypotheses:

$$\begin{aligned} &H_1': \quad p = r_1, \\ &H_2': \quad p > r_0, \\ &H_3': \quad p < r_0, \\ &H_4': \quad p \neq r_0. \end{aligned} \tag{4.7}$$

Obviously, UMP unbiased tests do not exist for testing H_0 against these alternative hypotheses. However, since we have

$$H_i \subset H_i', \qquad i = 1, 2, 3, 4,$$

the following result is apparent.

THEOREM 4.2. *Under the assumptions of Theorem* 3.1 *and assuming Q remains fixed, the tests* φ_1, φ_2, φ_3 *are admissible for testing* H_0 *against the alternative hypotheses* H_1', H_2', H_3' *and* H_4', *respectively, and have the same power functions just as those for testing* H_0 *against* H_1, H_2, H_3 *and* H_4 *respectively.*

It, perhaps, should be mentioned that in Theorem 4.2 we do not require the $2s$ nuisance parameter distributions to remain the same under both the null and alternative hypotheses.

5. Some model statistics. Since the distribution of S_{2T} is binomial under either the null hypothesis or one of the various alternative hypotheses, it is rather convenient to use in applications. Furthermore, the statistic S_{2T} may be considered as a generalization of some well known statistics, especially for the case

$$m_{11} = m_{21} = \cdots = m_{s1} = 1. \tag{5.1}$$

The following are several examples.

(i) *The "sign test" model.* If (5.1) is true and $a(u, 1) = 1$, $u = 1, 2, \cdots, s$, then S_{2T} reduces to

$$S' = (\textstyle\sum_{t=1}^{T} Z_{t1}, T - \sum_{t=1}^{T} Z_{t1}),$$

and the corresponding test may be considered as a generalization of the "sign test", since, for the case $s = 2$, it is just the well known "sign test".

(ii) *The "total number of runs" model.* If (5.1) is true and $a(u, 1) = u$, $u = 1, \cdots, s$, then S_{2T} reduces to

$$S'' = (\textstyle\sum_{t=1}^{T} \sum_{u=1}^{s} Z_{t-1u}Z_{tu}, T - \sum_{t=1}^{T} \sum_{u=1}^{s} Z_{t-1u}Z_{tu}).$$

This is just the statistic "total number of runs" of the s states for our Markov chain (see chapter 11 of [5], pp. 155–6 of [8] and also [9]), since

$$T + 1 - \textstyle\sum_{t=1}^{T} \sum_{u=1}^{s} Z_{t-1u}Z_{tu}$$

is exactly the total number of runs of the s states. This statistic was first introduced by Goodman (pp. 189–91 of [6]) and later used by Barton, David and Fix [2]. We notice that the second component of S'' is used by Goodman in formula

(16) of [6] and the first and second components of S'' are called T and L by Barton, David and Fix [2].

Example (persistence in a chain of multiple events). Suppose, in a certain city, it is known that, according to past experience, the chance is $\frac{1}{3}$ for a randomly selected summer day to be one of the three types of days: rainy, cloudy or sunny. If one wishes to test the hypothesis that the chance for two consecutive days to be of the same type is $\frac{1}{3}$ (randomness) against the alternative hypothesis that it is greater than $\frac{1}{3}$, one may then denote the three types of days by $z_1 = (1, 0, 0)$, $z_2 = (0, 1, 0)$ and $z_3 = (0, 0, 1)$ and use S'' as the test statistic. In this case, one may choose the initial distribution to be $Q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and start the chain by selecting an arbitrary summer day as the observed value of Z_0.

We note that this example is a special case of a more general problem studied in the above mentioned paper by Barton, David and Fix [2] and also by Goodman [7].

(iii) *The "cyclical random walk" model.* If (5.1) is true and $a(u, 1) = u + 1$, $u = 1, 2, \cdots, s - 1$ and $a(s, 1) = 1$, then S_{2T} reduces to

$$S''' = (\textstyle\sum_{t=1}^{T} \sum_{u=1}^{s} Z_{t-1u} Z_{tu+1},\ T - \sum_{t=1}^{T} \sum_{u=1}^{s} Z_{t-1u} Z_{tu+1}),$$

where $Z_{ts+1} = Z_{t1}$. This is a UMP unbiased test statistic for testing either a one-sided or a two-sided hypothesis about p in a Markov chain with, say, the 4×4 transition probability matrix

$$P = \begin{Vmatrix} 0 & p & 0 & q \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ p & 0 & q & 0 \end{Vmatrix}$$

which is just the stochastic matrix for a cyclical random walk (see, for example, p. 386 of [4]).

(iv) *A "mixed trend" model.* We shall illustrate this model by a simple example. Let the transition probability matrix of a Markov chain with three possible states be given by

$$P = \begin{Vmatrix} \alpha_1 p & (1 - \alpha_1)p & q \\ q & \alpha_2 p & (1 - \alpha_2)p \\ q & \alpha_3 p & (1 - \alpha_3)p \end{Vmatrix}.$$

If one wishes to test the hypothesis of randomness

$$H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \tfrac{1}{2} \quad \text{and} \quad p = \tfrac{2}{3}$$

against the alternative hypothesis of "mixed trend"

$$H_1 : p > \tfrac{2}{3},$$

then one may use the test statistic

$$S^{iv} = (Y, T - Y),$$

where

$$Y = \sum_{t=1}^{T} (Z_{t-11}Z_{t1} + Z_{t-11}Z_{t2} + Z_{t-12}Z_{t2} + Z_{t-12}Z_{t3} + Z_{t-13}Z_{t2} + Z_{t-13}Z_{t3}).$$

EXAMPLE. (Social mobility.) A problem of interest in sociology may be to test the effect of the social class of father on the social class of son. That this model may be appropriate for such a test is evidenced by the following observed transition relative frequency matrix of a Markov chain with three states (upper, middle and lower social classes):

$$P = \begin{Vmatrix} .448 & .484 & .068 \\ .054 & .699 & .247 \\ .011 & .503 & .486 \end{Vmatrix}$$

(see p. 257 of [10] and also [11]).

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