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SEQUENTIAL INTERVAL ESTIMATION FOR THE MEANS OF NORMAL POPULATIONS[1]

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1. Introduction and summary. The theory of sequential analysis was originally developed by Wald [12] in the context of testing a simple hypothesis against a specific alternative. Wald and Stein [11] subsequently considered the problem of finding a sequential confidence interval of prescribed width (w) and confidence coefficient $(1 - \alpha)$ for the mean (m) of a normal distribution with known variance (σ^2). Their basic result, to the effect that no sequential procedure existed for this problem which had an average sample size less than the number of measurements required by the classical single-sample procedure, implies a substantial limitation as to what sequential interval estimation for the mean can accomplish in the case of a normal distribution. However, as we hope to show in this paper, by suitably redefining the problem, we can find sequential procedures either for obtaining a confidence interval for the difference in the means of two normal populations or for obtaining simultaneous confidence intervals for the means of k normal populations, which promise to be useful in some applications.

The basic idea in the reformulation of the problem is that the requirement to be put on the width of the confidence interval should depend on the location of the confidence limits, either with respect to some standard value or with respect to the confidence limits for the means of other populations. When this reformulation is appropriate to the problem at hand, Tables I and II indicate that a substantial saving is possible with the sequential procedure developed here at the risk of a relatively small (about 15 or 20%) increase in the average sample size if the least favorable parameter configuration should occur.

In the next section we will find a sequence of random intervals (J_n) such that $P[m \, \varepsilon \, J_n \text{ for all } n, n_0 \leqq n \leqq T] \geqq 1 - \alpha$. When $T = \infty$, sequences of this type, which might be called "confidence sequences" seem to have been first introduced into statistics by Wald in Chapter 10 of [12]. Such sequences were used by the present writer [7], [8], [9] as an important tool in finding sequential solutions to problems involving the selection of one of a finite number of possible decisions. Recently Darling and Robbins [2], [3], [4], [5] Robbins and Siegmund [10] have introduced and studied the properties of some new types of confidence sequences. Their work will be commented on briefly at the end of the paper.

2. Derivation of sequential confidence limits. Let $X_1, X_2, \cdots$ be a sequence of independent and normally distributed random variables with mean m and

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variance σ^2. Let

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt,$$

and let

$$Z(p) \quad \text{be defined by the relation} \quad \Phi(Z(p)) = 1 - p.$$

We define T_1 to be the smallest integer $\geqq 4\sigma^2[Z(\alpha/4)]^2/w^2$, and let $A_1 = wT_1/2$. We will now show that

$$(1) \quad P[-A_1 \leqq \textstyle\sum_{i=1}^{n} (X_i - m) \leqq A_1 \quad \text{for all} \quad n, \quad 1 \leqq n \leqq T_1] \geqq 1 - \alpha.$$

By considering the complement of the event in (1) we have

$$P[-A_1 \leqq \textstyle\sum_{i=1}^{n} (X_i - m) \leqq A_1 \quad \text{for all} \quad n, \quad 1 \leqq n \leqq T_1]$$
$$\geqq 1 - 2P[\textstyle\sum_{i=1}^{n} (X_i - m) > A_1 \quad \text{for at least one} \quad n, \quad 1 \leqq n \leqq T_1].$$

Now if $X(t)$ denotes a Wiener process such that $X(0) = 0$ and $X(t)$ has zero mean and variance $= \sigma^2 t$, it is obvious that

$$P[\textstyle\sum_{i=1}^{n} (X_1 - m) > A_1 \quad \text{for at least one} \quad n, \quad 1 \leqq n \leqq T_1]$$
$$(2) \qquad \leqq P[X(t) > A_1 \quad \text{for at least one} \quad t, \quad 0 < t < T_1]$$
$$= 2[1 - \Phi(A_1/\sigma T_1^{\frac{1}{2}})]$$

by a standard result (see page 221 of [1]). Collecting these facts, we have

$$P[-A_1 \leqq \textstyle\sum_{i=1}^{n} (X_i - m) \leqq A_1 \quad \text{for all} \quad n, \quad 1 \leqq n \leqq T_1]$$
$$\geqq 1 - 4[1 - \Phi(A_1/\sigma T_1^{\frac{1}{2}})] \geqq 1 - 4[1 - \Phi(\alpha/4)] = 1 - \alpha.$$

If we now define $\mathfrak{U}_n^{(1)}$ and $\mathfrak{L}_n^{(1)}$ by $\mathfrak{U}_n^{(1)} = \min_r (1 \leqq r \leqq n)[\bar{x}_r + A_1/r]$ and $\mathfrak{L}_n^{(1)} = \max_r (1 \leqq r \leqq n)[\bar{x}_r - A_1/r]$, then it follows from (1) and the definitions of A_1 and T_1 that $\mathfrak{U}_{T_1}^{(1)} - \mathfrak{L}_{T_1}^{(1)} \leqq w$ and

$$P[\mathfrak{L}_n^{(1)} \leqq m \leqq \mathfrak{U}_n^{(1)} \quad \text{for all} \quad n, \quad 1 \leqq n \leqq T_1] \geqq 1 - \alpha.$$

For the case when σ is unknown, let s^2 be an estimate of σ^2 with f degrees of freedom such that s is independent of $\sum_{i=1}^{n_0} X_i$ and fs^2/σ^2 has the chi-square distribution with f degrees of freedom. Let $t(p, f)$ denote the value of 'Student's' distribution with f degrees of freedom which is exceeded with probability p and let T_2 denote the smallest integer $\geqq \max\{n_0, 4s^2[t(\alpha/4, f)]^2/w^2\}$, and let $A_2 = wT_2/2$.
We will now show that

$$(3) \quad P[-A_2 \leqq \textstyle\sum_{i=1}^{n} (X_i - m) \leqq A_2 \quad \text{for all} \quad n, \quad n_0 \leqq n \leqq T_2] \geqq 1 - \alpha.$$

Since $\sum_{i=1}^{n_0} X_i$ was assumed to be independent of s^2, it follows by essentially repeating the preceding analysis that for a fixed value of s^2

$$P[-A_2 \leqq \textstyle\sum_{i=1}^{n} (X_i - m) \leqq A_2 \quad \text{for all} \quad n, \quad n_0 \leqq n \leqq T_2 \mid s^2]$$
$$\geqq 1 - 4\{1 - \Phi[(s/\sigma)t(\alpha/4, f)]\}.$$

Taking expectations, we obtain for the unconditional probability

$$P[-A_2 \leqq \sum_{i=1}^{n} (X_i - m) \leqq A_2 \quad \text{for all} \quad n, \qquad n_0 \leqq n \leqq T_2] \geqq 1 - 4\{1 - E\Phi[(s/\sigma)t(\alpha/4, f)]\} = 1 - 4P[t_f > t(\alpha/4, f)] = 1 - \alpha,$$

where the random variable t_f is distributed as Student's t with f degrees of freedom. (For values of α or f for which $t(\alpha/4, f)$ is not tabulated, the approximation $t(p,f) = Z(p)\{2f/\{2f - 1 - [Z(p)]^2\}\}^{\frac{1}{2}}$ given by Elfving [6] should ordinarily be accurate enough for most application.)

We now define $\mathcal{L}_n^{(2)}$ and $\mathcal{U}_n^{(2)}$ for $n = n_0, n_0 + 1, \cdots, T_2$ by

$$\mathcal{U}_n^{(2)} = \min_{(r)} (n_0 \leqq r \leqq n)[\bar{X}_r + A_2/r]$$

and

$$\mathcal{L}_n^{(2)} = \max_{(r)} (n_0 \leqq r \leqq n)[\bar{X}_r - A_2/r].$$

Then $\mathcal{U}_{T_2}^{(2)} - \mathcal{L}_{T_2}^{(2)} \leqq w$ and from (3), we have

$$(4) \qquad P[\mathcal{L}_n^{(2)} \leqq m \leqq U_n^{(2)} \quad \text{for all} \quad n, \qquad n_0 \leqq n \leqq T_2] \geqq 1 - \alpha.$$

3. The sequential interval estimation procedures.

3.1 *Interval estimation for the difference of two means. Let* X_{1r} and X_{2r} denote the measurements taken at the rth stage of the experiment ($r = 1, 2, \cdots$) with categories Π_1 and Π_2. We assume that all measurements are normally and independently distributed and that X_{ir} has mean m_i and variance σ_i^2 ($i = 1, 2$). Let $\bar{x}_{in} = \sum_{r=i}^{n} X_{ir}/n$, and let $X_r = X_{1r} - X_{2r}$, so that the variables X_r ($r = 1, 2, \cdots$) are normally distributed with mean $m = m_1 - m_2$ and variance $\sigma^{*2} = \sigma_1^2 + \sigma_2^2$.

In finding a confidence interval for $m = m_1 - m_2$, it would seem reasonable in some applications to insist that the width of the confidence interval for $m_1 - m_2$ should not exceed a pre-assigned value w as long as the value $m = 0$ is included in the confidence interval, but to progressively relax this requirement as the distance between zero and the confidence interval increases. These general considerations will now be made more precise. Suppose that $\{(\mathcal{L}_n, \mathcal{U}_n)\}$ denotes a confidence sequence for $m = m_1 - m_2$ that is, $P[\mathcal{L}_n \leqq m_1 - m_2 \leqq \mathcal{U}_n$ for all $n]$ $\geqq 1 - \alpha$. Let λ denote a positive constant which together with w and α is selected in advance of the experiment on the basis of practical considerations. When Π_1 and Π_2 both represent experimental categories, the following solution to the problem of obtaining a confidence interval for $m_1 - m_2$ would appear to be useful in some applications: stop the experiment and decide that $\mathcal{L}_n \leqq m_1 - m_2 \leqq \mathcal{U}_n$ as soon as one of the inequalities

(a) $\mathcal{U}_n - \mathcal{L}_n \leqq w$,

(b) $\mathcal{L}_n > 0$ and $\mathcal{U}_n - \mathcal{L}_n \leqq w + \lambda\mathcal{L}_n$,

(c) $\mathcal{U}_n < 0$ and $\mathcal{U}_n - \mathcal{L}_n \leqq w - \lambda\mathcal{U}_n$

are satisfied. When Π_1 represents an experimental category and Π_2 represents a standard or control, suppose because of cost consideration, side effects, etc. that the experimental category is not of practical value unless $m_1 - m_2 \geqq \Delta$, where Δ

is a non-negative constant also specified in advance of the experiment. Then the following procedure would seem to be a reasonable solution: stop the experiment and decide that $\mathcal{L}_n \leqq m_1 - m_2 \leqq \mathcal{U}_n$ as soon as one of the inequalities

(a₁) $\mathcal{U}_n - \mathcal{L}_n \leqq w$

(b₁) $\mathcal{L}_n \geqq \Delta$ and $\mathcal{U}_n - \mathcal{L}_n \leqq w + \lambda(\mathcal{L}_n - \Delta)$

(c₁) $\mathcal{U}_n < \Delta$

are satisfied.

When $\sigma^{*2} = \sigma_1^2 + \sigma_2^2$ is assumed known, we take $\mathcal{U}_n = \min_{(r)} (1 \leqq r \leqq n) \cdot [(\bar{x}_{1r} - \bar{x}_{2r}) + A_1/r]$ and $\mathcal{L}_n = \max_{(r)} (1 \leqq r \leqq n)[(\bar{x}_{1r} - \bar{x}_{2r}) - A_1/r]$ where $A_1 = wT_1/2$ and T_1 is the smallest integer $\geqq 4\sigma^{*2}[Z(\alpha/4)]^2/w^2$. With this choice of $\mathcal{L}_n$ and $\mathcal{U}_n$, it follows directly from the results of Section 2 that the sequential procedure which results from taking pairs of measurements until (a), (b) or (c) occurs (or (a₁), (b₁), (c₁) when comparing an experimental category with a control) is a closed sequential procedure with $n \leqq T_1$ and at the termination of the experiment

$$P[\mathcal{L}_n \leqq m_1 - m_2 \leqq \mathcal{U}_n] \geqq 1 - \alpha.$$

TABLE I

Empirical Results of the Sequential Confidence Interval Procedure for m[2, 3]

$\alpha = .05$,	$w = .392$	$\Delta = 0$,	$\lambda = .7$	$\alpha = .01$,	$w = .515$,	$\Delta = .5$,	$\lambda = 1$
m	$\bar{n}$	$\bar{\mathcal{L}}_n$	$\bar{\mathcal{U}}_n$	m	$\bar{n}$	$\bar{\mathcal{L}}_n$	$\bar{\mathcal{U}}_n$
−.5	51	−1.06	−.01	0	63	−.53	.49
0	122	−.20	.19	.5	114	.25	.76
.5	90	.23	.77	1.0	90	.66	1.33
1.0	63	.60	1.40	1.5	61	1.00	2.00
1.5	50	.98	2.03	2.0	46	1.35	2.68
2.0	39	1.36	2.67	2.5	37	1.68	3.32

A slight complication is caused by the possibility that at the termination of the experiment we might have $\mathcal{L}_n > \mathcal{U}_n$, in which case the resulting confidence interval is empty. This event never occurred in any of the empirical sampling studies and it seems that the probability that it will occur is very small. However, if it should happen, we would at present recommend repeating the experiment.

To get some comparative idea of the average sample size of this sequential procedure as contrasted to the classical single-stage procedure with fixed width w, a number of sampling experiments were carried out for the situation where π_1 is the experimental and π_2 the control category.

The results are summarized in Table I.

[2] For $\alpha = .05$ and $m = 0, .5, 1.0$ and 1.5, the values of $\bar{n}$, $\bar{\mathcal{L}}_n$, $\bar{\mathcal{U}}_n$ are means, each based on 200 trials. For each of the other cases, 100 trials are used.

[3] The value $\sigma = 1$ was used throughout. The corresponding fixed sample size for both combinations of α and w was $N = 100$.

Now we consider the case when σ^* is unknown. First a sample of n_0 pairs of measurements $(X_{11}, X_{21}), (X_{12}, X_{22}), \cdots, (X_{1n_0}, X_{2n_0})$ is taken. If we know *a priori* that $\sigma_1 = \sigma_2$, we estimate $\sigma^{*2} = \sigma_1^2 + \sigma_2^2$ by

$$s^2 = [\sum_{r=1}^{n_0} (X_{1r} - \bar{x}_{1n_0})^2 + \sum_{r=1}^{n_0} (X_{2r} - \bar{x}_{2n_0})^2]/(n_0 - 1)$$

with $f = 2(n_0 - 1)$ degrees of freedom; otherwise we estimate σ^{*2} by

$$s^2 = \sum_{r=1}^{n_0} [(X_{1r} - X_{2r}) - (\bar{x}_{1n_0} - \bar{x}_{2n_0})]^2/(n_0 - 1)$$

with $f = (n_0 - 1)$ degrees of freedom. Next, we take $\mathfrak{U}_n = \min_{(r)} (n_0 \leqq r \leqq n) \cdot [(\bar{x}_{1r} - \bar{x}_{2r}) + A_2/r]$ and $\mathfrak{L}_n = \max_{(r)} (n_0 \leqq r \leqq n)[(\bar{x}_{1r} - \bar{x}_{2r}) - A_2/r]$, where $A_2 = wT_2/2$ and T_2 is the smallest integer $\geqq \max \{n_0, 4s^2[t(\alpha/4, f)]^2/w^2\}$. After the sample of n_0 pairs is taken, one pair of measurements is to be taken at a time until one of the inequalities (a), (b), (c) (or (a_1), (b_1), (c_1)) are satisfied. The resulting sequential procedure is closed in the sense that at most $T_2 - n_0$ additional pairs will be needed to terminate the experiment.

The choice of n_0 when σ^* is unknown is obviously important in determining the efficiency of the sequential procedure. It seems to be difficult to give a precise rule. The problem here is similar to the determination of the size of the first sample in Stein's well known two-sample test. When there is little or no *a priori* information available about the value of σ^* we might select n_0 as the smallest integer so that $t(\alpha/4, f)$ is within a specified percent (perhaps ten percent) of its limiting value $Z(\alpha/4)$ corresponding to the situation when σ^* is known. When a moderately accurate estimate $\hat{\sigma}$ of σ^* is known *a priori*, this can be used to see that n_0 is not too long or too small in relation to the single-sample size that would be required for a confidence interval of specified width w with confidence coefficient $1 - \alpha$ if $\sigma^* = \hat{\sigma}$.

3.2 *Simultaneous confidence intervals for the means of k populations.* In this section we consider the situation where we are dealing with k categories $\pi_1, \pi_2, \cdots, \pi_k$. Let X_{ir} denote the rth measurement with category π_i and assume that all measurements are normally and independently distributed with a common variance σ^2, and the measurements $\{X_{ir}\} (r = 1, 2, \cdots)$ from π_i have a mean m_i $(i = 1, 2, \cdots, k)$. For simplicity we will suppose the greater the value of the mean, the more valuable the corresponding category.

In applications, it would seem that the width of the confidence interval for a particular category is unimportant if that category is clearly inferior to the other categories. This suggests that in practice we can relax the requirement that the statistical procedure provide a simultaneous confidence interval of width $\leqq w$ for all k categories and only require that the width of the confidence interval for the 'superior' categories be $\leqq w$ (while keeping the requirement that the overall confidence coefficient is $\geqq 1 - \alpha$).

A sequential procedure which meets the revised requirement is easily obtained using the results of Section 2. First, let n_i denote the number of measurements taken with category π_i, and say that category π_i is 'eliminated' when we stop taking any more measurements with this category. We now consider the case when

σ is known *a priori*. Let $\alpha_1 = 1 - (1 - \alpha)^{1/k}$, $A_1^* = wT_1^*/2$ where T_1^* is the smallest integer $\geqq 4\sigma^2[Z(\alpha_1/4)]^2/w^2$, let $\bar{x}_{ir} = \sum_{s=1}^{r} X_{is}/r$, and for each i we define $\mathfrak{L}_r^{(i)}$ and $\mathfrak{U}_r^{(i)}$ by

$$\mathfrak{U}_r^{(i)} = \min (\nu)_{1 \leqq \nu \leqq r} [\bar{x}_{i\nu} + (A_1^*/\nu)] \quad \text{for} \quad r \leqq n_i,$$
$$\mathfrak{U}_r^{(i)} = \mathfrak{U}_{n_i}^{(i)} \quad \text{for} \quad r > n_i,$$
$$\mathfrak{L}_r^{(i)} = \max (\nu)_{1 \leqq \nu \leqq r} [\bar{x}_{i\nu} - (A_1^*/\nu)] \quad \text{for} \quad r \leqq n_i,$$
$$\mathfrak{L}_r^{(i)} = \mathfrak{L}_{n_i}^{(i)} \quad \text{for} \quad r > n_i.$$

Then we have the following sequential procedure. At the first stage, we take one measurement with each of the k categories. At the rth stage ($r = 2, \cdots$) we take one measurement with each category not yet eliminated after the first $(r - 1)$ stages, and then eliminate any category π_i for which either $\mathfrak{U}_r^{(i)} - \mathfrak{L}_r^{(i)} \leqq w$ or

TABLE II

Average Total Sample Size for Simultaneous Confidence Limits for 5 Means[4]

Parameter Configuration $(m_1, m_2, m_3, m_4, m_5)$	Average Total Sample Size for Sequential Procedure	Total Sample Size for Single-Stage Procedure
(0,0,0,0,0)	606	530
(0,0,0,0,1)	376	530
(0,0,0,2,2)	334	530

$\mathfrak{U}_r^{(i)} < \mathfrak{L}_r^{(j)}$ for some $j \neq i$. The experiment is concluded as soon as all categories are eliminated with the statement that $\mathfrak{L}_{n_i}^{(i)} \leqq m_i < \mathfrak{U}_{n_i}^{(i)}$ for each i, $i = 1, 2, \cdots, k$. From the results of Section 2 it follows that for each $i, n_i \leqq T_1^*$, and $P[\bigcap_{i=1}^{k} \{\mathfrak{L}_{n_i}^{1} \leqq m_i \leqq \mathfrak{U}_{n_i}^{(i)}\}] \geqq 1 - \alpha$. In order to obtain some idea of the efficiency of this sequential procedure as compared to the corresponding single-stage procedure where all k confidence intervals have width $= w$, a number of sampling experiments were carried out and the results are summarized in Table II.

When σ is unknown, we start by taking a sample of n_0 measurements with each category, and estimate σ^2 by

$$s^2 = \sum_{i=1}^{k} \sum_{r=1}^{n_0} (X_{ir} - \bar{x}_{in_0})^2/k(n_0 - 1) \quad \text{with} \quad f = k(n_0 - 1)$$

degrees of freedom. Let $\alpha_2 = \alpha/k$, $A_2^* = wT_2^*/2$ and $T_2^* =$ the smallest integer $\geqq \max \{n_0, 4s^2[t(\alpha_2/4, f)]^2/w^2\}$. We now define $\mathfrak{L}_r^{(i)}$ and $\mathfrak{U}_r^{(i)}$ for each $i (i = 1, 2, \cdots, k)$ by

$$\mathfrak{U}_r^{(i)} = \min (\nu)_{n_0 \leqq \nu \leqq r} [\bar{x}_{i\nu} + (A_2^*/\nu)] \quad \text{for} \quad n_0 \leqq r \leqq n_i,$$
$$\mathfrak{U}_r^{(i)} = \mathfrak{U}_{n_i}^{(i)} \quad \text{for} \quad r > n_i,$$
$$\mathfrak{L}_r^{(i)} = \max (\nu)_{n_0 \leqq \nu \leqq r} [\bar{x}_{i\nu} - (A_2^*/\nu)] \quad \text{for} \quad n_0 \leqq r \leqq n_i,$$
$$\mathfrak{L}_r^{(i)} = \mathfrak{L}_{n_i}^{(i)} \quad \text{for} \quad r > n_i.$$

[4] The values $k = 5$, $\sigma = 1$, $\alpha = .05$ and $w = .5$ were used throughout. Each entry in the second column is based on the results of 500 trials.

After taking the sample of n_0 measurements with each category, at each successive stage of the experiment we take one measurement with each category not yet eliminated, where any category π_i is eliminated as soon as either $\mathfrak{U}_r^{(i)} - \mathfrak{L}_r^{(i)} \leqq w$ or $\mathfrak{U}_r^{(i)} < \mathfrak{L}_r^{(j)}$ for any $j \neq i$ $(r = n_0, n_0 + 1, \cdots, T_2^*)$. As before, we conclude that $\mathfrak{L}_{n_i}^{(i)} \leqq m_i \leqq \mathfrak{U}_{n_i}^{(i)}$ for $i = 1, 2, \cdots, k$ and $P[\bigcap_{i=1}^k \{\mathfrak{L}_{n_i}^{(i)} \leqq m_i \leqq \mathfrak{U}_{n_i}^{(i)}\}] \geqq 1 - \alpha$. When the experiment is concluded, we can consider the categories $\{\pi_i\}$ for which $\mathfrak{U}_{n_i}^{(i)} - \mathfrak{L}_{n_i}^{(i)} \leqq w$ as forming the 'superior' group and the remaining categories as forming the 'inferior' group.

4. Concluding remarks. Different confidence sequences lead in general to different solutions to the statistical problem. There are an infinite number of different confidence sequences available, and the selection of one which is optimum in some reasonable sense is still an unsolved problem. The particular confidence sequence developed in Section 2 was used in preference to an earlier confidence sequence for the mean given in [8], since it seems to come fairly close to minimizing the average sample size in the least favorable situation when the population means are equal while still allowing a substantial saving when the parameter configuration is favorable.

Recently some new types of confidence sequences have been developed by Darling and Robbins [2], [3], [4], [5], and Robbins and Siegmund [10]. Their sequences have the desirable property that the width of the confidence interval approaches 0 as the sample size increases, and hence do not require that the constant w be specified in advance of the experiment. A preliminary calculation indicates that these confidence sequences lead to sequential procedures for the problems of the present paper which would require a great increase in the average sample size in the least favorable situation when all the means are equal. However, the Darling-Robbins confidence sequences are still in the process of being improved and a definitive verdict on their value in statistical applications is best left to the future.

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