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ON A CLASS OF CONDITIONALLY DISTRIBUTION-FREE TESTS FOR INTERACTIONS IN FACTORIAL EXPERIMENTS[1]

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1. Summary and introduction. Nonparametric analysis of variance tests for factorial experiments, available in the literature, relate mostly to main effects in two-way layout linear models without interactions. Although the approach of Lehmann (1964) (see also Puri and Sen (1967)) can be adapted to construct tests for interactions, such tests are rather tedious to apply and are only asymptotically distribution-free. To obviate these drawbacks, the theory of permutationally distribution-free rank order tests for main effects, developed independently by the two authors, Sen (1968), Mehra (1968) (see also Mehra and Sarangi (1967)), is further extended in this paper to provide suitable tests for interactions in factorial experiments. The asymptotic power-efficiencies of the proposed tests are also studied.

[The results of this paper were obtained independently by the two authors using, respectively, extensions of the Hájek (1962) and the Chernoff-Savage (1958) theorems on the asymptotic normality of rank order statistics.]

2. Preliminaries. To avoid notational complexities, we shall consider in detail only the case of replicated two-factor experiments with one observation per cell. The procedures and arguments below, being perfectly general, can be extended to cover the case of several factors and/or observations per cell. Let the random observations Y_{ijk}, denoting the yield in the (j, k)th cell and the ith replicate follow the fixed effects model:

$$Y_{ijk} = \mu_i + \nu_j + \tau_k + \gamma_{jk} + U_{ijk}, \quad 1 \leqq i \leqq n, 1 \leqq j \leqq p, 1 \leqq k \leqq q, \tag{2.1}$$

where the replication effects μ_i's, and the main effects ν_j's, and τ_k's, and the interactions γ_{jk}'s for the two-factors satisfy the side conditions: $\mu. = \nu. = \tau. = \gamma_{j\cdot} = \gamma_{\cdot k} = 0$ ($\cdot$ stands for the average over the subscript replaced by $\cdot$). It is assumed that $(U_{i11}, \cdots, U_{ipq})$, $i = 1, 2, \cdots, n$, are independently distributed with common continuous (joint) distribution function (cdf) $G(x_{11}, \cdots, x_{pq})$ which is symmetric in the arguments and satisfies the property that $P[U_{ijk} = U_{ij'k'}] = 0$ for any two distinct pairs (j, k) and (j', k'). This includes the conventional assumptions of independence, continuity, and identity of the distributions of all U_{ijk}'s. We want to test

$$H_0: \mathbf{\Gamma} = (\gamma_{jk}) = \mathbf{0}^{p\times q} \tag{2.2}$$

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against alternatives that $\mathbf{\Gamma}$ is non-null. By using the following intra-block transformations,

$$\mathbf{Z}_i = (\mathbf{I}_p - p^{-1}\mathbf{1}_p'\mathbf{1}_p)\mathbf{Y}_i(\mathbf{I}_q - q^{-1}\mathbf{1}_q'\mathbf{1}_q), \qquad i = 1, \cdots, n, \tag{2.3}$$

$$\mathbf{E}_i = (\mathbf{I}_p - p^{-1}\mathbf{1}_p'\mathbf{1}_p)\mathbf{U}_i(\mathbf{I}_q - q^{-1}\mathbf{1}_q'\mathbf{1}_q), \qquad i = 1, \cdots, n, \tag{2.4}$$

where $\mathbf{Y}_i = (Y_{ijk}), \mathbf{U}_i = (U_{ijk}), \mathbf{Z}_i = (Z_{ijk}), \mathbf{E}_i = (E_{ijk}), 1 \leqq i \leqq n$ are $p \times q$ matrices, $\mathbf{I}_t$ is the identity matrix of order t, and $\mathbf{1}_t$ is the (row) t-vector with each element equal to unity, we obtain

$$\mathbf{Z}_i = \mathbf{\Gamma} + \mathbf{E}_i, \qquad i = 1, 2, \cdots, n. \tag{2.5}$$

In the sequel we shall work with model (2.5) which is free of the nuisance parameters μ_i's, ν_j's, and τ_k's. Also we will consider only the case where $p, q \geqq 3$; for otherwise, because of the side conditions, the problem is easily seen to reduce to the one discussed in Sen (1968) or Mehra (1968).

3. The basic permutation principle. The cdf $G(\mathbf{x}_{11}, \cdots, x_{pq})$ of $(U_{i11}, \cdots, U_{ipq})$ is symmetric in its pq arguments. Hence, from (2.3) and some straightforward computations, (cf. [12]) it follows that the joint distribution of $(E_{i11}, \cdots, E_{ipq})$ (defined by (2.4),) remains invariant under any permutations of the labels $(1, \cdots, p)$ of the first factor and the labels $(1, \cdots, q)$ of the second factor. This yields a finite group $\mathcal{G}$ of $p!q!$ transformations on the variables $(E_{i11}, \cdots, E_{ipq})$ which maps the sample space onto itself and leaves the distribution invariant. Thus, considering the n independent matrices $\mathbf{E}_i$, $i = 1, \cdots, n$, and using the same group of transformations for each of them, we get a compound group $\mathcal{G}_n^*$ which contains $(p!q!)^n$ transformations, and each of these transformations maps the sample space (of npq dimension) on to itself and leaves the distribution of the sample point invariant. Hence, when H_0 in (2.2) holds, so that $\mathbf{Z}_i = \mathbf{E}_i$ for all $i = 1, \cdots, n$, the same group of transformation $\mathcal{G}_n^*$ also works on $\mathbf{Z}_1, \cdots, \mathbf{Z}_n$. Considering then the set $\mathbf{z}_n$ of all possible $(p!q!)^n$ realizations of $(\mathbf{Z}_1, \cdots, \mathbf{Z}_n)$ obtained by permuting the rows and columns of each $\mathbf{Z}_i$, we conclude from the above discussions that under H_0 in (2.2), the conditional distribution of $(Z_1, \cdots, Z_n)$ over $\mathbf{z}_n$ is uniform, each realization having the (conditional) probability $(p!q!)^{-n}$. Let us denote this conditional (permutational) probability measure by $\mathcal{P}_n$. Since $\mathcal{P}_n$ is completely specified, proceeding as in Hoeffding ((1952), pp. 169–170), the existence of similar size α tests for H_0 in (2.2) may be proved. By nature these are conditional tests, based on the above permutation-invariance structure. In this paper, a general class of rank order tests which is conditionally distribution-free in the above sense is studied.

4. Formulation of the rank order tests. Let r_{ijk} denote the rank of Z_{ijk} in a combined ranking of the totality of $N = pqn$ aligned observations Z_{ijk}'s, where on account of the assumptions in (2.1), ties among Z_{ijk}'s may be ignored with probability 1. Let $\{J_{N,1}, \cdots, J_{N,N}\}$ be a sequence of real numbers such that the function $J_N(H)$, $0 < H < 1$, as defined by Chernoff and Savage (1958) satisfies the conditions of their Theorem 1. Let $J_{N,r_{ijk}} = \eta_{ijk}$ and

$$\mathbf{T}_N = (T_{N,jk}); \qquad T_{N,jk} = n^{-1} \sum \eta_{ijk} = \eta_{\cdot jk}; \tag{4.1}$$
$$\mathbf{T}_N^* = (I_p - p^{-1}1_p'1_p)\mathbf{T}_N(I_q - q^{-1}1_q'1_q) = ((T^*_{N,jk})).$$

Then it is easily seen that

$$T^*_{N,jk} = n^{-1} \sum_{i=1}^{n} (\eta_{ijk} - \eta_{i\cdot k} - \eta_{ij\cdot} + \eta_{i\cdot\cdot}). \tag{4.2}$$

Let $r_i^{(1)}(r_i^{(2)})$ stand for the (observed) partition of the ranks in the ith replication into the $p(q)$ sets of $q(p)$ ordered elements and let the configuration $\{r_2^{(1)}, r_i^{(2)}: 1 \leqq i \leqq n\}$ be denoted by $\mathcal{E}$. We observe that $\mathcal{P}_N$ defined in Section 3 is simply the conditional probability measure given $\mathcal{E}$, and

$$E(\mathbf{T}_N^* \mid \mathcal{P}_N) = \mathbf{0}^{p\times q}, \tag{4.3}$$

$$nE(\mathbf{T}_N^{0*\prime}\mathbf{T}_N^{0*} \mid \mathcal{P}_N) = (\mathbf{I}_p - p^{-1}1_p'1_p) \otimes (\mathbf{I}_q - q^{-1}1_q'1_q)\sigma^2(\mathcal{P}_n), \tag{4.4}$$

where $\mathbf{T}_N^{0*} = (T^*_{N,11}, \cdots, T^*_{N,pq})$, and

$$\sigma^2(\mathcal{P}_N) = (n(p-1)(q-1))^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{q} (\eta_{ijk} - \eta_{i\cdot k} - \eta_{ij\cdot} + \eta_{i\cdot\cdot})^2. \tag{4.5}$$

Thus, considering the generalized inverse of the $pq \times pq$ matrix in (4.4) and employing it to construct a quadratic form in the elements of $\mathbf{T}_N^*$, we derive the following test statistic

$$\mathcal{L}_N = [n/\sigma^2(\mathcal{P}_N)] \sum_{j=1}^{p} \sum_{k=1}^{q} \{T^*_{N,jk}\}^2, \tag{4.6}$$

which is analogous to the classical parametric test based on the variance ratio criterion [cf. Scheffé (1959)].

For small n, p, and q the exact permutation distribution of $\mathcal{L}_N$ can be obtained by considering the $(p!q!)^n$ (conditionally) equally likely row and column permutations of the matrices $\mathbf{H}_i = (\eta_{ijk})$, $i = 1, \cdots, n$. This procedure becomes prohibitively laborious for large values of n, p, or q. For this reason we consider the following large sample approach.

Let $J(u) = \lim_{N\to\infty} J_N(u)$, $0 < u < 1$, and

$$\delta^2 = \int_{-\infty}^{\infty} J[H(x)]\, dH(x) - \int_{-\infty}^{\infty} J[H(x)]J[H(y)]\, d[H_{10}(x,y) + H_{01}(x,y) - H_{11}(x,y)], \tag{4.7}$$

where H_{01}, H_{10}, H_{11}, and H denote, respectively, the bivariate cdf's of $(Z_{ijk}, Z_{ijk'})$ for $k \neq k'$, $(Z_{ijk}, Z_{ij'k})$ for $j \neq j'$ and $(Z_{ijk}, Z_{ij'k'})$ for $j \neq j'$ and $k \neq k'$, and the common univariate cdf of the Z_{ijk}'s under the hypotheses H_0. Now if we assume that

$$P[\{J[H(Z_{jk})] - J[H(Z_{j'k})] - J[H(Z_{jk'})] + J[H(Z_{j'k'})]\} = \text{constant}] < 1 \tag{4.8}$$

for at least one pair $j \neq j'$ and $k \neq k'$, then as in Theorem 4.1 of Sen (1968), it can be shown that (4.7) is strictly positive. The main theorem of this section is the following:

THEOREM 4.1. *Under the conditions of Chernoff and Savage* (1958) *and* (4.8), *the conditional* (*permutational*) *distribution of* $\mathfrak{L}_N$ *under* H_0 *converges, in probability, to a chi-square distribution with* $(p-1)(q-1)$ *degrees of freedom.*

PROOF. By virtue of (4.3) to (4.6), it suffices to show that for any non-null $\mathbf{A} = (a_{jk})$, $Y_n = n^{\frac{1}{2}}\sum_{j=1}^{p}\sum_{k=1}^{q} a_{jk}T^{*}_{N,jk}$ converges in law (under $\mathcal{P}_N$) to a normal distribution as $n \rightarrow \infty$. By using the first two conditions of Chernoff and Savage (1958), (4.1) and (4.2), we write

$$Y_n = n^{-\frac{1}{2}}\sum_{i=1}^{n} Y_{ni} + o_p(1); \qquad Y_{ni} = \sum_{j=1}^{p}\sum_{k=1}^{q} a^{*}_{jk}J(R_{ijk}/(N+1)), \qquad i = 1, \cdots, n,$$

where a^{*}_{jk}'s are linear functions of a_{jk}'s and satisfy the constraints $a^{*}_{j\cdot} = a^{*}_{\cdot k} = 0$ for all j and k. By observing that $Y_{n1}, \cdots, Y_{nn}$ are stochastically independent under $\mathcal{P}_N$, we obtain $E[Y_{ni} \mid \mathcal{P}_N] = 0$ and

$$n^{-1}\sum_{i=1}^{n} E(Y^{2}_{ni} \mid \mathcal{P}_N) = (n(p-1)(q-1))^{-1}\sum_{i=1}^{n}\{\sum_{j=1}^{p}\sum_{k=1}^{q} J^{2}(R_{ijk}/(N+1)) - p^{-1}\sum_{k=1}^{q}[\sum_{j=1}^{p} J(R_{ijk})/(N+1))]^{2} - q^{-1}\sum_{j=1}^{p}[\sum_{k=1}^{q} J(R_{ijk}/(N+1))]^{2} + (pq)^{-1}[\sum_{j=1}^{p}\sum_{k=1}^{q} J(R_{ijk}/(N+1))]^{2}\} \rightarrow_p \delta^{2}\sum_{j=1}^{p}\sum_{k=1}^{q}(a^{*}_{jk})^{2} > 0, \tag{4.9}$$

where δ^2 is given by (4.7) and in proving (4.9) we use the arguments of Theorem 4.2 of Puri and Sen (1966) and some routine analysis. Similarly, using the growth condition of Theorem 1 of Chernoff and Savage (1958), it follows that

$$n^{-1}\sum_{i=1}^{n} E[|Y_{ni}|^{2+\eta} \mid \mathcal{P}_N] < \infty \qquad \text{for some} \quad \eta > 0. \tag{4.10}$$

By the Berry-Esseen theorem (cf. [4]), p. 288), the asymptotic normality of Y_n follows from (4.9) and (4.10); hence the theorem.

REMARK. Theorem 4.1 remains true also under the conditions (2.3a) (2.3b) of Mehra (1968) with $J(u)$ in place of $\xi(u)$, $0 < u < 1$, and if $\xi(u)$ is strictly monotonic everywhere, the result holds uniformly in the configuration $\mathcal{E}$.

Theorem 4.1 simplifies the large sample test based on $\mathfrak{L}_N$ and suggests that the chi-square distribution with $(p-1)(q-1)$ degrees of freedom (d.f) can be used to compute the critical values of $\mathfrak{L}_N$.

5. Asymptotic efficiency. For studying the asymptotic efficiency of the test based on $\mathfrak{L}_N$, we consider the sequence of Pitman alternatives $\{H_N\}$ where

$$H_N: \mathbf{\Gamma} = \mathbf{\Gamma}_n = n^{-\frac{1}{2}}\Lambda, \qquad \Lambda = (\lambda_{jk}), \tag{5.1}$$

where λ_{jk} are real and finite and satisfy $\lambda_{j\cdot} = \lambda_{\cdot k} = 0$ for each $1 \leqq j \leqq p, \leqq k \leqq q$. Further, set

$$(5.2) \quad B(H) = \int_{-\infty}^{\infty} \{(d/dx)J[H(x)]\} \, dH(x),$$

$$(5.3) \quad A^2 = \int_0^1 J^2(u) - [\int_0^1 J(u) \, du]^2,$$

$$(5.4) \quad \rho_{ij} = A^{-2}[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J[H(x)]J[H(y)] \, dH_{ij}(x, y) - (\int_0^1 J^2(u) \, du)^2]$$

for $(i, j) = (0, 1), (1, 0), (1, 1)$.

The proof of the following theorem can be accomplished by directly extending the proof of Theorem 5.1 of Sen (1967).

THEOREM 5.1. *If* (i) $\{H_N\}$ *in* (5.1) *holds,* (ii) *the conditions of Theorem* 1 *of Chernoff and Savage* (1958) *hold and* (iii) *the conditions of Lemma* 7.2 *of Puri* (1964) *hold,* $\mathfrak{L}_N$, *defined by* (4.10), *has asymptotically a non-central chi-square distribution with* $(p - 1)(q - 1)$ d.f. *and non-centrality parameter*

$$\Delta_{\mathfrak{L}} = [(1/pq) \sum_{j=1}^{p} \sum_{k=1}^{q} \lambda_{jk}^2]\cdot[B^2(H)/A^2(1 - \rho_{10} - \rho_{01} + \rho_{11})].$$

PROOF. By using the same technique as in the proof of Theorem 5.1 of Sen (1967), it follows that, under H_N,

$$n^{\frac{1}{2}}E\{\mathbf{T}_N^* \mid H_N\} = [(pq)^{-\frac{1}{2}}B(H)]\mathbf{\Lambda} + o(1),$$

$$(5.5) \quad nE\{\mathbf{T}_N^{0*\prime}\mathbf{T}_N^{0*} \mid H_N\} = (\mathbf{I}_p - p^{-1}\mathbf{1}_p{}'\mathbf{1}_p) \oplus (\mathbf{I}_q - q^{-1}\mathbf{1}_q{}'\mathbf{1}_q) \cdot A^2(1 - \rho_{10} - \rho_{01} + \rho_{11}) + o(1)$$

and, similarly as in (4.9), that under H_N, $\sigma^2(\mathcal{P}_N) \rightarrow_p A^2(1 - \rho_{10} - \rho_{01} + \rho_{11})$ where $\sigma^2(\mathcal{P}_N)$, $B(H)$, and ρ_{ij}'s are defined, respectively, by (4.5) and (5.2) to (5.4). The asymptotic normality of $(n^{\frac{1}{2}}T_{N,jk})$ and (5.5) complete the proof of the theorem.

REMARK. Theorem 5.1 also remains true if the conditions (2.3a), (2.3b), (4.1) and (4.2) of Mehra (1968) hold with a differentiable $J(u)$ in place of $\xi(u)$, $0 < u < 1$, F in place of G, and $t = (p - 1)(q - 1)$, provided the errors U_{ijk} in (2.1) are all mutually independent.

Let σ_e^2 denote the variance of e_{ijk}. Then as can easily be seen, the classical ANOVA test statistic for H_0 has asymptotically, under H_N, a non-central chi-square distribution with $(p - 1)(q - 1)$ degrees of freedom and the non-centrality parameter

$$(5.6) \quad \Delta_Q = [(1/pq) \sum_{j=1}^{p} \sum_{k=1}^{q} \lambda_{jk}^2]/[pq\sigma_e^2/\{(p - 1)(q - 1)\}].$$

We obtain from (5.6) and Theorem 5.1 the following:

THEOREM 5.2. *When the conditions of Theorem* 5.1 *hold, the asymptotic relative efficiency* (ARE) *of the* $\mathfrak{L}_N$*-test with respect to the classical analysis of variance test is given by*

$$(5.7) \quad e_{(\{\mathfrak{L}_N\},\{Q_N\})} = [pq/\{(p - 1)(q - 1)(1 - \rho_{10} - \rho_{01} + \rho_{11})\}][\sigma_e^2B^2(H)/A^2].$$

From Lemmas 4.4 and 4.5 of Sen (1968), it follows that $\rho_{10} \geqq -(p - 1)^{-1}$

and $\rho_{01} \geqq -(q-1)^{-1}$ with equality if and only if $J = H^{-1}$ (apart from an additive constant), so that from (5.6) we obtain that

$$e_{(\{\mathcal{L}_N\},\{Q_N\})} \geqq [1-(pq)^{-1}(1-(p-1)(q-1)\rho_{11})]^{-1}[\sigma_e^2 B^2(H)/A^2] \tag{5.8}$$

which leads to the following:

COROLLARY 5.2.1. *A sufficient condition for* $e_{(\{\mathcal{L}_N\},\{Q_N\})}$ *to be at least as large as* $[\sigma_e^2 B^2(H)/A^2]$ *is that* $\rho_{11} \leqq 1/[(p-1)(q-1)]$.

For the normal scores version of $\mathcal{L}_N$, say $\mathcal{L}_N^*$, it follows, using the well known [cf. Chernoff and Savage (1958)] lower bound of 1 for $[\sigma_e^2 B^2(H)/A^2]$, that

$$\begin{aligned} e_{(\{\mathcal{L}_N^*\},\{Q_N\})} &\geqq [1-(pq)^{-1}(1-(p-1)(q-1)\rho_{11})]^{-1} \\ &\geqq 1, \quad \text{if } \rho_{11} \leqq [(p-1)(q-1)]^{-1}, \\ &\geqq \tfrac{1}{2}, \quad \text{in general} \end{aligned} \tag{5.9}$$

(using the fact that $\rho_{11} \leqq 1$). When the parent cdf, F, is normal, the left hand side of (5.8) takes the value 1, so $\{\mathcal{L}_N^*\}$ and $\{Q_N\}$ are asymptotically equally efficient. Similarly for the Wilcoxon version, $\{W_N\}$ of $\{\mathcal{L}_N\}$,

$$e_{(\{W_N\},\{Q_N\})} \geqq (.864)/[1-(pq)^{-1}(1-(p-1)(q-1)\rho_{11})] \geqq .432 \tag{5.10}$$

in general. For normal F, it is known that $\rho_{10} = 6\pi^{-1}\operatorname{Sin}^{-1}(-1/2(p-1))$, $\rho_{01} = 6\pi^{-1}\operatorname{Sin}^{-1}(-1/2(q-1))$, $\rho_{11} = 6\pi^{-1}\operatorname{Sin}^{-1}(1/2(p-1)(q-1))$, and hence, from (5.7), we obtain that for normal cdf's, $e_{(\{W_N\},\{Q_N\})}$ is equal to

$$\begin{aligned} &3pq\{\pi(p-1)(q-1)[1+6\pi^{-1}\{\operatorname{Sin}^{-1}(1/2(p-1)) \\ &\qquad + \operatorname{Sin}^{-1}(1/2(q-1)) + \operatorname{Sin}^{-1}(1/2(p-1)(q-1))\}]\}^{-1}. \end{aligned} \tag{5.11}$$

Actual computation shows that (5.11) is bounded below by 0.955 and above by 0.975.

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