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EXISTENCE OF AN INVARIANT MEASURE AND AN ORNSTEIN'S ERGODIC THEOREM[1]

By Michel Metivier

Cornell University and University of Rennes

0. Introduction. T being a Markovian positive operator acting on $L_1(\lambda)$, and prolongated to the space $M^+(\lambda)$ of all λ-equivalence classes of positive functions, we are looking for finite $f \varepsilon M^+(\lambda)$ such that $Tf = f$. By using a very skillful and deep construction due to Ornstein (cf. [8], Part III), we give an existence theorem of such a T-invariant f, T belonging to a suitable class of conservative Markovian operators (Theorem 1), and we make clear a general setting in which an ergodic theorem proved by D. Ornstein for random walks ([8]) can be stated. Namely, for any bounded function h with a suitably "bounded" support and verifying $\int h \, d\lambda = 0$, the function $\sup_n |\sum_{i=1}^n T^i h|$ is bounded by pf, p constant.

Beside this situation, which for our convenience we call "the abstract case", we are looking in Section 3 at the problem of finding a σ-finite ν such that $\nu = \nu \underline{P}$ (cf. Definition in 1.7 through 1.9), where $\underline{P}$ is a Markov kernel, and state the previous ergodic theorem in this situation which we call "concrete case." We prove in 3.2 that our hypotheses in the "concrete case" are essentially equivalent to those of Harris' theorem on invariant measures. (Cf. [2] and our Theorem 4.) We give thus an alternate proof of the Harris theorem. Moreover, by introducing natural topological hypothesis, when E is locally compact, we can state that the invariant measure is a regular Borel measure, and we are allowed to use the word "bounded" above in the usual topological sense (i.e. with compact closure. Cf. Theorem 5).

The Part 4 is devoted to ratio limit theorems for Markov kernels, strengthening N. C. Jain's result insofar as we prove that, in some cases, "almost everywhere" can be replaced by "everywhere" in Jain's statements. In fact it is proved in [5] that, if $\underline{P}$ is any Harris Markovian kernel, and if λ is a $\underline{P}$-invariant measure, for any A and B measurable with $\lambda(B) < +\infty$,

$$\lim_{n\to\infty} \sum_{k=0}^n \underline{P}^k(x, A)/\sum_{k=0}^n \underline{P}^k(y, B) = \lambda(A)/\lambda(B) \qquad \text{for every } x \text{ and } y$$

outside a λ-null set (depending on A and B). We prove in fact (Theorem 6 and 7) that, if A and B are "bounded sets" (Cf. Definition 4.1) the above limit holds for every x and y.

The main statements of Part 4 were "strongly" suggested by D. Ornstein, to whom we are much indebted for helpful discussions throughout the writing of this paper.

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1. Preliminaries.

A. The state-space.

1.1. E is a set and $\mathcal{B}$ is a σ-field of subsets of E. In some cases, which will be specified, E will be a locally compact topological space denumerable at infinity (i.e. union of denumerably many compact subsets), and $\mathcal{B}$ will be the σ-field of Borel sets of E. In the latter case a regular Borel measure is a measure λ defined on $\mathcal{B}$, finite on compact subsets of E and such that for every $B \varepsilon \mathcal{B}$ $\lambda(B) = \sup \{\lambda(K): K \text{ compact}, K \subset B\}$.
1.2. $\mathfrak{M}^b$ (respectively $\mathfrak{M}^+$) denotes the vector space (respectively the convex cone) of bounded measures (respectively of positive measures).
1.3. $\mathcal{L}_e$ (respectively $\mathcal{L}_e^+$) is the vector space (respectively the convex cone) of bounded $\mathcal{B}$-measurable real functions on E (respectively bounded positive $\mathcal{B}$-measurable).
1.4. $\mathfrak{M}^+$ is the convex cone of positive (finite or not) measurable real functions on E. If ν is any σ-finite measure we call $M^+(\nu)$ the convex cone of equivalence classes of function in $\mathfrak{M}^+$ for the equivalence relation: "equal ν almost everywhere."

B. Operators T and T^*.

1.5. λ being a σ-finite measure on $(E, \mathcal{B})$ a positive-Markovian operator T is a positive bounded operator on the space $L_1(\lambda)$ such that $\int Th\, d\lambda = \int h\, d\lambda$ for every $h \varepsilon L_1^+(\lambda)$.

The adjoint mapping $L_\infty^+(\lambda)$ in $L_\infty^+(\lambda)$ will be noted T^*.
1.6. We recall the following (cf. [7], Chapter V): For every increasing sequence (g_n) in $L_1^+(\lambda)$ such that $\sup_n g_n = g \varepsilon L_1^+(\lambda)$, we have $\sup_n Tg_n = Tg$. And using a standard procedure T can be prolongated into a linear mapping of $M^+(\lambda)$ into $M^+(\lambda)$ still denoted by T, such that for every increasing sequence (g_n) in $M^+(\lambda)$ we have

$$T(\sup_n g_n) = \sup_n T(g_n).$$

C. Markovian transition kernel and associated Markovian operator.

1.7. A Markovian transition kernel $\underline{P}$ is a positive real function defined on $E \times \mathcal{B}$, such that for every $x \varepsilon E$ $\underline{P}(x, \cdot)$ is a probability on $\mathcal{B}$, and for every $B \varepsilon \mathcal{B}$ $P(\cdot, B)$ is a $\mathcal{B}$-measurable function.
1.8. As usual we denote by $\nu \to \nu\underline{P}$ the positive-linear operator defined on $\mathfrak{M}^b$ and $\mathfrak{M}^+$ by:

$$\nu\underline{P}(B) = \int \nu(dx)P(x, B)$$

and we denote by $h \to \underline{P}h$ the positive-linear operator defined on $\mathcal{L}_e$ and $\mathfrak{M}^+$ by

$$\underline{P}h(x) = \int h(y)\underline{P}(x, dy).$$

1.9. If λ is a σ-finite measure on $(E, \mathcal{B})$ such that $\lambda\underline{P}$ is absolutely continuous with respect to λ and σ-finite, it is immediately seen from the Radon-Nikodym theorem that for every $g \varepsilon L^1(\lambda)$ the P-transform $(g\cdot\lambda)\underline{P}$ of the measure $g\cdot\lambda$ (measure defined by the density g with respect to λ), is of the form $g'\cdot\lambda$ where the class of g' in $L_1(\lambda)$ depends only on the class g in $L_1(\nu)$. So that we can define an operator T in $L_1(\lambda)$ such that

$$(Tg)\cdot\lambda = (g\cdot\lambda)\underline{P}.$$

It is at once verified that T is Markovian and that T^* is such that if $\tilde{h} \varepsilon L_\infty(\lambda)$, $T^*\tilde{h}$ = equivalence class in $L_\infty(\lambda)$ of any $\underline{P}h$ for $h \varepsilon \tilde{h}$.

T will be called the Markovian operator associated with the Markovian transition kernel $\underline{P}$.

1.10. For our convenience, when dealing with a Markovian operator in $L^1(\lambda)$, we say that we are in the "abstract case," while when dealing with the study of the operator $\underline{P}$ on $\mathfrak{M}^+$ and $\mathfrak{M}^b$ we say that we are in the "concrete case."

Conforming ourselves to a traditional abuse of language we call functions the elements of $L^1(\lambda)$ and write $f = g$ λ a.e. when f is a function belonging to the equivalence class $g \varepsilon L^1(\lambda)$.

2. T-invariant functions and Ornstein's ergodic theorem (**abstract case**), **for some conservative-operators in $L_1(\lambda)$.** In this paragraph we deal with the abstract case: T is a Markovian operator on $L_1(\lambda)$, λ being a σ-finite measure on $(E, \mathcal{B})$.

We are first looking for a T-invariant function, i.e.: an $f \varepsilon M^+(\lambda)$ such that $Tf = f$, f being finite λ a.e.

We will use an hypothesis, namely (ii), the analogous of which in the concrete case, will appear as an hypothesis on the "absolutely continuous part" of the kernel. (Cf. 3.3 below).

2.1. THEOREM 1. *Let us suppose that T is a positive Markovian operator on $L_1(\lambda)$, verifying the hypothesis:*

(i) *For every $g \varepsilon L_1(\lambda)$, $g \geqq 0$ and $\lambda[g > 0] > 0$,*

$$\sum_{k=0}^{\infty} T^k g = +\infty \qquad \lambda \text{ a.e.}$$

(i.e. *T is conservative and with only $\varnothing$ and E as invariant sets* cf. [7]).

(ii) *For each measurable set A of finite measure and each real $\epsilon > 0$, there exists a measurable $K \subset A$ with $\lambda(A/K) < \epsilon$, an integer k and a real $\alpha > 0$ such that for every measurable $J \subset K$*

$$\sum_{j=0}^{k} T^{*j} 1_J(x) \geqq \alpha\lambda(J) \qquad \lambda \text{ a.e. } on \quad K.$$

Then, there exists a T-invariant finite $f \varepsilon M^+(\lambda)$. This f is strictly positive λ a.e., *and $f\cdot\lambda$ is a σ-finite measure on $(E, \mathcal{B})$. The only T-invariant functions in $M^+(\lambda)$ are the functions $p\cdot f$ where p is a positive constant.*

PROOF. 1°. We begin with constructing a sequence (B_n) of sets of finite measure, such that $\lambda(E/\bigcup_n B_n) = 0$, for each B_n exists an integer k_n and a real

$\alpha_n > 0$ such that for every measurable $J \subset B_n$

(2.1.1) $$\sum_{j=0}^{k_n} T^{*j} 1_J(x) \geqq \alpha_n \lambda(J) \qquad \lambda \text{ a.e. on } B_n$$

and

(2.1.2) $$\forall n, m, k \quad \text{ess sup}_{y \in B_m} T^k 1_{B_n}(y) < \infty.$$

λ being σ-finite we can, using (ii), construct an increasing sequence (K_n) such that $\lambda(E/\bigcup_n K_n) = 0$ and for which the property (2.1.1) is true. There exists a $g \in L_1(\lambda)$ such that $g \geqq \eta_n \cdot 1_{K_n}$ for suitable real numbers $\eta_n > 0$.

In view of the integrability of g and $T^i g$ for every i, there exists $A_n \subset K_n$ such that $\lambda(K_n/A_n) \leqq 1/n \cdot 2^n$ and

(2.1.3) $$\sup_{y \in A_n} \sum_{i=0}^{n} T^i g(y) < +\infty \qquad \lambda \quad \text{a.e.}$$

If we write

$$B_n = \bigcap_{k \geqq n} A_k ,$$

we see easily that (2.1.1) and (2.1.2) are true, with $\lambda(E/\bigcup_n B_n) = 0$.

2°. *Construction of f.* Let u and v be two disjoint measurable sets of strictly positive λ-measure included in some of the preceding B_n. Let u and v be two positive bounded functions null outside U and V respectively and such that

(2.1.4) $$\int u \, d\lambda = \int v \, d\lambda.$$

We denote by w a bounded function null outside B_n and such that

(2.1.5) $$w \geqq \sup(u, v).$$

Such a system (U, V, u, v, w) always exists as a consequence of the hypothesis and properties of (B_n).

The following construction comes from [8]. We write

(2.1.6) $$\begin{aligned} u_1 &= u, & \tilde{v}_1 &= 0, \\ u_2 &= Tu_1 - \tilde{v}_2 , & \tilde{v}_2 &= \inf(Tu_1 , v), \\ &\vdots & &\vdots \\ u_n &= Tu_{n-1} - \tilde{v}_n , & \tilde{v}_n &= \inf\left(Tu_{n-1} , v - \textstyle\sum_{j=1}^{n-1} \tilde{v}_j\right), \end{aligned}$$

and

(2.1.7) $$\begin{aligned} v_1 &= v, & \tilde{u}_1 &= 0, \\ &\vdots & &\vdots \\ v_n &= Tv_{n-1} - \tilde{u}_n , & \tilde{u}_n &= \inf\left(Tv_{n-1} , u - \textstyle\sum_{j=1}^{n-1} \tilde{u}_j\right). \end{aligned}$$

From the definition of $\tilde{v}_n$ and $\tilde{u}_n$ we get at once:

(2.1.8) $$\forall n \quad \sum_{j=1}^{n} \tilde{v}_j \leqq v \qquad \text{and} \qquad \sum_{j=1}^{n} \tilde{u}_j \leqq u$$

and by recurrence on n, it is immediately seen that u_n and $\tilde{v}_n$ are in $L_1^{+}(\lambda)$.

As T is Markovian we get from (2.1.6) and (2.1.7)

$$\int u_n \, d\lambda = \int u \, d\lambda - \int \sum_{i=1}^{n} \tilde{v}_i \, d\lambda, \tag{2.1.9}$$

$$\int v_n \, d\lambda = \int v \, d\lambda - \int \sum_{i=1}^{n} \tilde{u}_i \, d\lambda. \tag{2.1.10}$$

The formula (2.1.6) gives too:

$$T^j\left(\sum_{i=1}^{n} u_i\right) = T^{j-1}\left(\sum_{i=2}^{n+1} u_i\right) + T^{j-1}\left(\sum_{i=2}^{n+1} \tilde{v}_i\right). \tag{2.1.11}$$

Because of (2.1.8) and by recurrence on j, we get by using (2.1.11),

$$T^j\left(\sum_{i=1}^{n} u_i\right) \leqq \sum_{i=1}^{j-1} T^i w \qquad \lambda \text{ a.s. on } \quad S_{n+j} = \left[\sum_{i=1}^{n+j} u_i = 0\right]. \tag{2.1.12}$$

We define

$$f = \sum_{i=1}^{\infty} u_i + v_i \,. \tag{2.1.13}$$

The sum being taken in the λ a.e. sense, which is meaningful since the u_i and v_i are $\geqq 0$.

3°. *We prove now that f is λ-integrable on each B_n.* Let B be any set of the sequence (B_n) such that $B \supset U \cup V$. We assume first that $\forall n \quad \lambda(B \cap S_n) \neq 0$. Then, according to (2.1.12):

$$\forall n \quad \operatorname{ess\,inf}_{y \in B \cap S_{n+k}} \sum_{j=0}^{k} T^j\left(\sum_{i=1}^{n} u_i\right)(y) \leqq \operatorname{ess\,sup}_{y \in B \cap S_{n+k}} \sum_{j=1}^{k} \sum_{i=0}^{j-1} T^i w(y). \tag{2.1.14}$$

If k, α and K are as in (ii) for every g in $M^+(\lambda)$ we have $\forall J$ measurable $\subset K$

$$\int_J \sum_{j=0}^{k} T^j g \, d\lambda = \int g \cdot \left(\sum_{j=0}^{k} T^{*j} 1_J\right) d\lambda \geqq \alpha \lambda(J) \int_K g \, d\lambda$$

which proves that for every $g \in M^+(\lambda)$

$$\sum_{j=0}^{k} T^j g(x) \geqq \alpha \int_K g \, d\lambda \qquad \lambda \text{ a.e. on } \quad K. \tag{2.1.15}$$

If we apply this to B and $g = \sum_{i=1}^{n} u_i$, we get the existence of an integer k, and a real $\alpha > 0$ such that, under the assumption $\lambda(B \cap S_n) \neq 0$ for every n

$$\forall n \quad \int_B \sum_{i=1}^{n} u_i \, d\lambda \leqq \alpha^{-1} M_B \tag{2.1.16}$$

where

$$\operatorname{ess\,inf}_{y \in B \cap S_{n+k}} \sum_{j=1}^{k} \sum_{i=0}^{j-1} T^i w(y) \leqq \operatorname{ess\,sup}_{y \in B} \sum_{j=1}^{k} \sum_{i=0}^{j-1} T^i w(y) = M_B < +\infty. \tag{2.1.17}$$

Suppose now on the contrary that $\lambda(B \cap S_{n_0}) = 0$ for some n_0. According to the definition (2.1.6) this means that on $V \subset B$ we have $v \leqq \sum_{j=1}^{n_0} \tilde{v}_j$ which, with (2.1.8), implies $v = \sum_{j=1}^{n_0} \tilde{v}_j \quad \lambda$ a.e. So that, as a consequence of (2.1.9) $u_{n_0+k} = 0 \quad \lambda$ a.e. for all $k > 0$, and

$$\int \sum_{i=1}^{\infty} u_i(x)\lambda(dx) \leqq \sum_{i=1}^{n_0} \int u_i(x)\lambda(dx) \leqq n_0 \int u(x)\lambda(dx). \tag{2.1.18}$$

We do the same reasoning for $\sum_{i=1}^{\infty} v_i$, and prove thus that in any case $\sum_{i=1}^{\infty} (u_i + v_i)$ is integrable on each B_m.

The finiteness λ almost everywhere of f and the σ-finiteness of $f\cdot\lambda$ result from this.

4°. *We prove now that* $f = \sum_{i=1}^{\infty} (u_i + v_i)$ *is T-invariant.* According to the definition

$$T\sum_{i=1}^{n} (u_i + v_i) = \sum_{i=1}^{n+1} u_i + v_i - (u + v) + \sum_{i=1}^{n+1} (\tilde{u}_i + \tilde{v}_i).$$

As T commutes with the increasing limits and as $(u + v) - \sum_{i=1}^{\infty} \tilde{u}_i + \tilde{v}_i \geqq 0$. We have

$$Tf \leqq f. \tag{2.1.19}$$

But, by the following standard argument we deduce that $Tf = f$.

As a consequence of hypothesis (i) we have either $\lambda[f - Tf > 0]) = 0$ or

$$\int (f - Tf) \sum_{k=0}^{\infty} T^{*k} 1_{B_n}\, d\lambda = +\infty. \tag{2.1.20}$$

But as this last equality is contradicted by

$$+\infty > \int f\cdot 1_{B_n}\, d\lambda \geqq \lim_k \int (f - T^{k+1}f)\cdot 1_{B_n}\, d\lambda$$
$$= \lim_k \int (f - Tf)\left(\sum_{j=0}^{k} T^{*j} 1_{B_n}\right) d\lambda \geqq 0,$$

we have $\lambda([f - Tf > 0]) = 0$.

5°. *We prove that* $\lambda[f = 0] = 0$. For every B_n, according to the reasoning leading to formula (2.1.15),

$$\alpha \int_{B_n} f(x)\lambda(dx) \leqq \operatorname{ess\,inf}_{y\in B_n} \sum_{j=0}^{k} T^k f(y) = \operatorname{ess\,inf}_{y\in B_n} k\cdot f(y) \tag{2.1.21}$$

for some (α, k) depending only on B_n, with $\alpha > 0$.

As $f \geqq u$ the left member of the last inequality is never zero for sufficiently great B_n. So that for every n $\operatorname{ess\,inf}_{y\in B_n} f(y) > 0$.

6°. *We prove the unicity of f up to a constant factor.* Consider the T-invariant function f just constructed, and let ν be the measure $f\cdot\lambda$. As f is strictly positive and finite λ a.e., we have $M^+(\nu) = M^+(\lambda)$ and $L_\infty(\nu) = L_\infty(\lambda)$. We can define a positive operator $\tilde{T}$ in $M^+(\nu)$ by

$$h \varepsilon M^+(\nu) \qquad \tilde{T}h = f^{-1}T(h\cdot f). \tag{2.1.22}$$

If $h \varepsilon L_1(\nu)$, that is $f\cdot h \varepsilon L_1(\lambda)$ we have

$$\int \tilde{T}h\, d\nu = \int T(h\cdot f)\, d\lambda = \int h\cdot f\, d\lambda = \int h\cdot d\nu. \tag{2.1.23}$$

This proves that $\tilde{T}$ restricted to $L_1(\nu)$ is a Markovian operator. Moreover, from

$$\int \tilde{T}h\cdot g\, d\nu = \int T(h\cdot f)\cdot g\, d\lambda = \int h\cdot f\cdot T^*g\, d\lambda,$$

$$\forall h \varepsilon M^+(\nu) \quad \int \tilde{T}h\cdot g\, d\nu = \int h\cdot T^*g\, d\nu, \tag{2.1.24}$$

we deduce that

$$\tilde{T}^* = T^*. \tag{2.1.25}$$

But if $h \varepsilon L^\infty(\nu)$

$$\tilde{T}h \leqq f^{-1}T(\|h\|_\infty \cdot f) = \|h\|_\infty$$

which proves that $\tilde{T}$ acts continuously on $L^\infty(\nu)$. Therefore $\tilde{T}^*$ which acts on $L^1(\nu)$ (see for example (2.1.24) above) acts also continuously on $L^1(\nu)$. Interchanging the roles of $\tilde{T}^*$ and $\tilde{T}$, and noting that $\tilde{T}^*$ is conservative with only an invariant set, we deduce from a classical result that the only invariant functions for $\tilde{T}$ are the constants (cf. [7] Chapter V). But then the only invariant functions for T are, according to (2.1.22), of the form $p \cdot f$ where p is a constant.

2.2. THEOREM 2. (Ornstein's Ergodic Theorem—Abstract case.) *Let T be a positive Markovian operator on $L^1(\lambda)$ for which the following hypotheses are true:*

(i) *For every* $g \varepsilon L^1(\lambda)$, $g \geqq 0$ and $\lambda[g > 0] > 0$,

$$\sum_{k=0}^{\infty} T^k g = +\infty \qquad \lambda \text{ a.e.}$$

(ii) *For every measurable set A of finite measure $\lambda(A)$ and every $\epsilon > 0$, there exists a measurable $K \subset A$ with $\lambda(A/K) < \epsilon$, a real number $\alpha > 0$ and an integer k such that:*

$$\forall J \text{ measurable} \subset K \quad \sum_{j=0}^{k} T^{*j} 1_J \geqq \alpha\lambda(J) \qquad \lambda \text{ a.e. on } K.$$

1°. *Then, there exists an increasing sequence (B_n) of measurable sets of finite measure such that $\lambda(E/\bigcup_n B_n) = 0$ and such that for every g bounded null outside some B_n, with $\int g\, d\lambda = 0$,*

$$\sup_m \left|\sum_{k=1}^{m} T^k g\right| \leqq p \cdot f \qquad \lambda \text{ a.e.} \tag{2.2.1}$$

where $f \varepsilon M^+(\lambda)$ is finite and T-invariant and p a constant.

2°. *The inequality in* (2.2.1) *holds uniformly with the same p and f for all g verifying $\int g\, d\lambda = 0$ and $|g| \leqq 1_{B_n}$.*

PROOF. 1°. We consider the sequence (B_n) constructed at the beginning of the proof of Theorem 1. Then we consider g null outside some B_n. We call B this B_n. We suppose $\int g\, d\lambda = 0$ and write $g = g^+ - g^-$, $g^+ = u$, $g^- = v$ and denote by w a bounded function null outside B and greater than or equal to $|g|$. We consider the functions u_1, $\tilde{v}_i$, v_i, $\tilde{u}_i$ as defined in the proof of Theorem 1 and write (2.1.6) in the following way:

$$\begin{aligned} T^n u_1 &= T^n u, \\ T^{n-1} u_2 &= T^n u_1 - T^{n-1}\tilde{v}_2, \\ &\vdots \\ u_{n+1} &= T u_n - \hat{v}_{n+1}. \end{aligned} \tag{2.2.2}$$

We get

$$T^n u = u_{n+1} + \sum_{k=0}^{n-1} T^k \tilde{v}_{n-k+1} \tag{2.2.3}$$

so that, just as in [8], we write

$$\sum_{i=1}^{n} T^i g^+ = \sum_{i=2}^{n+1} u_i + \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} T^k \tilde{v}_{i-k+1},$$
$$\sum_{i=1}^{n} T^i g^- = \sum_{i=2}^{n+1} v_i + \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} T^k \tilde{u}_{i-k+1}.$$

But as

$$\sum_{i=k+1}^{n} \tilde{v}_{i-k+1} \leqq v = g^- \quad \text{and} \quad \sum_{i=k+1}^{n} \tilde{u}_{i-k+1} \leqq u = g^+,$$

we have

$$\sum_{i=1}^{n} T^i g^+ \leqq \sum_{i=2}^{n+1} u_i + \sum_{k=0}^{n} T^k g^- \quad \text{and}$$
$$\sum_{i=1}^{n} T^i g^- \leqq \sum_{i=2}^{n+1} v_i + \sum_{k=0}^{n} T^k g^+.$$

So that

$$|\sum_{i=1}^{n} T^i g| \leqq \sum_{i=1}^{\infty} u_i + v_i \tag{2.2.4}$$

where it is known from Theorem 1 that $\sum_{i=1}^{\infty} u_1 + v_i$ is necessary of the form $p \cdot f$ where f is a constant and p a T-invariant finite function.

2°. If we go back to the definition of S_n in (2.1.12) we see that $\lambda(B \cap S_n) = 0$ would imply $V \subset \bigcup_{k=1}^{n}[u_k > 0]$ and then $v = \sum_{i=1}^{n} \tilde{v}_i$ λ a.e. According to (2.1.9) this would give $\mu_n = 0$ λ a.e. and then $S_n = S_{n-1}$ so that $\lambda(B \cap S_{n-1}) = 0$. As $\lambda(B \cap S_1) > 0$ by hypothesis, we see by recurrence on n that $\lambda(B \cap S_n) \neq 0$ for every n. The inequality (2.1.16) shows that

$$\int_B (\sum_{i=1}^{\infty} u_i + v_i)\, d\lambda \leqq \alpha^{-1} M_B \tag{2.2.5}$$

where α depends only on B and $M_B = \text{ess sup}_{y \varepsilon B} \sum_{i=1}^{k} T^i 1_B(y)$, k depending only on B too. The T-invariant function $p \cdot f$ on the right side of (2.2.4) must verify $p \int_B f a \lambda \leqq \alpha^{-1} M_B$. This proves that all the p's corresponding to all the considered functions g are smaller than $\alpha^{-1} M_B / \int_B f\, d\lambda$, which finishes up the proof.

2.3. THEOREM 2.*bis*. *Let us suppose that T is a positive Markovian operator on $L^1(\lambda)$ for which hypotheses* (i) *and* (ii) *of Theorem* 2 *are true.*

1°. *Then, there exists an increasing sequence (B_n) of measurable sets of finite measure λ such that $\lambda(E/\bigcup_n B_n) = 0$ and such that for every g bounded null outside some B_k with $\int g \cdot f\, d\lambda = 0$*

$$\sup_n |\sum_{k=1}^{n} T^{*k} g| \leqq p \qquad \lambda \text{ a.e.} \tag{2.3.1}$$

where p is a constant.

2°. *k being fixed, the inequality* (2.3.1) *holds uniformly with the same p, for all g verifying $\int g \cdot f\, d\lambda = 0$ and $|g| \leqq 1_{B_k}$.*

PROOF. Denote by f a T-invariant function in $M^+(\lambda)$ and $\nu = f \cdot \lambda$, and $\tilde{T}$ is the operator on $L_1(\nu)$ defined in the part 6° of the proof of Theorem 1.

We first note that we have the following equivalent conditions:

$$\forall J \varepsilon \mathcal{B}, \qquad J \subset K \quad \sum_{j=0}^{k} T^{*j} 1_J \geqq \alpha \lambda(J) \qquad \lambda \text{ a.e. on } K; \tag{2.3.2}$$

(2.3.3) $\forall h$ and g measurable bounded null outside K

$$\int \left(\sum_{j=0}^{k} T^{*j} h \right) \cdot g \, d\lambda \geqq \alpha \int h \, d\lambda \cdot \int g \, d\lambda;$$

(2.3.4) $\forall J \varepsilon \mathcal{B}, \quad J \subset K \quad \sum_{j=0}^{k} T^{j} 1_J \geqq \alpha \lambda(J) \quad \lambda$ a.e. on K.

It is, in fact, easily seen that (2.3.3) is equivalent to (2.3.2) and in the same way, as $\int (\sum_{j=0}^{k} T^{*j} h) \cdot g \, d\lambda = \int h \cdot (\sum_{j=0}^{k} T^{j} g) d\lambda$, it is evident that (2.3.3) is equivalent to (2.3.4). As T^* is a Markovian operator on $L_1(\nu)$ such that $\tilde{T} = (T^*)^*$, we see from this equivalence that if (ii) is true for (K, ν, T^*) it is equally true for $(K, \nu, \tilde{T})$.

But f being finite λ a.e., we can construct an increasing sequence (K_n) of sets of finite λ-measure, such that $\lambda(E/K_n) = 0$ and such that on each K_n, (ii) is true while f is bounded on K_n. We then have

$$\forall J \varepsilon \mathcal{B}, \quad J \subset K_n \quad \sum_{j=0}^{k} T^{*} 1_J \geqq \alpha \int_J f^{-1} \, d\nu \geqq \alpha' \nu(J) \qquad \nu \text{ a.e. on } K_n$$

where α' is another strictly positive constant.

The equivalence of (2.3.2) and (2.3.3) applied to the operator T^* on $L^1(\nu)$ (instead of T on $L^1(\lambda)$) proves that condition (ii) is true for $\tilde{T} = (T^*)^*$ and ν. We can then apply Theorem 2.

3. Invariant measures and Ornstein's ergodic theorem for Markovian kernels (concrete case). In this paragraph we consider a Markovian kernel $\underline{P}(x, A)$ defined on $(E, \mathcal{B})$ (cf. 1.7.), such that the successive iterates $\underline{P}^i$ of $\underline{P}$ can be written

(3.1) $$\underline{P}^i(x, A) = \int_A d^{(i)}(x, y) \lambda(dy) + s^{(i)}(x, A).$$

λ being a σ-finite measure on $(E, \mathcal{B})$, the $d^{(i)}$ being measurable on $(E \times E, \mathcal{B} \otimes \mathcal{B})$, and for each x, the measure $s^{(i)}(x, \cdot)$ being singular with respect to λ.

3.1. THEOREM 3. *Let us suppose that λ is a σ-finite measure such that $\lambda(A) < +\infty$ implies $\lambda\underline{P}(A) < +\infty$ and $\lambda\underline{P}$ is absolutely continuous with respect to λ. We suppose moreover that*

(j) *$\forall A$ measurable $\lambda(A) > 0 \Rightarrow \sum_{j=0}^{\infty} \underline{P}^j(x, A) = +\infty \quad \lambda$ a.e.*

(jj) *For every A measurable of finite λ-measure, there exists a measurable set $K \subset A$, with $\lambda(A/K) < \epsilon$, an integer k and a real number $\alpha > 0$ such that*

$$\{y: \textstyle\sum_{i=0}^{k} d^{(i)}(x, y) > \alpha\} \supset K \quad \text{for } \lambda \text{ almost all } x \,\varepsilon\, K.$$

Then

1°. *There exists a $\underline{P}$-invariant σ-finite measure $f \cdot \lambda$.*

2°. *There exists an increasing sequence (B_n) of sets of finite measure λ such that $\lambda(E/\bigcup_n B_n) = 0$, and for every $g \,\varepsilon\, L_\infty(\lambda)$ null outside some B_k with $\int g \, d\lambda = 0$:*

(3.1.1) $$\sup_n \left| \sum_{k=1}^{n} (d/d\lambda)(g \cdot \lambda)\underline{P}^k \right| \leqq p \cdot f \qquad \lambda \text{ a.e.}$$

3°. *k being kept fixed, the inequality in (3.1.1) holds uniformly with the same p for all g verifying $\int g \, d\lambda = 0$ and $|g| \leqq 1_{B_k}$.*

PROOF. As λP is by hypothesis absolutely continuous with respect to λ, we can associate with $\underline{P}$ the Markovian operator T on $L_1(\lambda)$ as defined in 1.9. As (j) and (jj) imply trivially the conditions (i) and (ii) of Theorems 1 and 2 we deduce at once the actual theorem from the previous ones.

THEOREM 3. *bis. Under the assumptions of Theorem* 3 ((j) *and* (jj)), *for every g bounded null outside some* B_k *and for which* $\int g \cdot f \, d\lambda = 0$

$$\text{(3.1.2)} \qquad \sup_n \left| \sum_{k=1}^{n} \int g(y) \underline{P}(x, dy) \right| \leqq p \qquad \lambda \text{ a.e.}$$

where p is a constant. k being kept fixed, the inequality in (3.1.2) *holds uniformly with the same p for all g verifying* $\int g \cdot f \, d\lambda = 0$ *and* $|g| \leqq 1_{B_k}$.

PROOF. This is a trivial consequence of Theorem 2.bis.

3.2. *Ergodic theorem for Harris' case.* We recall that in [2] Harris proved the existence of a $\underline{P}$-invariant σ-finite measure in the concrete case, if there exists a σ-finite measure m such that (3.1) holds with m instead of λ and if the following assumption (c) is true.

$$\text{(c)} \qquad m(A) > 0 \Rightarrow \forall x \in E \quad \underline{P}_x\left[\sum_{i=0}^{\infty} 1_A(X_i) = +\infty\right] = 1$$

where $\underline{P}_x$ is the probability law of the Markov process (X_n) associated with the kernel $\underline{P}$ and starting from x.

It is clear that the same result is true if we replace (c) by

(c$'$) there exists an $E' \subset E$ with $m(E/E') = 0$, $\underline{P}(x, E/E') = 0$ for every $x \in E'$ and

$$m(A) > 0 \Rightarrow \forall x \in E' \quad \underline{P}_x\left[\sum_{i=1}^{\infty} 1_A(X_i) = +\infty\right] = 1$$

we describe this situation [(3.1) and (c$'$)] as the *"essential Harris' case."*

It has been proved by Jain [4] (c.f. also [1] and [3]) that the "essential Harris' case" is equivalent to the following: There exists a σ-finite measure λ such that $\lambda(A) < +\infty$ implies $\lambda\underline{P}(A) < +\infty$, $\lambda\underline{P}$ is absolutely continuous with respect to λ and the following two hypotheses hold:

(NS) (non singularity). (3.1) holds and there exists x and i such that $\lambda\{y : d^i(x, y) > 0\} > 0$, (c_2) = hypothesis (j) of Theorem 3.

We have then the following:

THEOREM 4. *The hypotheses of Theorem* 3 *are equivalent to the "essential Harris' case."*

PROOF. It is trivial that hypotheses (j) and (jj) imply (NS) and (c_2). We have then only to prove that in the Harris' case (jj) is true. Let A be a set such that $0 < \lambda(A) < +\infty$. It is proved in [2], Lemma 2, that for every $\epsilon > 0$ one can find $C \subset A$, $\delta > 0$, and an integer r such that, $\lambda(A/C) \leqq \epsilon/4$ and (writing $a = \lambda(C) > 0$)

$$\text{(3.2.1)} \quad \forall x \in C \quad \lambda\left\{y : \sum_{j=0}^{r} d^{(j)}(x, y) > \delta\right\} \geqq \lambda(A) - \epsilon/4 \geqq a - \epsilon/4.$$

Let us denote

$$S = \left\{(x, y) : (x, y) \in C \times C, \quad \sum_{j=0}^{r} d^{(j)}(x, y) > \delta\right\};$$

$$S_x^1 = \{y : (x, y) \in S\}; \qquad S_y^2 = \{x : (x, y) \in S\};$$

we then have

(3.2.2) $\lambda \otimes \lambda(S) = \int_C \lambda(S_\eta^2)\lambda(d\eta) = \int_C \lambda(S_\xi^1)\lambda\, d(d\xi) \geqq a(a - \epsilon/4).$

From this and from $\lambda(S_\eta^2) \leqq a$ we deduce

(3.2.3) $(2\epsilon/4)\cdot\lambda\{\eta:\lambda(S_\eta^2) < 2\epsilon/4\} + a\cdot\lambda\{\eta:\lambda(S_\eta^2) \geqq 2\epsilon/4\} \geqq a(a - \epsilon/4)$

so that

(3.2.4) $$\lambda\{\eta:\lambda(S_\eta^2) \geqq 2\epsilon/4\} \geqq a - 3\epsilon/4.$$

We write $B = \{y:y \in C, \lambda(S_y^2) \geqq 2\epsilon/4\}$, we then have

(3.2.5) $$B \subset A \quad \text{and} \quad \lambda(B) \geqq \lambda(A) - \epsilon.$$

As for every $x \in B \subset C$ we have $\lambda(S_x^1) \geqq \lambda(C) - \epsilon/4$, it comes from the definitions of B and S:

(3.2.6) $$\lambda(S_y^2 \cap S_x^1) \geqq \epsilon/4$$

so that, using (3.2.1) and (3.2.6), $\forall(x, y) \in B \times B$

$$\begin{aligned}\sum_{i=1}^{2r} d^{(i)}(x, y) &\geqq r^{-2}\int \sum_{i=1}^{r} d^{(i)}(x, \eta)\lambda(d\eta) \sum_{j=1}^{r} d^{(j)}(\eta, y)\\ &\geqq r^{-2}\int_{S_x^1 \cap S_y^2} \sum_{i=1}^{r} d^{(i)}(x\cdot\eta)\lambda(d\eta)\sum_{j=1}^{r} d^{(j)}(\eta, y)\\ &\geqq r^{-2}\epsilon\delta^2/4.\end{aligned}$$

If we write $2r = k\alpha = \frac{1}{4}\epsilon\delta^2 r^{-2}$, the last inequality and (3.2.5) prove that (jj) is true.

REMARK. The proof of Theorem 1 gives then an alternate proof of Harris' theorem on the existence of an invariant measure.

3.3. *Topological case.* We now state a theorem under some natural topological assumptions. The measure λ we are considering is supposed to be a regular Borel measure on E (see 1.1) which is a locally compact topological space. To avoid useless complications we assume E is denumerable at infinity. In this topological situation we will always assume that the Markovian kernel $\underline{P}$ on $(E, \mathcal{B})$ possesses the following properties:

For every $x \in E$, $\underline{P}(x, \cdot)$ is a regular Borel measure in (E, B). This implies that for each regular Borel measure μ on (E, B) $\mu\underline{P}$ is regular Borel as soon as it is finite on the compact subsets of E.

We assume (cf. 3.2 above for the generality of this assumption) that λP is absolutely continuous with respect to λ, and moreover that λP is finite on the compacts. (Then it is regular Borel.)

THEOREM 5. *Let E be a locally compact topological space, and λ and $\underline{P}$ as previously described. We suppose that*

(j′) $\forall A$ *Borel measurable*

$$\lambda(A) > 0 \Rightarrow \sum_{j=0}^{\infty} \underline{P}^j(x, A) = +\infty.$$

(j′j′) *The functions $d^{(i)}(x, y)$ of formula* (3.1) *are lower semicontinuous on $E \times E$ and*

$$\forall x \in E \quad \bigcup_{i=0}^{\infty}\{y: d^{(i)}(x, y) > 0\} \supset \text{support } (\lambda).$$

(j″) *For every compact* $K \subset E$ *and any integer* k, *the density* $d(\lambda \underline{P}^k)/d\lambda$ *is bounded on* K.
Then

1°. *There exists a* $\underline{P}$*-invariant regular Borel measure* $\nu = f \cdot \lambda$.

2°. *If* g *is any bounded function, null outside some compact set* K *with* $\int g\, d\lambda = 0$, *then*

$$\text{(3.3.1)} \qquad \sup_n |(d/d\lambda) \textstyle\sum_{i=1}^n (g \cdot \lambda)\underline{P}^i| \leqq p \cdot f \qquad \lambda \text{ a.e.}$$

3°. *If* g *is any bounded function, null outside some compact set* K, *with* $\int g \cdot f\, d\lambda = 0$, *then*

$$\text{(3.3.2)} \qquad \sup_n |\textstyle\sum_{i=1}^n \underline{P}^i g| \leqq c.$$

In these formulas p *and* c *are constants, depending only on* K *and* $\|g\|_\infty$.

Proof. Let T_λ be the operator on $L_1(\lambda)$ associated with P and λ (if 1.9). We prove only that we can apply Theorem 1, 2 and 2 bis, as in the proof of Theorems 3 and 3 bis, and that we can moreover take for a sequence (B_n) any increasing sequence of compact sets (K_n) such that $\lambda(E/\bigcup_n K_n) = 0$. That is to say, according to (2.1.1) and (2.1.2), for any compact $K \subset \text{Supp}(\lambda)$,

$$\text{(3.3.3)} \quad \exists k \text{ and } \alpha > 0 \text{ such that } \forall J \subset K \quad \textstyle\sum_{i=0}^k \underline{P}^{(i)}(x, J) \geqq \alpha\lambda(J),$$

$$\text{(3.3.4)} \qquad \forall k \quad \text{ess sup}_K\, T_\lambda{}^k 1_K < +\infty.$$

This last inequality is a trivial consequence of (j″).

As to prove (3.3.3) we proceed as follows: According to the hypothesis (j′j′) every compact $K \subset \text{Supp}(\lambda)$ can be covered by the open sets $\{y: \sum_{i=0}^n d^{(i)}(x, y) > 0\}$. We can then find $n(x)$ and $\alpha_x > 0$ such that

$$K \subset \{y: \textstyle\sum_{i=0}^{n(x)} d^{(i)}(x, y) > \alpha_x\}.$$

But, because of the lower semi-continuity of $(x', y) \to \sum_{i=0}^{n(x)} d^{(i)}(x', y)$ we see that, for every x' in a suitable neighborhood of x,

$$K \subset \{y: \textstyle\sum_{i=0}^{n(x)} d^{(i)}(x', y) > \alpha_x\}$$

so that by the compacity of K again, we can find a k and an $\alpha > 0$ such that

$$\text{(3.3.5)} \qquad K \subset \{y: \textstyle\sum_{i=0}^k d^{(i)}(x, y) > \alpha\} \qquad \text{for every} \quad x \in K.$$

(3.3.3) is then an immediate consequence of (3.3.5). And this proves the theorem.

The proof of the theorem proves that

Corollary. *In the statement of Theorem* 5 *we can replace* (j′, j′) *by the following*: (j″j″) *There exist lower semi-continuous functions*

$$\delta^{(i)}(x, y) \leqq d^{(i)}(x, y) \quad \text{such that}$$

$$\forall x \in E \quad \textstyle\bigcup_{i=0}^\infty \{y: \delta^{(i)}(x, y) > 0\} \supset \text{Supp}\, \lambda.$$

3.4. Remark. The condition (j″j″) is in particular true for the following convolution operators in R^n

$$\underline{P}(x, A) = \textstyle\int_A d(y - x)dy + \mathcal{S}(A - x) = \mu(A - x)$$

where $\mathcal{S}$ is a measure, singular with respect to the Lebesgue measure. We can in fact take $\delta^{(i)}(x, y) = \mu^{*(i-2)} * d * d(y - x)$ and even $\delta^{(i)}(x, y) = d^{*i}(y - x)$.

4. Further limit theorems in the concrete case. In this section we deal either with the Harris' case or with the more precise topological case described by the hypothesis of Theorem 5. As $\lambda \underline{P}$ is always supposed to be absolutely continuous with respect to λ, we denote by T the operator on $L_1(\lambda)$ associated with $\underline{P}$. For convenience we introduce the following:

4.1. DEFINITION. A set $B' \subset E$ will be said "bounded" if it is included in a measurable B such that

(4.1.1) $$\forall k \quad \text{ess sup}_{y \varepsilon B}\, T^k 1_B(y) < +\infty$$

and

(4.1.2) $$\exists k, \alpha > 0 \quad \text{such that} \quad \forall J \subset B$$
$$\text{ess inf}_{x \varepsilon B} \textstyle\sum_{j=0}^{k} T^{*j} 1_J(x) \geqq \alpha\lambda(J).$$

It has been proved that in the Harris' case, E *is* λ a.e. *the union of an increasing sequence of bounded sets*, and, in the topological case that, under the hypothesis of Theorem 5, *every compact subset of E is "bounded" in this sense.*

Moreover, if λ is invariant, the condition (4.1.1) is trivial for any set B, because then $T^k 1_B(y) \leqq 1 \quad \lambda$ a.e.

4.2. THEOREM 6. *Let* $\underline{P}$ *a Markovian kernel, verifying the Harris' hypotheses, or the stronger topological hypothesis of Theorem* 5, λ being supposed from now on a P-invariant measure. *Let* μ *and* μ' *be two finite measures supported by "bounded" sets and with* $\|\mu\| = \|\mu'\|$. *Then, for any "bounded" set A*

(4.2.1) $$\lim_{n\to\infty} \left[\textstyle\sum_{k=1}^{n} \mu \underline{P}^k(A) / \sum_{k=1}^{n} \mu' \underline{P}^k(A)\right] = 1.$$

COROLLARY. *Let* $\underline{P}$ *be a Markovian kernel verifying the hypothesis of Theorem* 6. *Then, for every* $x \varepsilon E'$ *and* $y \varepsilon E'$, *where* E' *is the union of "bounded" sets, and any "bounded" set A*

(4.2.2) $$\lim_{n\to\infty} \left[\textstyle\sum_{k=1}^{n} \underline{P}^k(x, A) / \sum_{k=1}^{n} \underline{P}^k(y, A)\right] = 1.$$

PROOF. The theorem and its corollary are an immediate consequence of the following proposition.

PROPOSITION 1. *Let P be a Markovian kernel as in Theorem* 6, *and* λ *a* $\underline{P}$-*invariant measure.*

Let μ *and* ν *two positive finite measures with the following properties:*

(A') *There exist* 2 *"bounded sets" U and V such that* $\mu(U) = \nu(V) = \|\mu\| = \|\nu\|$.

(B') $\nu = v \cdot \lambda$ *where* $v \varepsilon L_\infty(\lambda)$ *and* $(d\mu/d\lambda) \varepsilon L_\infty(\lambda)$, $d\mu/d\lambda$ *is the Radon Nikodym derivative of the absolutely continuous part of* μ *with respect to* λ. *Then* (4.2.1) *holds with* $\mu' = \nu$.

PROOF. The proof of the proposition rests upon the following construction and sequence of lemmas. Let us suppose that the conditions of Proposition 1 are fulfilled. We generalize the construction of 2° of the proof of Lemma 1 by reasoning with the vector lattice $\mathfrak{M}^{\mathfrak{C}}$ instead of $L_1(\lambda)$. We define, as in (2.1.6),

$$\begin{aligned} \mu_1 &= \mu - \tilde{\nu}_1, & \tilde{\nu}_1 &= \inf(\mu, \nu), \\ \mu_2 &= \mu_1\underline{P} - \tilde{\nu}_2, & \tilde{\nu}_2 &= \inf(\mu_1\underline{P}, \nu - \tilde{\nu}_1), \qquad (4.2.3) \\ \mu_n &= \mu_{n-1}\underline{P} - \tilde{\nu}_{n-1}, & \tilde{\nu}_n &= \inf(\mu_{n-1}\underline{P}, \nu - \textstyle\sum_{j=1}^{n-1} \tilde{\nu}_j). \end{aligned}$$

We also write, because $\tilde{\nu}_n$ is necessarily absolutely continuous with respect to λ, $\tilde{\nu}_n = \tilde{v}_n \cdot \lambda$. The measures $\nu_n = v_n \cdot \lambda$ and $\tilde{\mu}_n$ are defined in a similar way.

We get immediately the following extensions of formulas (2.1.8) through (2.1.10):

$$\forall n \quad \textstyle\sum_{j=1}^{n} \tilde{\nu}_j \leqq \nu \quad \text{and} \quad \sum_{j=1}^{n} \tilde{\mu}_j \leqq \mu; \tag{4.2.4}$$

$$\|\mu_n\| = \|\mu\| - \|\textstyle\sum_{i=1}^{n} \tilde{\nu}_i\|; \tag{4.2.5}$$

$$\|\nu_n\| = \|\nu\| - \|\textstyle\sum_{i=1}^{n} \tilde{\mu}_i\|. \tag{4.2.6}$$

But as $\|\mu\| = \|\nu\|$ and because of (4.2.4)

$$\|\mu_n\| = \|\nu - \textstyle\sum_{i=1}^{n} \tilde{\nu}_i\| = \int (v - \sum_{i=1}^{n} v_i)\, d\lambda; \tag{4.2.5'}$$

$$\|\nu_n\| = \|\mu - \textstyle\sum_{i=1}^{n} \tilde{\mu}_i\|. \tag{4.2.6'}$$

LEMMA 1. *Under the previous conditions, for every B "bounded" and for every integer l*

$$\textstyle\sum_{i=1}^{\infty} \mu_i \underline{P}^l(B) < \infty.$$

We have even the following:

$$\textstyle\sum_{i=1}^{\infty} \mu_i \underline{P}^l(B) \leqq K \cdot \|v\|_\infty \tag{4.2.7}$$

where the constant K depends only on B and l, not on μ.

PROOF. We have only to extend to the present situation the part 3° of the proof of Theorem 1. We prove first, that, for every n, there exists $J_n \subset V = [v > 0]$ with $\lambda(J_n) > 0$ and $\sum_{i=1}^{n} \mu_i(J_n) = 0$. Suppose that for some n, $\forall J \subset V = [v > 0]$, $\lambda(J) > 0$ would imply $\sum_{i=1}^{n} \mu_i(J) > 0$. This expresses that ν, and therefore all measures $\tilde{\nu}_i$, $i = 1, \cdots, n$, and $\mu_{i-1}\underline{P}$, $i = 1, \cdots, n$, would be absolutely continuous with respect to $m = \sum_{i=1}^{n} \mu_i$. We can then read the equalities (4.2.3) replacing all $\mu_1, \cdots, \mu_n, \tilde{\nu}_1, \cdots, \tilde{\nu}_n$ by their densities and deduce that for m almost all $x \in V$ we have $d\nu/dm = \sum_{j=1}^{n} (d\tilde{\nu}_j/dm)$. So that $\nu = \sum_{j=1}^{n} \tilde{\nu}_j$. But, according to (4.2.5′) we would have $\|\mu_n\| = 0$ and $\sum_{i=1}^{n-1} \mu_i(J) = \sum_{i=1}^{n} \mu_i(J) > 0$ as soon as $\lambda(J) > 0$. Then, if there exists $J_{n-1} \subset V = [v > 0]$ with $\lambda(J_{n-1}) > 0$ and $\sum_{i=1}^{n-1} \mu_i(J_{n-1}) = 0$ there exists $J_n \subset V = [v > 0]$ with $\lambda(J_n) > 0$ and $\sum_{i=1}^{n} \mu_i(J_n) = 0$. As, by hypothesis, $\mu_1(V) = 0$ and $\lambda(V) > 0$, we see, by recurrence on n, that the property is true for all n.

So, let us write $J_n \subset V = [v > 0]$ with $\lambda(J_n) > 0$ and $\sum_{i=1}^{n} \mu_i(J_n) = 0$.

As in (2.1.12) we obtain that, on J_{n+j}, we have the following inequality between measures:

(4.2.8) $$\sum_{i=1}^{n} \mu_i \underline{P}^j \leqq \sum_{r=1}^{j-1} \left(\sum_{i=1}^{n+j-r} \tilde{\nu}_i \underline{P}^r\right) \leqq \sum_{r=1}^{j-1} \nu \underline{P}^r.$$

Let k, α be such that

$$\forall J \subset B \quad \sum_{j=1}^{k} \underline{P}^j(x, J) \geqq \alpha\lambda(J) \qquad \lambda \text{ a.e on } B.$$

Then

$$\forall J \subset J_{n+l+k} \quad \sum_{j=l}^{l+k} \sum_{i=1}^{n} \mu_i \underline{P}^j(J) = \int \sum_{j=0}^{k} \sum_{i=1}^{n} \underline{P}^j(x, J)\mu_i \underline{P}^l(dx) \geqq \alpha\lambda(J) \sum_{i=1}^{n} \mu_i \underline{P}^l(B)$$

which proves that

$$\alpha \sum_{i=1}^{n} \mu_i \underline{P}^l(B) \leqq \operatorname{ess\,inf}_{x \varepsilon B \cap J_{n+l+k}} (d/d\lambda)\left(\sum_{j=1}^{l+k} \sum_{i=1}^{n} \mu_i \underline{P}^j\right)(x).$$

But (4.2.8) then gives

$$\sum_{i=1}^{n} \mu_i \underline{P}^l(B) \leqq \alpha^{-1} \operatorname{ess\,inf}_{x \varepsilon B_n \cap J_{n+l+k}} (d/d\lambda)\left(\sum_{j=1}^{l+k} \sum_{r=1}^{j-1} \nu \underline{P}^r\right)(x)$$

and as $\nu \underline{P}^r$ is absolutely continuous with density $T^r v$ with respect to λ and as λ is P-invariant $\|T^r v\|_\infty = \|v\|_\infty$. So that

(4.2.9) $$\sum_{i=1}^{n} \mu_i \underline{P}^l(B) \leqq \alpha^{-1}(l + k)^2 \|v\|_\infty.$$

This proves Lemma 1.

LEMMA 2. *Suppose that μ and ν are as in Proposition 1 except that possibly $\|\mu\| \neq \|\nu\|$. Then for every "bounded" A*

(4.2.10) $$\sum_{i=1}^{n} \mu \underline{P}^i(A) \leqq \|\mu\| \, \|\nu\|^{-1}\left(\sum_{i=1}^{n} \nu \underline{P}^i(A) + K_\nu\right),$$

the constant K_ν being an increasing function of ν, independent of μ, and depending only on a fixed bounded set containing A.

PROOF. Let us write $\mu' = \|\nu\| \, \|\mu\|^{-1}\mu$. So that μ' and ν verify the hypothesis of Proposition 1. We can then get inequalities analogous to those preceding (2.2.4). In particular,

$$\sum_{i=1}^{n} \mu' \underline{P}^i(A) \leqq \sum_{i=2}^{n+1} \mu_i'(A) + \sum_{k=0}^{n} \nu \underline{P}^k(A)$$

which gives, using Lemma 1,

$$\sum_{i=1}^{n} \mu \underline{P}^i(A) \leqq \|\mu\| \, \|\nu\|^{-1}\left(\sum_{i=1}^{n} \nu \underline{P}^i(A) + \|\nu\| + \sum_{i=1}^{\infty} \mu_i'(A)\right) \leqq \|\mu\| \, \|\nu\|^{-1}\left(\sum_{i=1}^{n} \nu \underline{P}^i(A) + \|\nu\| + K \cdot \|v\|_\infty\right),$$

K depending only on a bounded set $B \supset A$. This proves the lemma.

LEMMA 3. *Let us suppose that μ and ν are as in Proposition 1. Let us write $\tilde{\nu}^L = \sum_{i=1}^{L} \tilde{\nu}_i$. Then for every A "bounded"*

$$\lim_n \left[\sum_{i=1}^{n} \mu \underline{P}^i(A) \Big/ \left(\sum_{i=1}^{n} \left(\tilde{\nu}^L \underline{P}^i(A) + \mu_L \underline{P}^i(A)\right)\right)\right] = 1 \textit{ for each } L.$$

PROOF. We have

$$\sum_{j=1}^{L} \tilde{\nu}_j \underline{P}^k + \mu_L \underline{P}^k = \sum_{j=1}^{L-1} \tilde{\nu}_j \underline{P}^k + \mu_{L-1} \underline{P}^{k+1},$$

so that

$$\sum_{k=1}^{n} \left(\sum_{j=1}^{L} \tilde{\nu}_j \underline{P}^k + \mu_L \underline{P}^k\right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{L-1} \tilde{\nu}_j \underline{P}^k + \mu_{L-1} \underline{P}^k\right) + \mu_{L-1} \underline{P}^{n+1} - \mu_{L-1} \underline{P}.$$

We deduce by recurrence on L

$$\sum_{k=1}^{n} \left(\sum_{j=1}^{L} \tilde{\nu}_j \underline{P}^k + \mu_L \underline{P}^k\right) = \sum_{k=1}^{n} \mu \underline{P}^k + \sum_{j=1}^{L-1} (\mu_j \underline{P}^{n+1} - \mu_j \underline{P}).$$

And for each $A \subset B$ bounded

$$\left(\sum_{i=1}^{n} \tilde{\nu}^L \underline{P}^i(A) + \mu_L \underline{P}^i(A)\right) / \sum_{i=1}^{n} \mu \underline{P}^i(A) = 1 + \left(\sum_{j=1}^{L-1} \mu_j \underline{P}^{n+1}(A) - \mu_j \underline{P}(A)\right) / \sum_{i=1}^{n} \mu \underline{P}^i(A).$$

But the second term on the right hand of this equality tends to zero for each L according to:

$$\left(\sum_{j=1}^{L-1} \mu_j \underline{P}^{n+1}(A) - \mu_j P(A)\right) / \sum_{i=1}^{n} \mu \underline{P}^i(A) \leqq L\|\mu\| / \sum_{i=1}^{n} \mu \underline{P}^i(A)$$

and the recurrence condition on P (condition (j)).

LEMMA 4. *Under hypothesis of Proposition* 1,

$$\nu = \sum_{i=1}^{\infty} \tilde{\nu}_i \quad \textit{and} \quad \lim_{k\to\infty} \|\mu_k\| = 0.$$

PROOF. Let D be the set $[\nu > \sum_{i=1}^{\infty} \tilde{\nu}_i]$, and let us denote by $\underline{P}_D$ the operator on $\mathfrak{M}^b$ defined by $\mu \to (1_{D^c} \cdot \mu)\underline{P}$ where D^c is the complementary set of D. Because of the recurrence involved in the Harris condition,

$$\text{if } \lambda(D) > 0, \quad \|\mu\| = \sum_{j=0}^{\infty} \mu \underline{P}_D{}^j(D).$$

As on D we have $\nu - \sum_{i=1}^{n} \tilde{\nu}_j > \tilde{\nu}_{n+1}$ for every n, we have $\tilde{\nu}_{n+1} = \mu_n \underline{P}$ on D and thus $\mu_{n+1}(D) = 0$ for every n. We then deduce by recurrence on k

$$\forall J \subset D \quad \mu_k(J) \leqq \mu \underline{P}_D{}^k(J).$$

So that

$$\|\mu_k\| = \sum_{j=0}^{\infty} \mu_k \underline{P}_D{}^j(D) \leqq \sum_{j=k}^{\infty} \mu \underline{P}_D{}^j(D).$$

As the series on the right hand converges we have

$$\lim_{k\to\infty} \|\mu_k\| = \int \left(\nu - \sum_{i=1}^{n} \tilde{\nu}_i\right) d\lambda = 0$$

which contradicts $\lambda(D) > 0$ and proves that in any case the two relations in the lemma are true.

As a consequence of Lemma 2 and of $\tilde{\nu}^L \leqq \nu$ for every L,

$$0 \leqq \sum_{k=1}^{n} \mu_L \underline{P}^k(A) / \sum_{k=1}^{n} \tilde{\nu}^L \underline{P}^k(A) \leqq \|\mu_L\| \, \|\tilde{\nu}^L\|^{-1}\left(1 + K_\nu / \sum_{k=1}^{n} \tilde{\nu}^L \underline{P}^k(A)\right).$$

As $\tilde{\nu}^L$ is increasing and $\sum_{k=1}^{\infty} \tilde{\nu}^L \underline{P}^k(A) = +\infty$ we see that as a consequence of Lemma 4

(4.2.11) $\lim_{L\to\infty} [\sum_{k=1}^{n} \mu_L \underline{P}^k(A) / \sum_{k=1}^{n} \tilde{\nu}_L \underline{P}^k(A)] = 0$ uniformly in n.

In the same way, as $\lim_{L\to\infty} \|\nu - \tilde{\nu}^L\| = \lim_{L\to\infty} \|\mu_L\| = 0$, we get from Lemma 2

(4.2.12) $\lim_{L\to\infty} [\sum_{k=1}^{n} \tilde{\nu}^L \underline{P}^k(A) / \sum_{k=1}^{n} \nu \underline{P}^k(A)] = 1$ uniformly in n.

If we next consider

$$\begin{aligned} &\textstyle\sum_{k=1}^{n} \mu \underline{P}^k(A) / \sum_{k=1}^{n} \nu \underline{P}^k(A) \\ (4.2.13)\qquad &= [\textstyle\sum_{k=1}^{n} \mu \underline{P}^k(A) / \sum_{k=1}^{n} \tilde{\nu}^L \underline{P}^k(A) + \mu_L \underline{P}^k(A)] \\ &\quad \cdot [(\textstyle\sum_{k=1}^{n} \tilde{\nu}^L \underline{P}^k(A) + \mu_L \underline{P}^k(A)) / \sum_{k=1}^{n} \tilde{\nu}^L \underline{P}^k(A)] \\ &\quad\quad \cdot [\textstyle\sum_{k=1}^{n} \tilde{\nu}^L \underline{P}^k(A) / \sum_{k=1}^{n} \nu \underline{P}^k(A)], \end{aligned}$$

as a consequence of (4.2.12) and (4.2.13), we can choose L_ϵ such that the two last ratios, on the right hand side of (4.2.13), take their values in $[1 - \epsilon, 1 + \epsilon]$.

From Lemma 3 we can then find n_ϵ such that for each $n > n_\epsilon$, the first ratio on the right hand side of (4.2.12) takes its values in $[1 - \epsilon, 1 + \epsilon]$. As a consequence:

$$\forall n > n_\epsilon \qquad \textstyle\sum_{k=1}^{n} \mu \underline{P}^k(A) / \sum_{k=1}^{n} \nu \underline{P}^k(A) \ \varepsilon\ [(1 - \epsilon)^3, (1 + \epsilon)^3].$$

The proof of Theorem 6 and its corollary are thus completed.

4.3. THEOREM 7. *Let* $\underline{P}$ *be a Markovian kernel verifying the Harris' hypothesis, or the stronger topological hypothesis of Theorem* 5. *If* λ *is an invariant measure for* $\underline{P}$, *then for any "bounded" sets* A *and* B,

$$(4.3.1)\quad \lim_{n\to\infty} [\textstyle\sum_{k=1}^{n} \underline{P}^k(x, A) / \sum_{k=1}^{n} \underline{P}^k(y, B)] = \lambda(A)/\lambda(B) \qquad \forall (x, y)\ \varepsilon\ E^1 \times E^1.$$

PROOF. It is sufficient to prove when $A \cap B = \varnothing$. Let ν be any measure $v \cdot \lambda$ where $\int v \cdot d\lambda = 1$ and $v \leqq k \cdot 1_C$ where k is a constant and C a "bounded" set with $x \not\varepsilon C$ and $y \not\varepsilon C$.

By applying Theorem 3-bis to the function $g = 1_A - (\lambda(A)/\lambda(B))1_B$ we get immediately

$$\sup_n |\textstyle\sum_{k=1}^{n} \underline{P}^k(x, A) - (\lambda(A)/\lambda(B))\underline{P}^k(x, B)| \leqq p \qquad \lambda\text{a.e.}$$

and then

$$\sup_n |\textstyle\sum_{k=1}^{n} \nu \underline{P}^k(A) - (\lambda(A)/\lambda(B))\nu \underline{P}^k(B)| \leqq p\|\nu\| < \infty$$

so that

$$\lim_n [\textstyle\sum_{k=1}^{n} \nu P^k(A) / \sum_{k=1}^{n} \nu \underline{P}^k(B)] = \lambda(A)/\lambda(B).$$

Using now the Proposition 1 we get immediately (4.3.1).

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