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CROSS SPECTRAL ANALYSIS OF A GAUSSIAN VECTOR PROCESS IN THE PRESENCE OF VARIANCE FLUCTUATIONS

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1. Summary. Let

$$x'(t) = (x_1(t), x_2(t)), \qquad (t = 1, 2, \cdots)$$

be a two dimensional, Gaussian, vector process. Let the process $x'(t)$ have the representation

$$x'(t) = \sum_{m=0}^{p} B_m y(t - m), \tag{1.1}$$

where

$$B_m = \{b_{ijm}\,;\, i, j = 1, 2\};$$
$$y'(t) = (y_1(t), y_2(t)); \tag{1.2}$$
$$y_l(t) = \sigma_l(t)\epsilon_l(t) \qquad (l = 1, 2).$$

The random variables $\epsilon_l(t)$ are independently and normally distributed with mean zero and variance unity. p is a finite positive integer. The coefficients $B_m = (b_{ijm})_{2\times 2}$ are finite real constants, and $\sigma_l^2(t)$ are non-random sequence of positive numbers which are not, in general equal, but do satisfy the conditions

$$N^{-1}\sum_{t=1}^{N} \sigma_l^2(t) = \nu_l < \infty \qquad (\text{as } N \to \infty), \tag{1.3}$$

and

$$L \leqq \sigma_l^2(t) \leqq U < \infty \qquad (t = 1, 2, \cdots).$$

The relation (1.1) is a multivariate representation of a finite moving average process with time trending coefficients. Consider the matrix

$$F(\lambda) = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix} = G(\lambda)\begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} G^{*\prime}(\lambda), \tag{1.4}$$

where $G(\lambda) = \sum_{m=0}^{p} B_m e^{im\lambda}$ and $G^*(\lambda)$ is its complex conjugate. Under the condition (1.3), Herbst [1] has defined $f_{11}(\lambda)$ and $f_{22}(\lambda)$ as the spectral densities of the processes $x_1(t)$ and $x_2(t)$ respectively, and considered their estimation.

Here we generalize Herbst [1] results to a vector process and show that under the conditions (1.3) and (3.3) $f_{12}(\lambda)$, which is defined as the cross spectral density of the process $x_1(t)$ and $x_2(t)$, can consistently be estimated.

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2. Generalization of a cross periodogram. Let $x'(t) = (x_1(t), x_2(t))(t = 1, 2, \cdots, N)$ be a sample of size N from the process considered above. Consider the quantity

$$J_x(\lambda) = (2\pi N)^{-\frac{1}{2}} \sum_{t=1}^{N} x(t) e^{it\lambda}. \tag{2.1}$$

Substitute the expression for $x(t)$ from (1.1) in (2.1); then following Herbst [1] it can be shown that the periodogram matrix $F^{(N)}(\lambda)$ is

$$\begin{aligned} F^{(N)}(\lambda) &= J_x(\lambda) J_{x'}^{*}(\lambda) \\ &\cong G(\lambda) F_y^{(N)}(\lambda) G^{*\prime}(\lambda) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} F_y^{(N)}(\lambda) &= \{f_{ij,y}^{(N)}(\lambda); i, j = 1, 2\}, \\ f_{12,y}^{(N)}(\lambda) &= c_{12,y}^{(N)}(\lambda) + i q_{12,y}^{(N)}(\lambda), \\ q_{jj,y}^{(N)}(\lambda) &\equiv 0 \qquad (j = 1, 2), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} G(\lambda) &= (g_{ij}(\lambda); i, j = 1, 2), \\ g_{ij}(\lambda) &= \sum_{m=0}^{p} b_{ijm} e^{im\lambda}. \end{aligned}$$

The expressions for the coperiodogram $c_{12}^{(N)}(\lambda)$ and quadrature periodogram $q_{12}^{(N)}(\lambda)$ which are, respectively, the real and imaginary parts of the element $f_{12}^{(N)}(\lambda)$ of the matrix $F^{(N)}(\lambda)$ can be shown to be

$$\begin{aligned} c_{12}^{(N)}(\lambda) &= \sum_{l=1}^{2} \sum_{l'=1}^{2} [H_{ll'}^{R}(\lambda) c_{ll',y}^{(N)}(\lambda) - H_{ll'}^{I}(\lambda) q_{ll',y}^{(N)}(\lambda)], \\ q_{12}^{(N)}(\lambda) &= \sum_{l=1}^{2} \sum_{l'=1}^{2} [H_{ll'}^{R}(\lambda) q_{ll',y}^{(N)}(\lambda) + H_{ll'}^{I}(\lambda) c_{ll',y}^{(N)}(\lambda)], \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} H_{ll'}^{R}(\lambda) &= \sum_{j=0}^{p} \sum_{j'=0}^{p} b_{1lj} b_{2l'j'} \cos \lambda(j - j'), \\ H_{ll'}^{I}(\lambda) &= \sum_{j=0}^{p} \sum_{j'=0}^{p} b_{1lj} b_{2l'j'} \sin \lambda(j - j'). \end{aligned} \tag{2.5}$$

From (2.2) it follows that

$$E(F^{(N)}(\lambda)) \cong (2\pi)^{-1} G(\lambda) \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} G^{*\prime}(\lambda), \tag{2.6}$$

and hence

$$\begin{aligned} E(c_{12}^{(N)}(\lambda)) &\cong (2\pi)^{-1} \sum_{l=1}^{2} \nu_l H_{ll}^{R}(\lambda) = \phi(\lambda), \\ E(q_{12}^{(N)}(\lambda)) &\cong (2\pi)^{-1} \sum_{l=1}^{2} \nu_l H_{ll}^{I}(\lambda) = \psi(\lambda). \end{aligned} \tag{2.7}$$

In the case of a stationary Gaussian vector process, Rao [3] has shown that the quantities

$$(2\pi)^{-1} \sum_{l=1}^{2} H_{ll}^{R}(\lambda) \qquad \text{and} \qquad (2\pi)^{-1} \sum_{l=1}^{2} H_{ll}^{I}(\lambda)$$

(then the limits of j and j' of the expressions (2.5) will be from 0 to ∞) are respectively the cospectral density and quadrature spectral density of the process $x'(t)$.

When $\sigma_l(t)$ of (1.2) is independent of t (say $\sigma_l^2(t) = \sigma^2$), $\sigma^2 (2\pi)^{-1} \sum_l H_{ll}^R(\lambda)$ measures the dependence of inphase harmonics of $x_1(t)$ and $x_2(t)$, and $\sigma^2(2\pi)^{-1} \sum_l H_{ll}^I(\lambda)$ measures the dependence of out of phase components. Even if $\sigma_l^2(t)$ is not independent of t this interpretation holds. It is therefore reasonable to regard $\phi(\lambda)$ and $\psi(\lambda)$ as cospectral density and quadrature spectral density, respectively, of the process $x'(t)$.

The sampling properties of $c_{12}^{(N)}(\lambda)$ and $q_{12}^{(N)}(\lambda)$ are given in Theorem 1.

THEOREM 1. *Let the Gaussian vector process $x'(t)$ have the representation* (1.1) *and let $\sigma_l^2(t)$ satisfy the conditions* (1.3). *Then for $\lambda_1 \neq \lambda_2$,*

$$(2.8)\quad \begin{aligned} &\text{(i)} && \operatorname{cov}(c_{12}^{(N)}(\lambda_1), c_{12}^{(N)}(\lambda_2)) = O(N^{-2}),\\ &\text{(ii)} && \operatorname{cov}(q_{12}^{(N)}(\lambda_1), q_{12}^{(N)}(\lambda_2)) = O(N^{-2}),\\ &\text{(iii)} && \operatorname{cov}(c_{12}^{(N)}(\lambda_1), q_{12}^{(N)}(\lambda_2)) = O(N^{-2}). \end{aligned}$$

PROOF. From (2.4) we get

$$(2.9)\quad \begin{aligned} &\operatorname{cov}(c_{12}^{(N)}(\lambda_1), c_{12}^{(N)}(\lambda_2))\\ &\quad = \textstyle\sum_{l,l',m,m'} \{H_{ll'}^R(\lambda_1)H_{mm'}^R(\lambda_2) \operatorname{cov}(c_{ll',y}^{(N)}(\lambda_1); c_{mm',y}^{(N)}(\lambda_2))\\ &\quad - H_{ll'}^R(\lambda_1)H_{mm'}^I(\lambda_2) \operatorname{cov}(c_{ll',y}^{(N)}(\lambda_1), q_{mm',y}^{(N)}(\lambda_2))\\ &\quad - H_{ll'}^I(\lambda_1)H_{mm'}^R(\lambda_2) \operatorname{cov}(q_{ll',y}^{(N)}(\lambda_1), c_{mm',y}^{(N)}(\lambda_2))\\ &\quad + H_{ll'}^I(\lambda_1)H_{mm'}^I(\lambda_2) \operatorname{cov}(q_{ll',y}^{(N)}(\lambda_1), q_{mm',y}^{(N)}(\lambda_2))\}. \end{aligned}$$

There will be sixteen combinations of indices l, l', m, m' in each term of (2.9). For illustration we consider the evaluation of the first term. By definition we have

$$(2.10)\quad \begin{aligned} c_{ll',y}^{(N)}(\lambda) &= (2\pi N)^{-1} \textstyle\sum_{t=1}^N \sum_{s=1}^N \sigma_l(t)\sigma_{l'}(s)\epsilon_l(t)\epsilon_{l'}(s) \cos \lambda(s - t)\\ &= (2\pi N)^{-1}\epsilon_l Q_{ll'}(\lambda)\epsilon_{l'}', \end{aligned}$$

where $\epsilon_l = (\epsilon_l(1), \epsilon_l(2), \cdots, \epsilon_l(N))(l = 1, 2)$ and $\sigma_l(t)\sigma_{l'}(s) \cos \lambda(s - t)$ is the (s, t)th term of the matrix $Q_{ll'}(\lambda)$. Using the result

$$\operatorname{cov}(xPy', zQw') = \operatorname{tr} P'\Sigma^{xz}Q\Sigma^{wy} + \operatorname{tr} P\Sigma^{yz}Q\Sigma^{wx},$$

where $\Sigma^{\alpha\beta} = E(\alpha'\beta)$, it can be shown that

$$\begin{aligned} &\operatorname{cov}(c_{ll',y}^{(N)}(\lambda_1), c_{mm',y}^{(N)}(\lambda_2))\\ &\quad = A_{12}(c, \lambda_1, \lambda_2) \quad \text{if} \quad (1)\ l = 1, l' = 2, m = 2, m' = 1,\\ &\qquad\qquad\qquad\qquad\qquad\quad (2)\ l = 1, l' = 2, m = 1, m' = 2,\\ &\qquad\qquad\qquad\qquad\qquad\quad (3)\ l = 2, l' = 1, m = 1, m' = 2, \end{aligned}$$

$$(2.11)\qquad (4)\ l = 2, l' = 1, m = 2, m' = 1,$$
$$= 2A_{11}(c, \lambda_1, \lambda_2) \quad \text{if} \quad l = m = l' = m' = 1,$$
$$= 2A_{22}(c, \lambda_1, \lambda_2) \quad \text{if} \quad l = m = l' = m' = 2,$$
$$= 0 \quad \text{otherwise,}$$

where

$$(2.12)\qquad A_{ll'}(c, \lambda_1, \lambda_2) = \tfrac{1}{4}\pi^{-2} \textstyle\sum_{t=-(N-1)}^{N-1} \cos t\lambda_1 \cos t\lambda_2 S_{ll'}^2(t),$$
$$A_{ll'}(s, \lambda_1, \lambda_2) = \tfrac{1}{4}\pi^{-2} \textstyle\sum_{t=-(N-1)}^{N-1} \sin t\lambda_1 \sin t\lambda_2 S_{ll'}^2(t);$$
$$(2.13)\qquad S_{ll'}^2(t) = N^{-2} \textstyle\sum_{j=1}^{N-t} \sigma_l^2(j + t)\sigma_{l'}^2(j) \quad \text{if} \quad t > 0$$
$$= N^{-2} \textstyle\sum_{j=1}^{N+t} \sigma_{l'}^2(j - t)\sigma_l^2(j) \quad \text{if} \quad t < 0.$$

The expression for cov $(q_{12,y}^{(N)}(\lambda_1), q_{12,y}^{(N)}(\lambda_2))$ can be obtained similarly from (2.11) by replacing $A(c, \lambda_1, \lambda_2)$ by $A(s, \lambda_1, \lambda_2)$.

From (1.3), (2.12) and (2.13) we have

$$A_{ll'}(c, \lambda_1, \lambda_2) = \tfrac{1}{4}\pi^{-2} \textstyle\sum_{t=-(N-1)}^{N-1} \cos t\lambda_1 \cos t\lambda_2 S_{ll'}^2(t)$$
$$\leqq \tfrac{1}{4}U^2\pi^{-2}N^{-2} \textstyle\sum_{t=-(N-1)}^{N-1} (N - |t|) \cos t\lambda_1 \cos t\lambda_2$$
$$(2.14)\qquad = \tfrac{1}{8}U^2\pi^{-2}N^{-2}[\sin^2 \tfrac{1}{2}N(\lambda_1 + \lambda_2)/\sin^2 \tfrac{1}{2}(\lambda_1 + \lambda_2)$$
$$+ \sin^2 \tfrac{1}{2}N(\lambda_1 - \lambda_2)/\sin^2 \tfrac{1}{2}(\lambda_1 - \lambda_2)]$$
$$= O(N^{-2}) \quad \text{if} \quad \lambda_1 \neq \lambda_2;$$
$$A_{ll'}(s, \lambda_1, \lambda_2) = O(N^{-2}) \quad \text{if} \quad \lambda_1 \neq \lambda_2,$$

and

$$\textstyle\sum_{t=-(N-1)}^{N-1} \cos t\lambda_1 \sin t\lambda_2 S_{ll'}^2(t) = 0 \qquad \text{for all} \quad \lambda_1 \quad \text{and} \quad \lambda_2.$$

The result (i) of (2.8) follows from (2.9), (2.11) and (2.14), for the first term and the last term of (2.9) are of order $O(N^{-2})$ (if $\lambda_1 \neq \lambda_2$) and the two middle terms of (2.9) are zero. Similarly the results (ii) and (iii) of (2.8) can be obtained.

COROLLARY 1. *Let the conditions of Theorem 1 be satisfied. Then*

$$(2.15)\qquad \text{(i)}\ \operatorname{Var}(c_{12}^{(N)}(\lambda)) \geqq \tfrac{1}{8}L^2\pi^{-2}Z_{12}(\lambda),$$
$$\text{(ii)}\ \operatorname{Var}(q_{12}^{(N)}(\lambda)) \geqq \tfrac{1}{8}L^2\pi^{-2}Z_{12}(\lambda),$$

where

$$Z_{12}(\lambda) = (H_{12}^R(\lambda) + H_{21}^R(\lambda))^2 + 2H_{11}^{R^2}(\lambda) + 2H_{22}^{R^2}(\lambda)$$
$$+ (H_{12}^I(\lambda) + H_{21}^I(\lambda))^2 + 2H_{11}^{I^2}(\lambda) + 2H_{22}^{I^2}(\lambda).$$

PROOF. Put $\lambda_1 = \lambda_2 = \lambda$ in (2.10). Then by using (2.11) we get

$$\operatorname{Var}(c_{12}^{(N)}(\lambda)) = A_{12}(c, \lambda, \lambda)(H_{12}^R(\lambda) + H_{21}^R(\lambda))^2$$

$$(2.16) \quad + 2A_{11}(c, \lambda, \lambda)H_{11}^{R^2}(\lambda) + 2A_{22}(c, \lambda, \lambda)H_{22}^{R^2}(\lambda) + A_{12}(s, \lambda, \lambda)(H_{12}^{I}(\lambda) + H_{21}^{I}(\lambda))^2 + 2A_{11}(s, \lambda, \lambda)H_{11}^{I^2}(\lambda) + 2A_{22}(s, \lambda, \lambda)H_{22}^{I^2}(\lambda).$$

The result (i) of (2.15) can be obtained from (2.16) and the inequalities

$$(2.17) \quad \sum_{t=-(N-1)}^{N-1} \cos^2 t\lambda S_{ll'}^2(t) \geqq \tfrac{1}{8}L^2\pi^{-2} \qquad (l, l' = 1, 2),$$
$$\sum_{t=-(N-1)}^{N-1} \sin^2 t\lambda S_{ll'}^2(t) \geqq \tfrac{1}{8}L^2\pi^{-2} \qquad (l, l' = 1, 2).$$

Similarly the result (ii) can be obtained.

These results show that $c_{12}^{(N)}(\lambda)$ and $q_{12}^{(N)}(\lambda)$ do not provide consistent estimates of $\phi(\lambda)$ and $\psi(\lambda)$ respectively.

3. Heuristic treatment of sampling properties of the cross spectral estimate. Though $c_{12}^{(N)}(\lambda)$ and $q_{12}^{(N)}(\lambda)$ are respectively, asymptotically, unbiased estimates of $\phi(\lambda)$ and $\psi(\lambda)$, it follows from Theorem 1 and Corollary 1 that they are not consistent estimates of them. Hence, to ensure consistency, we consider the estimates (Rosenblatt, [2]),

$$(3.1) \quad c_{12}^*(\lambda) = \int_0^\pi w_N(\eta; \lambda)c_{12}^{(N)}(\eta)\, d\eta \cong 2\pi N^{-1} \sum_{j=0}^{[N/2]} w_N(\lambda, \omega_j)c_{12}^{(N)}(\omega_j),$$

$$(3.2) \quad q_{12}^*(\lambda) = \int_0^\pi w_N(\eta; \lambda)q_{12}^{(N)}(\eta)\, d\eta \cong 2\pi N^{-1} \sum_{j=0}^{[N/2]} w_N(\lambda, \omega_j)q_{12}^{(N)}(\omega_j),$$

where $\omega_j = 2\pi j/N (j = 0, 1, 2, \cdots, [N/2])$ and the weight function $w_N(\eta; \lambda)$ is assumed to satisfy the conditions (Rosenblatt [2], p. 253).

(3.3) (1) $\int_0^\pi w_N(y; \lambda)\, dy = 1, \qquad \int_0^\pi w_N^2(y; \lambda)\, dy < \infty,$

(2) Given any $\epsilon > 0$, $w_N(y; \lambda) \to 0$ uniformly in y for $|y - \lambda| \geqq \epsilon$.

It has to be noted that $q_{12}^*(\lambda) = 0$ at $\lambda = 0$ or π.

Using (2.7) and the conditions (3.3) it can be shown that, as $N \to \infty$,

$$(3.4) \quad E(c_{12}^*(\lambda)) \cong \int_0^\pi w_N(\lambda, \eta)\phi(\eta)\, d\eta \cong \phi(\lambda),$$
$$E(q_{12}^*(\lambda)) \cong \int_0^\pi w_N(\lambda, \eta)\psi(\eta)\, d\eta \cong \psi(\lambda).$$

THEOREM 2. *Let the conditions of Theorem 1 be satisfied. Also let the weight function $w_N(y, \lambda)$ satisfy the conditions (3.3). Then*

(i) $\operatorname{Var}(c_{12}^*(\lambda))$

$$(3.5) \quad \cong 2\pi N^{-1}\{A_{12}(c, \lambda, \lambda)(H_{12}^{R}(\lambda) + H_{21}^{R}(\lambda))^2 + 2A_{11}(c, \lambda, \lambda)H_{11}^{R^2}(\lambda) + 2A_{22}(c, \lambda, \lambda)H_{22}^{R^2}(\lambda) + A_{12}(s, \lambda, \lambda)(H_{12}^{I}(\lambda) + H_{21}^{I}(\lambda))^2 + 2A_{11}(s, \lambda, \lambda)H_{11}^{I^2}(\lambda) + 2A_{22}(s, \lambda, \lambda)H_{22}^{I^2}(\lambda)\} \int_0^\pi w_N^2(\omega, \lambda)\, d\omega;$$

(ii) Var $(q_{12}^{*}(\lambda))$

$$
\begin{aligned}
&\cong 2\pi N^{-1}\{A_{12}(s, \lambda, \lambda)(H_{12}^{R}(\lambda) + H_{21}^{R}(\lambda))^{2} \\
&\quad + 2A_{11}(s, \lambda, \lambda)H_{11}^{R^2}(\lambda) + 2A_{22}(s, \lambda, \lambda)H_{22}^{R^2}(\lambda) \\
&\quad + A_{12}(c, \lambda, \lambda)(H_{12}^{I}(\lambda) + H_{21}^{I}(\lambda))^{2} + 2A_{11}(c, \lambda, \lambda)H_{11}^{I^2}(\lambda) \\
&\quad + 2A_{22}(c, \lambda, \lambda)H_{22}^{I^2}(\lambda)\} \int_0^{\pi} w_N^{2}(\omega, \lambda)\, d\omega. \\
&= 0 \quad \text{if} \quad \lambda = 0 \quad \text{or} \quad \pi.
\end{aligned}
\tag{3.6}
$$

The right hand side expression of (i) *has to be doubled when* $\lambda = 0$ *or* π.

PROOF. Since $c_{12}^{(N)}(\lambda_1)$ and $c_{12}^{(N)}(\lambda_2)$, for $\lambda_1 \neq \lambda_2$, are asymptotically uncorrelated, we can write

$$\operatorname{Var}(c_{12}^{*}(\lambda)) \cong 4\pi^2 N^{-2} \sum_{j=0}^{[N/2]} w_N^{2}(\lambda, \omega_j) \operatorname{Var}(c_{12}^{(N)}(\omega_j)). \tag{3.7}$$

Substitute the expression for Var $(c_{12}^{(N)}(\omega_j))$ from (2.16) in (3.7). Then by using the conditions (3.3) one can arrive at the result (i). Similarly the result (ii) can be obtained.

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