*The Annals of Mathematical Statistics*
1969, Vol. 40, No. 3, 1142–1161

# ABSTRACTS OF PAPERS

*(An abstract of a paper presented at the European meeting, Amsterdam, Netherlands, September 2–7, 1968. Additional abstracts appeared in earlier issues.)*

**15. Inference in multivariate normal populations with structure** (preliminary report). GEORGE P. H. STYAN, University of Minnesota, Minneapolis.

We consider inference based on observations from a multivariate normal population with unspecified mean vector; the covariance matrix is assumed to have a specified form or structure. We suppose that the correlation coefficients are all known and the variances completely unknown. We estimate the variances by the method of maximum likelihood and obtain a closed form for the resulting equations using the notion of Hadamard product of matrices. These equations form a set of simultaneous nonhomogeneous quadratic equations and in general cannot be solved analytically. We show that they have a unique real solution and obtain an approximation to this by the Newton-Raphson technique. We prove that the first iterate based on a consistent trial solution is a best asymptotically normal estimate. The asymptotic efficiency of the sample variances over the maximum likelihood estimates is computed and bounds obtained. The Fréchet-Cramér-Rao inequality leads to some interesting results in matrix algebra concerning Hadamard products of positive definite matrices and their inverses. We evaluate likelihood ratio tests for a given correlation structure in a general multivariate normal population and for equality of variances given the correlation coefficients. These results are specialized to the situation where all correlation coefficients are equal and known. (Received 28 February 1969.)

*(Abstracts of papers presented at the Central Regional meeting, Iowa City, Iowa, April 23–25, 1969. Additional abstracts have appeared in earlier issues.)*

**15. Nonparametric tests for multivariate two-way layouts.** RASHID AHMAD, University of Michigan.

For the case of one observation per cell the hypothesis "no row effect" following Friedman (1937) form can be stated $\Omega(H_{0R}) = \{F_{ij} = \cdots = F_{rj}$ for all $j\}$ where the $F_{ij}$ are $p$-dimensional continuous distributions. Using the ideas of Pitman [(1937), (1938)], Scheffé (1943), Lehmann and Stein (1949), and extending the results of Bell, Deshpandé and Geller (1968) to multivariate case one proves: THEOREM 1. (a) *If $T$ is a statistic distribution-free* (DF) wrt $\Omega(H_{0R})$, *then $T$ has a discrete $H_0$-distribution with probabilities integral multiples of* $(r!)^{-c}$, (b) *$T$ is* DF wrt $\Omega(H_{0R})$ *if and only if $T$ is a function of permutation statistic, and* (c) *for each discrete distribution $G$ with probabilities which are integral multiples of* $(r!)^{-c}$ *there exists a* DF *statistic $T$ with $H_0$-distribution $G$.* THEOREM 2. *The most powerful* DF *test against a specific alternative is based on the permutation statistic of the likelihood function.* THEOREM 3. *Each randomized test based on a similar partition is* DF; *and for each preassigned distribution $G$, there exists a randomized statistic with $H_0$-distribution $G$.* Analogous results hold for "no column effect," that is $\Omega(H_{0C})$. Finally one tests "no interaction" by independent contrasts. (Received 3 March 1969.)

**16. Some aspects of semi-Bayesian discrimination.** PETER ENIS and SEYMOUR GEISSER, State University of New York at Buffalo.

When classifying an observation into one of two multivariate normal populations $N(\mu_1, \Sigma_1)$ or $N(\mu_2, \Sigma_2)$ where $\mu_1$ and $\Sigma_1$ $(i = 1, 2)$ are known, the discriminant which minimizes the (total) probability of misclassification is based on the ratio of the pdf's of the populations. When the parameters are unknown, the "usual" procedure, in the case of a linear discriminant (i.e., $\Sigma_1 = \Sigma_2$), has been to substitute the sample estimates for these unknown parameters and use the sample discriminant thus obtained as a basis for assigning observations. This sample discriminant, however, does not retain (from the sampling theory point of view) the property of minimizing the probability of misclassification. We obtain here a Bayesian justification for this procedure by showing that the sample discriminant which minimizes the posterior squared error loss for estimation of the (true) population discriminant is, in fact, the "usual" sample discriminant plus a bias or "weighting" factor which vanishes for equal sample sizes. This is shown for both the linear and quadratic ($\Sigma_1 \neq \Sigma_2$) discriminant and for various assumptions concerning the parameters. The predictive errors of classification associated with the use of the above obtained quadratic sample discriminant (all parameters unknown) are investigated and estimates of these errors are obtained. For purposes of classification, a more appropriate criterion for choosing a sample discriminant is the minimization of the total predictive error of classification. In particular, it can be shown that when the prior probabilities and sample sizes are equal, the "usual" estimate of the true linear discriminant does, in fact, minimize the average probability of misclassification. Finally, it is pointed out that the sample discriminants which minimize the average probability of misclassification are essentially obtainable from the posterior odds ratio approach to classification as developed by Geisser. (Received 3 March 1969.)

**17. The admissibility of certain invariant statistical tests involving a $p$-dimensional location parameter.** MARTIN FOX, Michigan State University.

A theorem of Lehmann and Stein [*Ann. Math. Statist.* **24** (1953) 473–479] is extended to the case where $Y$ and, hence, the location parameter take values in $R^p$. The moment condition becomes $E_{10}\|Y\|^p < \infty$, $E_{20}\|Y\|^p < \infty$. All other conditions are unchanged. The example of Fox and Perng (Math. Research Center Summary Report #873 (1968)) is adapted to show that, if the moment condition fails, inadmissibility may result. Proofs are elementary extensions of those in the cited papers. This result contrasts sharply with the situation when the location parameter is to be estimated. In that case inadmissibility occurs under general conditions when $p \geqq 3$ [L. D. Brown, *Ann. Math. Statist.* **37** (1966) 1087–1136]. (Received 6 March 1969.)

**18. On asymptotically robust rank tests.** JOSEPH L. GASTWIRTH, Johns Hopkins University.

This paper continues the study of asymptotically robust rank tests initiated by the author, [*J. Amer. Statist. Assoc.* **61** (1966) 929–948]. For every density function there is an asymptotically most powerful rank test (amprt). For a family $\mathfrak{F}$ of density functions the maximum robust rank test, $R$, maximizes the minimum limiting Pitman efficiency relative to the amprt for each member of $\mathfrak{F}$. Sometimes two members of the family $\mathfrak{F}$ are "most different" and the maximin robust test $R^*$ for this extreme pair is the maximin robust test for the entire family $\mathfrak{F}$. Necessary and sufficient conditions are given for $R^*$ to be the maximin robust test for the family $\mathfrak{F}$. Examples of several families are discussed illustrating the theorem. In particular, numerical work indicates that the maximin robust test $R^*$ for observations from either the normal or Cauchy density is the maximin robust rank test for samples from any $t$-density. (Received 27 February 1969.)

**19. Stopping time of a rank order SPRT for symmetry based on Lehmann alternatives.** Z. GOVINDARAJULU, University of Kentucky. (By title)

Parent (see *Sequential Ranking Procedures*, Tech. Rep No. 80 (1965) Dept. of Statistics Stanford Univ.) developed sequential rank procedure and sequential signed rank procedure for Lehmann alternatives. Savage and Sethuraman (see *Ann. Math. Statist.* **37** (1966) 1154–

1160) showed, under general conditions, that the two-sample rank order SPRT based on Lehmann alternatives surely terminates and the moment-generating function of the sample size is finite. In this paper, a one-sample rank order SPRT for symmetry is derived on the basis of Lehmann alternatives. It is shown that, under very general conditions, this procedure terminates with probability one and the moments of the stopping time are finite for all alternatives. (Received 13 March 1969.)

**20. Nonparametric hypotheses: Independence, uniformity, and wreath products** (preliminary report). K. Guire and C. B. Bell, University of Michigan.

Let $\Omega(H_0) = \{F: F$, nonatomic on $Z$ and invariant wrt group $G\}$. Orbits are generated by the wreath product group $G \wr S_n$. For finite $G$ (e.g. univariate and multivariate symmetry) the usual Lehmann-Stein (1949) theory applies as in Bell and Haller (1969) for constructing all DF tests, MPDF tests, etc. Trivially, with $G = \{e\}$ this formulation includes the multivariate $k$-sample and randomness hypotheses (Bell and Smith (1969)). For infinite $G$ (as in spherical symmetry) one employs the Neyman structure theorem to get the desired results. To include independence hypotheses one needs the more general formulation $\Omega(\hat{H}_0) = \{F \times J: F$, nonatomic on $Z$ and $J$, uniform on $Z'\}$. Here $Z$ is the space of values of a minimal sufficient statistics, and $Z'$ is isomorphic to the group generating the sample space from $Z'$. This model includes all cases above but is limited to compact groups (Stapleton (1963)). One can construct all DF and MPDF tests, but additional assumptions are necessary to characterize rank tests. (Received 6 March 1969.)

**21. Structural inference for the linear model with auto-regressive error.** M. Safiul Haq, University of Western Ontario.

The responses from a linear model with first order autoregressive error have been considered to have been obtained from a set of realised error variables by a transformation discussed by Fraser (1968, *The Structure of Inference*, Wiley, Chapter III). The error variables depend on the autocorrelation parameter $\rho$ and avoiding the degenerate case (linear dependence) the model thus obtained is a composite response model. The orbit of the response describes certain characteristics of the error. For known value of $\rho$ the inference is based on the distribution of the error regression coefficients and the error residual standard deviation conditioned on the orbit, the known error characteristics. For uncorrelated standard normal error variable this probability element does not depend on the orbit [Fraser, *Ann. Math. Statist.* **38** (1967) 1456–65]. However, in this case it does depend on the orbit and the marginal likelihood of $\rho$ obtained from the marginal probability element for the orbit provides the basis of inference about $\rho$. When the error variables have a periodic structure the error variables can be transformed into uncorrelated variables by a suitable transformation depending on $\rho$. Naturally the orbit of the transformed response depends on $\rho$ and offers a useful mode of inference about $\rho$. (Received 17 February 1969.)

**22. An inequality of jointly normally distributed random variables.** Kumar Jogdeo, University of Illinois.

Let $X_1, \cdots, X_k$ be jointly normally distributed random variables centered at the expectations. It is shown that either $P[|X_i| < c_i; i = 1, \cdots, k] > \prod_{i=1}^k P[|X_i| < c_i]$, for every set of $k$ positive constants or equality holds for every set. This gives a characterization of independence in the family of multivariate normal distributions. The results constitute sharpening of the inequalities given recently by Sidak (*Ann. Math. Statist.* **39**). It is also shown that the above result is false if $|X|$ is replaced by $X$ and $k > 2$. (Received 5 March 1969.)

**23. Fractions obtained by omitting a $2^{n-k}$ replicate from a $2^n$ factorial design.** Peter W. M. John, University of Texas.

When a $2^{n-k}$ fraction is omitted from a $2^n$ factorial, the remaining points form $2^k - 1$ overlapping half replicates. If $Q$ is a defining contrast of the omitted fraction, one of the half replicates is defined by $I = -Q$. Let $P$ be an effect. A necessary and sufficient condition for $P$ to be estimable is that the alias set containing $P$ also contains at least one suppressed effect. The least squares estimate of $P$ is the unweighted average of the estimates of $P$ from each of the half replicates in which $P$ is estimable. Examples are given. (Received 3 March 1969.)

**24. Robustness and sensitivity analysis.** ROBERT A. LEW, University of Michigan. (Introduced by James G. Wendel.)

Let $h(\theta)$, $g_1(\theta)$, $g_2(\theta)$, $\cdots$, $g_n(\theta)$ be real valued Borel measurable functions of $\theta$. Let $C$ be the class of distribution functions $F$ satisfying constraints of the form $\int_{-\infty}^{\infty} g_i(\theta)\, dF(\theta) = c_i$, $i = 1, 2, \cdots, n$. The expectation of $h(\theta)$ under $F$, $E_F(h(\theta))$, is conditional on the data. The notation disregards this dependence as the data never enter the theory explicitly. *Sensitivity Analysis.* We prove the range of $E_F(h(\theta))$, the posterior expectation of $h(\theta)$ under prior $F$, over the class $C$ equals the range over $C_n \subseteq C$, the subclass of purely atomic priors with at most $n$ atoms. Depending on constraints, the range is computed directly or found by a computer search over a bounded surface. *Robustness* Let $E_{F_0}(h(\theta))$ reflect the effectiveness of a statistical procedure under an hypothesized distribution $F_0(\theta)$. The range of $E_F(h(\theta))$ over $C$ indicates the robustness of $F_0(\theta)$ for the procedure. When $h(\theta)$, $g_1(\theta)$, $g_2(\theta)$, $\cdots$, $g_n(\theta)$ form a Tchebycheff system the range is computed directly for procedures involving confidence intervals and ratio estimates. The above results generalize. The range is found over subclasses of $C$ restricted to symmetric and/or unimodal $F$. Expressions of the form $E_F(h_F(\theta))$ where $h_F(\theta)$ depends on $F$ (e.g., quadratic loss, variance) replace $E_F(h(\theta))$. (Received 20 February 1969.)

**25. Bayesian solutions for the Fieller-Creasy problem.** RUSSELL F. KAPPENMAN and SEYMOUR GEISSER, State University of New York at Buffalo.

Suppose we have a random sample from a bivariate normal population, $N(\mu, \Sigma)$, where $\mu' = (\mu_1, \mu_2)$. The elements of $\mu$ and of $\Sigma$ are assumed to be unknown. Two methods are given for finding Bayesian posterior limits for $\eta = \mu_2/\mu_1$. The prior density that is used for $\mu$ and $\Sigma$ is one which purports to reflect to a large degree prior ignorance. It is given by $g(\mu, \Sigma) \propto |\Sigma|^{3/2}$. The first method always yields a finite interval which may be centered at any desired point $c$, for $\eta$. The $\gamma$ posterior interval is $(c - d, c + d)$, where $d$ is determined as follows. Let $A = \{a_{ij}\}$ be the matrix of sums of squares and cross products of deviations about the sample mean. Then $d = 2(\delta^*)^{\frac{1}{2}}(1 - r)^{-\frac{1}{2}}[(a_{22}/a_{11})^{\frac{1}{2}} - r + (1 - r^2)^{\frac{1}{2}}c]$. $r$ is the sample correlation coefficient and $\delta^*$ is the $\gamma$th percentage point of a certain distribution which is asymptotically doubly noncentral $F$. The second method is based on a transformation which has been used in the finding of a fiducial (and Bayesian) solution to the problem when $\Sigma$ is known. The limits in this case may sometimes be the entire real line with possibly a finite interval excluded. In addition, the posterior density of $\eta$ has been found. This is an aid in determining $c$ for the first method and indicates why exclusive limits are inappropriate in a Bayesian framework. (Received 20 February 1969.)

**26. Some asymptotic simultaneous tests for multivariate moving average processes.** P. R. KRISHNAIAH and P. K. SEN, Aerospace Research Laboratories, Wright-Patterson AFB, and University of North Carolina.

For $i = 1, 2, \cdots, k$, let $t$ be discrete time parameter and $\{\mathbf{Y}_{it}\}$ be a $p$-variate moving average process such that $\mathbf{Y}_{it} = \mathbf{M}_{it} + \boldsymbol{\varepsilon}_{it}$, $t = 1, 2, \cdots, n$, where $\boldsymbol{\varepsilon}_{it} = \sum_{j=0}^{m} B_j \mathbf{n}_{i,t-j}$. In the above model, $\mathbf{M}_{it}$ are column vectors of unknown parameters, $B_0, \cdots, B_m$ are unknown matrices whose elements are real constants, and the $\mathbf{n}_{it}$ are independent and identically distributed random vectors with a continuous cumulative distribution function $F(\mathbf{x})$. We assume that $F(\mathbf{x})$ is diagonally symmetric about $\mathbf{0}$. Now, let $\mathbf{M}_t = (\mathbf{M}_{1t}, \cdots, \mathbf{M}_{kt})$ $\mathbf{c}_l' = (c_{l1}, \cdots, c_{lk})$ and $\boldsymbol{\mu}_{lt} = \mathbf{M}_t\mathbf{c}_l$ for $l = 1, 2, \cdots, q$. Also, let $H_l : \boldsymbol{\mu}_{lt} = \mathbf{0}$, $H = \bigcap_{l=1}^q H_l$, $A_l : \boldsymbol{\mu}_{lt} \neq \mathbf{0}$, $\boldsymbol{\mu}_{lt}' = (\mu_{lt1}, \cdots, \mu_{ltp})$, $A = \supset_{l=1}^q A_l$; for any given $l$, it is assumed that the directions of $\mu_{l1}, \cdots, \mu_{ln}$ are the same when $A_l$ is true. In this paper, we consider various procedures for testing $H_1, \cdots, H_q$ and $H$ simultaneously against $A_1, \cdots, A_q$ and $A$ respectively. The test statistics are based upon either the sample mean vectors, or the vectors of the sign or the signed-rank statistics. The asymptotic properties of these tests are compared. (Received 26 February 1969.)

**27. Optimum block designs.** Viktor Kurotschka, University of Wisconsin, Milwaukee.

Consider the general two way classification $Y_{ijk} = m + a_i + b_j + Z_{ijk}$, $i = 1, \cdots, I$; $j = 1, \cdots, J$; $k = 1, \cdots, n_{ij}$, $n = \sum_{i=I}^I \sum_{j=1}^J n_{ij}$ with uncorrelated $Z_{ijk}$ and $\operatorname{Var} Z_{ijk} = \sigma^2$, and designs $\{n_{ij}; i = 1, \cdots, I; j = 1, \cdots, J\} \varepsilon N: = \{\{n_{ij}\}; \sum_{i=1}^I n_{ij} = n_{\cdot j}, \sum_{j=1}^J n_{ij} = n_{i\cdot}\}$ with restriction on the number of observations by given $\{n_{i\cdot}, n_{\cdot j}; i = 1, \cdots, I, j = 1, \cdots, J\}$. It is proved that $\{n_{ij}^* : = n_{j\cdot}n_{\cdot j}/n\}$ minimizes among all $\{n_{ij}\} \varepsilon N$ the generalized variances of the systems of best unbiased estimators for any maximal system of estimable functions of $\{a_i\}$, $\{b_j\}$ and $\{m, a_i, b_j\}$. Under the usual assumption of normal distribution and restriction to the common invariant tests of $H$: $a_1 = \cdots = a_I = 0$, respectively, $b_1 = \cdots = b_J = 0$ the design $\{n_{ij}^*\}$ has all optimum properties described and discussed by Kiefer (1958). This is a consequence of the fact that $\{n_{ij}^*\}$ maximizes among all $\{n_{ij}\} \varepsilon N$ the power of the corresponding $F$-tests uniformly in all alternatives $a_i \neq 0$, respectively, $b_j \neq 0$. This optimum property (first introduced by Wald (1943) but at the same time rejected by him because of the assumed lack of realizations) of $\{n_{ij}^*\}$ provides optimum statement for balanced block designs by dropping some or all restrictions $\{n_{i\cdot}, n_{\cdot j}\}$ according to a fixed samplesize $n$. (Received 3 March 1969.)

**28. On a class of selection procedures including procedures for restricted families of distributions.** S. Panchapakesan, Purdue University.

Let $\pi_1, \cdots, \pi_k$ be $k$ continuous populations with associated distribution functions $F_{\lambda_i}$, $\lambda_i \varepsilon \Lambda$, an interval on the real line, $i = 1, \cdots, k$. A class of selection procedures is defined for selecting a non-empty subset of the $k$ populations containing the one with the largest or the smallest $\lambda_i$ subject to a certain probability requirement. A sufficient condition is obtained for the monotonicity of a certain probability integral leading to the evaluation of the infimum of the probability of a correct selection. Sufficient conditions are also obtained regarding the supremum of the expected size of the selected subset. Other properties of this class of procedures are also considered. More specific results are obtained in the case where the density $f_\lambda(x) = \sum_{j=0}^\infty w(\lambda, j)g_j(x)$, a convex mixture of the density functions $g_j(x)$. Selection procedures are considered also for restricted families of distributions. On the space of distributions a general partial ordering, which includes some of the orderings earlier considered by Barlow and Gupta (to appear in the *Ann. Math. Statist.*) as special cases, is considered. Results similar to those of Barlow and Gupta are obtained with this general ordering. (Received 24 February 1969.)

**29. A class of conjugate prior distributions, and optimal allocation.** Ronnie L. Morgan, University of Missouri, Columbia. (Introduced by H. D. Brunk.)

Let $f(x_j; \tau) = \exp[x_j\tau - \Theta(\tau)]f(x_j)$, $j = 1, 2, \cdots, n$, denote a probability density of a distribution belonging to a one-parameter exponential family. Thus $\Theta(\tau) = \log \int e^{x\tau} dF(x)$,

$E(\tilde{x} \mid \tau) \equiv \theta(\tau) = \Theta'(\tau)$, and Var $(\tilde{x} \mid \tau) = \theta'(\tau)$. A prior distribution is assumed on $\tau$ with density $g(\tau)\alpha \exp[a\tau - b\Theta(\tau)]$, thus yielding a posterior distribution with density $g(\tau \mid \tilde{r} = r) = \alpha \exp[a'\tau - b'\Theta(\tau)]$ where $r = \sum_{j=1}^{n} x_j$, $a' = a + r$, and $b' = n + b$. It is shown under certain regularity conditions that $E\{\text{Var}''[\theta(\tilde{\tau}) \mid \tilde{r}]\} = b \text{ Var}'[\theta(\tilde{\tau})]/(n + b)$, where Var$''$ and Var$'$ denote the variances with respect to the posterior and prior distributions respectively. This relationship is applied to the problem of optimum stratified sampling and generalizes a result of W. A. Ericson [*J. Amer. Statist. Assn.* **60** (1965) 750–771]. The solution is shown to be a generalization of the Neyman stratified allocation result. (Received 28 February 1969.)

**30. The distribution of frequency counts of the Poisson distribution.** Chong Jin Park, University of Nebraska.

Let $Y_1, Y_2, \cdots, Y_n$ be a random sample from the poisson distribution with parameter $\lambda$. Let $T = \sum_{i=1}^{n} Y_i$ and let $s_j$ denote the number of $Y_i$'s equal to $j$; $j = 0, 1, 2, \cdots, T$. In this paper we study the conditional joint distribution of $(s_0, s_1, \cdots, s_k)$, for a fixed $k$, given $T = t$. In particular we prove that the conditional joint distribution of $(s_0, s_1, \cdots, s_k)$ given $T = t$ is asymptotically jointly normal as $n$ and $t$ tend to infinity with $t/n \rightarrow \lambda$. In the proof we use the characteristic function. This result gives an alternate proof of the limiting normality of the distribution of classical occupany numbers. (Received 3 March 1969.)

**31. A duality between autoregressive and moving average processes concerning their least squares parameter estimates.** David A. Pierce, University of Missouri, Columbia.

The methods employed in least squares parameter estimation for moving average (MA) processes differ from those appropriate for autoregressive (AR) processes, as only the latter are linear in the parameters. There is nevertheless an interesting duality between these two classes of time series models: if AR and MA series, each of the same order and with the same parameter values, are generated from the same sequence of errors, then [to $O_p(1/n)$ for series of $n$ observations] the least squares estimates calculated from the MA series will *under*estimate the true parameter values by the same amount that those determined from the AR series will *over*estimate them. This result is established via a linear approximation of the moving average errors (considered as functions of the parameters for a given series) in a neighborhood of the true parameter values. Since the distribution of the AR estimates is normal [Mann and Wald, (1943) *Econometrica*] and therefore symmetric, this shows that the least-squares estimates from the MA series, and thus for any MA process, possess the same large-sample distribution as do those of the corresponding AR process, a result derived also by Box and Jenkins [*Statistical Models for Forecasting and Control*, Holden-Day (to appear)]. (Received 26 February 1969.)

**32. A special case of the distribution of the median.** Herman Rubin and Suresh R. Paranjape, Purdue University.

Let $t$ be the translation parameter of a process $X(t)$, $-\infty < t < \infty$. The likelihood ratio of the process $X(t)$ at $t$ against $t = 0$ can be written as $\exp[w(t) - \frac{1}{2}|t|]$, $-\infty < t < \infty$, where $w(t)$ is a standard Wiener process. For the absolute error-loss function the best invariant estimator of the translation parameter is the median of the *a posteriori* distribution. We treat a special case of the *a posteriori* distribution, when *a priori* distribution for $t$ is the Lebesgue measure on the real line and obtain the distribution of the median. (Received 6 March 1969.)

**33. On the sequential Wilcoxon test.** J. SETHURAMAN, Florida State University. (Invited)

Sequential tests based on the Wilcoxon statistic for the two sample problem have been proposed before by R. A. Bradley, L. J. Rhodes and F. Wilcoxon (1963), "Two sequential two-sample grouped rank tests with applications" (*Biometrics* **19** 58–84). These tests require that the observations be grouped. We describe a sequential Wilcoxon test which does not involve grouping of the observations and also show how to choose the initial constants of the test to achieve certain desired properties. Finally, we show that our test terminates with probability one and that the sample size at termination has a moment generating function. (Received 26 February 1969.)

**34. A class of multi-urn models.** NORMAN C. SEVERO, State University of New York at Buffalo.

Given $r$ urns and a total of $N$ balls, let $X_i(t)$ be the number of balls in the $i$th urn at time $t \geqq 0$, $\mathbf{x}(t)$ the $r$-tuple whose $i$th component is $X_i(t)$, $S_{N,r}$ the set of $r$-tuples $\mathbf{n}$ whose components are non-negative integers summing to $N$, $\lambda_{ij}(t)$ non-negative continuous functions of $t$ for $i \neq j$, and $\mathbf{e}_i$ the $r$-tuple with a one in the $i$th position and zeros elsewhere. ASSUMPTIONS. (1) If $\mathbf{x}(t) = \mathbf{n}$, then the probability that $\mathbf{x}(t + \Delta t) = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j$ for $\Delta t > 0$ and $i \neq j$ is $n_i \lambda_{ij}(t)\Delta t + o(\Delta t)$. (2) The probability of any transition other than as described in (1) is $o(\Delta t)$. (3) The probability of no change is $1 - \sum_{i \neq j} n_i \lambda_{ij}(t) + o(\Delta t)$. These assumptions determine transition probabilities $P(\mathbf{n}, t \mid \mathbf{m}, s)$ that $\mathbf{x}(t) = \mathbf{n}$ given that $\mathbf{x}(s) = \mathbf{m}$, where $0 \leqq s \leqq t$ and both $\mathbf{m}$ and $\mathbf{n}$ belong to $S_{N,r}$. We let $\mathbf{y}(m, \{p_j\})$ denote the multi-nomial random vector whose $r$ components sum to $m$ and for which $p_j$ is the probability of an increase of one unit in the $j$th component and $\sum p_j = 1$.

THEOREM. *For fixed $s$, $t$, and $\mathbf{m}$, the distribution $\{P(\mathbf{n}, t \mid \mathbf{m}, s), \mathbf{n} \varepsilon S_{N,r}\}$ is equal to the distribution of $\mathbf{z} = \sum_{i=1}^{r} \mathbf{y}_i(m_i, \{p_{ij}\})$, where the $\mathbf{y}_i$ aee statistically independent and $p_{ij} =$* $P(\mathbf{e}_j, t \mid \mathbf{e}_i, s)$. The Ehrenfest multi-urn model is a special case in which $\lambda_{ij}(t) \equiv \alpha_j$ for all $i$ and the $p_{ij}$ are known. See Karlin and McGregor [*J. Appl. Prob.* **2** (1965) 352–376). (Received 4 March 1969.)

**35. An asymptotic expansion for the nonnull distribution of Hotelling's generalized $T_0^2$ statistic.** MINORU SIOTANI, University of North Carolina and Nihon University. (By title)

Let $Z$ be a $p \times m$ matrix whose columns are independently distributed according to $p$-variate normal distributions $N_p(\mu_i, \Sigma)$, $i = 1, \cdots, m$, and let $nS_n$ be a $p \times p$ matrix which is independent of $Z$ and is subject to the central Wishart distribution $W_p(\Sigma, n)$. The asymptotic expansion for the non-null distribution of Hotelling's generalized statistic $T_0^2 = \operatorname{tr} S_n^{-1}ZZ'$; which is one of test criteria for testing the hypothesis $H$: $M = \{\mu_1, \cdots, \mu_m\} = \mathbf{0}$ against $K$; $M \neq \mathbf{0}$, was treated by Siotani [*Ann. Inst. Stat. Math.* **9** (1957)] and Ito [*Ann. Math. Statist.* **31** (1960)]. Unfortunately they contained only the terms up to order $n^{-1}$ and in addition somewhat inconvenient terms for numerical work. In this paper the following asymptotic expansion formula for the cdf $F(x; \Omega)$ of $T_0^2$ in the non-null case is obtained up to the terms of order $n^{-2}$ in the convenient form for numerical work; $F(x; \Omega) = G_{mp}(x; \omega^2) + (1/4n)\sum_{j=0}^{4} a_j(m, p; \Omega)G_{mp+2j}(x; \omega^2) + (1/96n^2)\cdot \sum_{k=0}^{8} b_k(m, p; \Omega)G_{mp+2k}(x; \omega^2) + O(n^{-3})$, where $\Omega = \sum^{-1}\mathrm{MM}'$, $\omega^2 = tr\Omega$, $G_f(x; \omega^2)$ is the df of the noncentral chi-square and $a_j$'s and $b_k$'s contain $\Omega$ in the form $\operatorname{tr} \Omega^j$, $j = 1, 2, \cdots$. (Received 17 February 1969.)

**36. Structure of tests of various symmetry hypotheses.** PAUL SMITH and C. B BELL, University of Michigan.

A unified treatment of the hypotheses of univariate symmetry about zero, multivariate sphericity and multivariate interchangeability is proposed. In each case, one tests the hypothesis that the unknown density function $f$ of the observations possesses certain symmetry properties. This symmetry can be expressed by saying that $f$ is invariant under $S$, a group of transformations of the sample space. All distribution-free tests of these symmetry hypotheses can be characterized as permutation tests with respect to the wreath product group $S \wr S_N$, where $S_N$ denotes the group of all permutations of $N$ objects and $N$ is the sample size. This characterization is similar to that of Bell and Smith (1969), but in this case the group $S$ may be infinite, as in the multivariate symmetry problem where $S$ is the $p$-dimensional orthogonal group. In each problem, there is a complete sufficient statistic and a most powerful test against a simple alternative. The problem of constructing invariant distribution-free tests is still open. For the univariate symmetry and sphericity problems on appropriate transformation group is known and leads to rank tests. For interchangeability problems, no such group has been found. (Received 4 March 1969.)

**37. The asymptotic value distribution of a 2 × 2 game with normally distributed payoffs.** DONALD J. SOULTS, Boeing Corporation and Iowa State University. (Introduced by H. T. David.)

Consider a $2 \times n$ game with matrix $\|a_{ij}\|$; $i$: 1, 2; $j$: 1, 2, $\cdots$, $n$. Suppose that the $2n$ payoffs $a_{ij}$ are iid, each standard normal, and denote by $V_n$ the (random) value corresponding to these $a_{ij}$. Let $\Phi$ denote the standard normal cdf and let $\{a(n)\}$ and $\{b(n)\}$ be a sequence of scale and location norming constants applicable to the minimum of $n$ iid standard normal variates. It is shown that $\Pr\{2^{\frac{1}{2}}V_n \leqq ta(n) + b(n)\}$ tends with $n$ to the cdf $1 - w(t)$, where $w(t)$ is given by $\int_{-\infty}^{+\infty} (2y\Phi(2^{\frac{1}{2}}y) + \pi^{-\frac{1}{2}} \exp[-y^2]) \exp[t + y^2 - \exp(t + y^2)]\, dy$. The case of uniform and essentially uniform $a_{ij}$ has been considered in Thomas, David R. and David, H. T. (1967). Game value distributions. I. *Ann. Math. Statist.* **38** 242–250. (Received 3 March 1969.)

**38. Partially balanced weighing designs.** K. V. SURYANARAYANA, University of North Carolina.

This work mainly extends the balanced weighing designs (BWD) of Bose, R. C. and Cameron, J. M. [*J. Res. Nat. Bur. Standards Sect. B.* **69** (1965) 323–332; *J. Res. Nat. Bur. Standards Sect. B* **71** (1967) 149–60] to one of "partial balance" based on a concept of association between different objects. A partially balanced weighing design (PBWD) with a given two associate class is an arrangement of $v$ objects to be weighed in $b$ weighings, with $p$ objects on each pan, the frequencies for an $i$th associate pair to appear together or in opposite pans being respectively $\lambda_{1i}$, $\lambda_{2i}$ $(i = 1, 2)$. The methods used to construct these designs involve the use of orthogonal arrays, association matrices and Partial Difference Sets. The efficiency of PBWD as compared to Balanced Weighing Designs is under investigation. (Received 6 March 1969.)

**39. Hadamard products and multivariate analysis.** GEORGE P. H. STYAN, University of Minnesota.

The *Hadamard product* of two matrices multiplied together elementwise seems to be a neglected concept in matrix theory and has found only brief and scattered application in statistical analysis. This paper surveys the known results on Hadamard products with historical annotations. These results all involve square matrices; we develop extensions to rectangular matrices and applications to multivariate statistical analysis. We write complicated equations and expressions in a more appealing manner and in closed form. These ideas lead to new results concerning Hadamard products of positive definite matrices and their inverses. Some special results for correlation matrices are also obtained. The paper ends with an exhaustive bibliography of books and articles which have mentioned the Hadamard product. (Received 17 February 1969.)

**40. Multi-stage interval estimations of the largest mean of $k$ normal populations.** YUNG LIANG TONG, University of Nebraska.

We have $k$ normal populations ($k \geqq 1$) with means $\theta_1, \theta_2, \cdots, \theta_k$ and a common unknown variance $\sigma^2$. A fixed-width confidence interval for $\theta_{[k]} = \max_{1 \leqq i \leqq k} \theta_i$ is desired so that the coverage probability is at least $\gamma$ (preassigned) for every $\theta = (\theta_1, \theta_2, \cdots, \theta_k)$. A class of multi-stage procedures is considered for the solution of this problem. It is shown that at least for $k \leqq 2$, the efficiencies of those procedures are always less than one. (Received 10 March 1969.)

**41. On strong consistency of density estimates.** J. VAN RYZIN, University of Wisconsin, Milwaukee.

Let $X_1, \cdots, X_n$ be a sample of $n$ independent observations of a random variable $X$ with distribution $F(x) = F(x_1, \cdots, x_m)$ on Euclidean $m$-space with Lebesgue density $f(x) = f(x_1, \cdots, x_m)$. This paper considers density estimates of the form $f_n(x) = n^{-1} \Sigma_{j=1}^{n} K_n(x, X_j)$, $K_n(x, X_j) = h_n^{-m}K(h_n^{-1}(x - X_j))$ for $K(u) = K(u_1, \cdots, u_m)$ a suitably chosen function on Euclidean $m$-space. Parzen [*Ann. Math. Statist.* **33** (1962) 1065–1076] and Cacoullos [*Ann. Inst. Stat. Math.* **18** (1966) 179–190] give conditions on $K(u)$ and the sequence $\{h_n\}$ in the cases $m = 1$ and $m > 1$ respectively such that (i) $f_n(x)$ converges to $f(x)$ in probability on the continuity set of $f(x)$ and (ii) $\sup_x |f_n(x) - f(x)|$ converges to zero in probability when $f(x)$ is uniformly continuous on Euclidean $m$-space. This paper gives conditions under which strong consistency holds in both cases (i) and (ii). The additional conditions given are minor additional conditions on the function $K(u)$ and the sequence $\{h_n\}$ *only* and not the unknown density $f(x)$. (Received 3 March 1969.)

**42. Sufficient conditions for a first passage time process to be that of Brownian motion.** M. T. WASAN, Queen's University, Kingston, Ontario. (By title)

In this paper we assign a set of conditions to a strong Markov process and arrive at a differential equation analogous to the Kolmogorov equation. However, in this case the duration variable is the net distance travelled and the state variable is a time, a situation precisely opposite to that of Brownian motion. Solving this differential equation under certain boundary conditions produces the density function of the first passage times of Brownian motion with positive drift, with the aid of which we define a new stochastic process and existence of which also is proved. (Received 19 February 1969.)

**43. Asymptotic properties of multivariate permutation tests.** JAMES E. WARNER, Case Western Reserve University and University of Michigan.

Let $X_1, \cdots, X_m$; $Y_1, \cdots, Y_n$ be independent random samples from continuous bivariate distribution functions $F$ and $G$ respectively. Consider the two-sample problem of testing $H_0 : F = G$ against the simple bivariate normal shift alternative $H_1 : F = \mathfrak{N}(0, \Sigma^0)$, $G = \mathfrak{N}(\Delta^0, \Sigma^0)$ where $\Sigma^0$ is known and $\Delta^0$ is a fixed non-zero constant vector. The most powerful distribution free (MPDF) test of $H_0$ against $H_1$ was obtained by C. B. Bell and P. J. Smith (1969), and this test was shown to depend on the values taken on by a permutation statistic over the orbit of the data. In this paper theorems are stated which give sufficient conditions similar to those of Wald and Wolfowitz (1944), Hájek (1962), under which the permutation statistic is asymptotically normally distributed both under $H_0$ and $H_1$. This test is also shown to be consistent. Currently these results are being extended to the general multivariate multisample problem using both normal and non-normal alternatives. (Received 28 February 1969.)

*(Abstracts of papers presented at the Western Regional meeting, Monterey, California, May 7–9, 1969. Additional abstracts appeared in the April issue.)*

**3. Estimation of the change point of a $U$-shaped generalized failure rate function.** S. ARUNKUMAR and R. E. BARLOW, University of California, Berkeley.

Window estimators are proposed for the change point of the generalized failure rate function, defined by $r(x) = f(x)/g(G^{-1}F(x))$, where $F(x)$ is the unknown distribution function with density $f(x)$ and $G(x)$ is any known distribution function, with density $g(x)$. In particular, $r(x) = f(x)$ when $G$ is the uniform distribution on $(0, 1)$ and $r(x) = f(x)/1 - F(x)$ when $G$ is the exponential distribution with mean one. Estimation of the mode of a density has been considered by Chernoff [*Ann. Inst. Statist. Math.* **16** (1964), 31–41] and Venter [*Ann. Math. Statist.* **38** (1967), 1446–1455]. Chernoff considers the case of fixed windows while in Venter's case, the windows are random and go to zero asymptotically. Since we are considering the change point of generalized failure rate functions, estimation of the mode of a density falls out as a special case. Whereas both Chernoff and Venter assume the order statistics to be the grid, it has been shown that the asymptotic results are not changed by choosing a wider grid. The latter fact may be important in practical cases, where the narrower the grid, the greater the cost of analyzing the data. Asymptotic distributions for the change point estimates are obtained using theorems on convergence of distributions of stochastic processes. (Received 25 February 1969.)

**4. Nonparametric signal detection: multivariate and stochastic process models** (preliminary report). C. B. BELL, University of Michigan.

(A) "*Known-Pure-Noise*" *Model*: The multivariate distribution of the $k$ measured characteristics of pure noise is a known monotone function. Using Rosenblatt's (1952) transformation, a random sample size $n$ of received data becomes a univariate uniform random sample of size $kn$. The NP detectors are based on tests of uniformity and independence (Bell and Smith (1969)). For various families of "signal + noise" distributions one designs minimum false alarm rate, etc., detectors. The results hold for appropriately continuous multivariate stationary stochastic processes. (B) "*Before-After*" *Model*: One compares known pure noise data with possible-signal data. "Standard" NP detectors must base decisions on permutations. Various "good" detectors are designed from optimal 2-sample multivariate statistics (Bell and Smith (1969)). Desirable distributions are obtained on introducing appropriate artificial noise. (C) "*Randomness*" *Model*: A time change of distribution indicates "signal present". For multivariate data, NP detectors base decisions on permutations. These results hold for exchangeable processes also. Goodness results here are comparable to those in (A) and (B). (Received 13 March 1969.)

**5. The number of $r \times c$ matrices which are monotonic.** W. J. CONOVER, Kansas State University.

A matrix is said to be monotonic if the elements of the matrix are in a strictly increasing order as one reads across each row and down each column. For a given set of elements of an

$r \times c$ matrix there are several arrangements of the elements into $r$ rows and $c$ columns which result in monotonic matrices. A formula for finding the number of monotonic matrices is derived under the assumption that the $r \cdot c$ elements are all unequal to each other. This formula enables one to compute the probability of obtaining a monotonic matrix of ranks from 1 to $r \cdot c$, under the null hypothesis of random assignment of ranks within the matrix. (Received 3 March 1969.)

**6. A test of temporal independence for partially homogeneous Gaussian random fields.** HERBERT T. DAVIS, Air Force Weapons Lab, Kirtland AFB. (By title)

A partially homogeneous random field is a family of random variables $\{X(P, t)\}$ where the following two conditions are true: First, for a given time $t$, the $X(P, t)$ are observations on a non-homogeneous random field where its second order properties are not a function of that time, i.e., $EX(P, t)\overline{X(Q, t)} = R(P, Q)$. Second, that the family of random variables is weakly stationary over time, i.e., $EX(P, t) = 0$ and $EX(P, t)\overline{X(Q, s)} = R(P, Q, t - s)$. In this paper we consider the Karhunen-Loève representation of the random field, together with necessary and sufficient conditions for a Gaussian random field to be temporally independent. It is shown that a set of weakly stationary principle components can be used to construct a multivariate test of independence. (Received 4 April 1969.)

**7. Approachability and the strong law of large numbers for stochastic vector payoffs.** T. F. HOU, Bell Telephone Laboratories, Holmdel, New Jersey.

In this paper we present a solution for an open problem proposed by Blackwell [*Proc. Internat. Congress Math.* **3** (1954) 336–338]. Consider a two person game with a payoff $r \times s$ matrix $M = \|M(i, j)\|$, where each element of $M$ is a probability distribution over a closed bounded convex set in Euclidean $k$-space. We show that the class of approachable (excludable) sets for a given $M$ depend only on the matrix of mean values of $M$. In a sense, this theorem is a particular form of the strong law of large numbers for two-person games with stochastic vector payoffs. (Received 20 February 1969.)

**8. Random effects model: nonparametric case.** Z. GOVINDARAJULU and JAYANT V. DESHPANDE, University of Kentucky and Case Western Reserve University.

The random effects models have been treated adequately in the literature in the normal theory case (see, for instance, Scheffé: *Analysis of Variance*, John Wiley and Sons, 1959). Greenberg has proposed some partially distribution-free tests for the one-way layout and nested designs with several factors (see Robust Inference in Some Experimental Designs, Ph.D. thesis, Univ. of Calif., 1964, section 5). In the present paper, we obtain locally most powerful rank tests for one, two and several sample problems of random location and random scale. The asymptotic of the derived test procedures are also considered. (Received 14 March 1969.)

**9. Non-null distribution of Wilks' likelihood ratio criterion in the complex case.** A. K. GUPTA, University of Arizona. (By title)

Khatri [*Ann. Math. Statist.* **36** (1965)] has shown that the distribution of Wilks' likelihood ratio criterion in the noncentral linear (i.e. when the alternative hypothesis is of unit rank) complex case is the same as the product of $p$ independent real beta variables $x_j (j = 1, 2,$

$\cdots, p$) given by (5·2·2) and (5·2·3) of his paper. Consequently $-\log \Lambda$ is distributed as a sum of $p$ independently distributed random variables. Then the process of successive convolution leads to the principal results of this paper, provided that the process yields expressions which can be easily integrated at each stage. Gupta (to appear), Pillai and Gupta [*Biometrika* **56** (1969)] and Schatzoff [*Biometrika* **53** (1966)] have shown that this is in fact the case. Hence, in this paper, we have obtained the distribution of Wilks' likelihood ratio criterion in the noncentral linear complex case, giving the specific expressions for the density and the distribution function for $p = 2$ and 3 for general $f_1$ and $f_2$. (Received 13 March 1969.)

**10. The non-existence of some linked block designs.** PETER W. M. JOHN, University of Texas.

The listings in the literature of partially balanced incomplete block designs with Latin square association schemes contain no linked block designs. The question then arises whether any such designs exist. It is shown here that there do not exist any linked block designs which are partially balanced with two associate classes and have the $L_i$ association scheme with $i = 2$ or 3. (Received 17 March 1969.)

**11. A moment problem for order statistics.** JOSEPH B. KADANE, Center for Naval Analyses and Yale University.

Necessary and sufficient conditions are given for a triangular array of numbers to be expectations of order statistics of some non-negative random variable. Using well-known recurrence relations, the expectations of all order statistics of the largest sample size, $n$, in the triangular array, or the expectations of the smallest of every sample size up to and including $n$ are sufficient. The former are reduced to a Stiljes moment problem, the latter to a Hausdorf moment problem. These results are applied to show that for every sample size, there is a positive random variable with geometrically increasing expectations of order statistics with arbitrary ratio and expectation of smallest order statistic. However, only the degenerate distributions have geometrically increasing expectations of order statistics for more than one sample size, even when the ratio and mean of the smallest order statistic can depend on the sample size. These results were required for a study of participation in discussion groups. (Received 25 March 1969.)

**12. A sufficient statistics characterization of the normal distribution.** DOUGLAS KELKER, Washington State University.

Basu [*Sankhyā* **15** (1955) 377–380] obtained the result that if $\sum b_i X_i$ is a boundedly complete sufficient statistic for a location parameter, based on an independent sample of size $\geqq 2$, then each $X_i$ is a normal random variable. We show that bounded completeness can be dropped from the hypothesis. A corresponding result is that if the sample mean and sample variance are jointly sufficient for a location and a scale parameter, based on a sample of size $\geqq 4$ of iid random variables, then each random variable is normal. Additional hypothesis are necessary for a similar result for the scale parameter case. Let independent $X_i$'s have distributions $F_i$'s with $F_i'$ existing in a neighborhood of the origin and with $F_i'$ non-zero and continuous at the origin. Then if $\sum X_i^2$ is sufficient for a scale parameter, each $X_i$ has a normal distribution. This result is a corollary of the result that $\sum X_i^2$ is a sufficient statistic for a scale parameter, based on an independent sample of size $\geqq 2$, only if $X_i^2$ has a gamma distribution. (Received 13 March 1969.)

**13. On stochastic games** (preliminary report). ASHOK MAITRA and T. PARTHASSARATHY, Indian Statistical Institute and Case Western Reserve University.

A stochastic game problem is determined by five objects $S$, $A$, $B$, $q$ and $r$. $A$ and $B$ are finite sets, $S$ is a compact subset of reals and $r$ is a jointly continuous function on $S \times A \times B$. We interpret $S$ as the state-space of some system and $A$, $B$ as the sets of actions available to players I and II respectively at each state. When the system is in state $s$ and I takes action $a$, II takes action $b$, the system moves to a new state $s'$ according to the distribution $q(\cdot \mid s, a, b)$ and I receives an immediate return $r(s, a, b)$. The process is then repeated from the new state $s'$ and I wants to maximize the total expected return over the infinite future and II wants to minimize the same. It is known that I and II have optimal strategies when $S$ is also finite. Even when $S$ is infinite I and II have optimal strategies and these strategies may not be measurable as a function of $s$. We are investigating conditions under which these are measurable. (Received 18 March 1969.)

**14. Use of sign statistic in problems concerning** $P(Y < X)$ (preliminary report). K. M. LAL SAXENA, University of Nebraska.

Let $X, Y_1, \cdots, Y_k$ be $k + 1$ independent random variables with continuous distribution functions $F_1, G_1, \cdots, G_k$, respectively. Denote the corresponding populations by $\pi_i$, $i = 0, 1, \cdots, k$. Denote $p_i = P(Y_i < X)$. Let $(X_{i1}, Y_{i1}), \cdots, (X_{in}, Y_{in})$ be $n$ pairs of observations from populations $\pi_0$ and $\pi_i$, $i = 1, \cdots, k$. Define the sign statistics $Z_i = \sum_{j=1}^{n} Z_{ij}$ where $Z_{ij} = 1(0)$ if $Y_{ij} < (\geqq) X_{ij}$. Then $\hat{p}_i = Z_i/n$ is an unbiased estimate of $p_i$. Two problems are considered. (A) For large sample sizes and preassigned $\gamma (0 < \gamma < 1)$, distribution free bounds for $\epsilon$ are obtained such that (i) $P(p_i < \hat{p}_i + \epsilon) > \gamma$ or $P(p_i > \hat{p}_i - \epsilon) > \gamma$, (ii) $P(|p_i - \hat{p}_i| < \epsilon) > \gamma$, $i = 1, \cdots, k$. These bounds are identical with those of Govindarajulu [*Ann. Inst. Statist. Math.* **20** (1968) 229–238] obtained by using the Mann Whitney statistics. (B) Let $p_i$'s be ordered as $p_{[1]} \leqq \cdots \leqq p_{[k]}$. There is no *a priori* knowledge about the ordering. Then, using sign statistics, it is shown that one can formulate the various specifications of selection and ranking problems in the framework of Sobel and Huyett [*Bell System Tech. J.* **36** (1957) 536–576]. Confidence bounds for $p_{[1]}$ and $p_{[k]}$ are also obtained. (Received 25 1969.)

**15. Multiple decision problems of type I.** WILLEM SCHAAFSMA, University of California, Berkeley, and University of Gronigen.

Let the decision space consist of $n + 1$ decisions $d_0, \cdots, d_n$. The parameter space is partitioned into $m + 1$ mutually exclusive subsets $\Omega_0, \cdots, \Omega_m$ where $\Omega_0$ is an indefiniteness zone which may be empty. The loss $L(\theta, d)$ is defined for all decisions and all $\theta$'s outside $\Omega_0$. For each $d$ the loss $L(\theta, d)$ considered as a function of $\theta$ is constant over each of the regions $\Omega_1, \cdots, \Omega_m$. The problem is said to be of type I if the subsets $\Omega_1, \cdots, \Omega_m$ have a common boundary point and some regularity conditions are satisfied. Type I problems behave in a degenerate way. One can characterize the class $M$ of minimax risk procedures and the class $W$ of all unbiased (in Lehmann's sense) procedures. Under certain conditions $W \subset M$. The results generalize Example 11 on p. 12 in Lehmann's *Testing Statistical Hypotheses* concerning $n = 1, m = 2$, and Chapter 3 of the author's *Hypothesis Testing Problems with the Alternative Restricted by a Number of Inequalities* (Noordhoff-Groningen) concerning $n = m = 2$. Classes of general type I problems were also studied by Lehmann (A theory of some multiple decision problems I and II, *Ann. Math. Statist.* **28** (1957) 1–25, 547–572). (Received 13 March 1969.)

**16. Sequential hierarchical multiresponse designs of the order and truncation types.** J. N. SRIVASTAVA, Colorado State University.

Hierarchical multiresponse (HM) designs have been considered earlier (e.g. see author's paper "On the extensions of Gauss-Markov Theorem to complex multivariate linear models," *Ann. Inst. Stat. Math.* (1968) and references therein). Briefly, let there be $p$ responses $V_1, \cdots, V_p$, and let $D_i$ denote the set of experimental units on which $V_i$ is measured. Then the design is an HM design iff $D_1 \geqq D_2 \geqq \cdots \geqq D_p$. In the earlier work, it was implicitly assumed that the choice of $D_i$ does not depend in any way on the observations on $V_1, \cdots, V_{i-1}$. In the present paper, a class of sequential HM designs is introduced in which this assumption is relaxed. These designs are, in particular, of two broad categories. The first one, called Order HM design, is such that first $V_1$ is observed on $D_1$. Next, these observations are arranged in order of magnitude, and the units corresponding to, say, the $m$ largest observations are marked out. These units form $D_2$. Similarly $D_3$ is formed using the observations on $V_2$ on the set $D_2$, and so on. The second category, called Truncation HM design is such that after $V_1$ is observed on $D_1$, $D_2$ is selected by retaining those units, on which the observations on $V_1$ are, in magnitude, larger than some preassigned number $C_1$. Similarly $D_3$ is selected from $C_2$ by retaining those units, the observations on $V_2$ on which is larger than some number $C_2$, and so on. Some aspects of analysis and applications of such designs are considered. (Received 7 March 1969.)

**17. Non-parametric estimation of $g$ when $G = g(F)$ (preliminary report).** GEORGE P. STECK, Sandia Laboratories, Albuquerque.

Let $X_1, X_2, \cdots, X_m$ be from a continuous distribution $F$ and let $Y_1, Y_2, \cdots, Y_n$ be from a continuous distribution $G = g(F)$ where $g$ is an absolutely continuous distribution function. Estimates $\hat{g}$ for $g$ are derived from estimates $\hat{F}$ and $\hat{G}$ through the defining equation $\hat{G} = \hat{g}(\hat{F})$. In particular, if $\hat{F}$ and $\hat{G}$ are the empirical cumulatives, the corresponding $\hat{g}$ is the MLE for $g$ and is a random walk whose $j$th step is $1/m$ to the right or $1/n$ up according as the $j$th element in the ordered combined sample is an $X$ or a $Y$. Confidence regions for $g$ are obtainable from the confidence regions for $F$ and for $G$. If $F_U$, $F_L$, $G_U$, $G_L$, $g_U$, $g_L$ denote, respectively, the upper and lower confidence curves for $F$, $G$ and $g$ and if $\alpha_F$, $\alpha_G$, $\alpha_g$ denote the corresponding confidences, then defining $g_U$ and $g_L$ by $G_U = g_U(F_L)$ and $G_L = g_L(F_U)$ gives $\alpha_g \geqq \alpha_F \alpha_G$. (Empirical studies indicate $\alpha_g$ is much nearer $1 - (1 - \alpha_F)(1 - \alpha_G)$.) The confidence curve $g_U$ is also obtained by shifting $g$ $h/m$ units left and $k/n$ units up where $k/m$ and $k/n$ are the corresponding shifts determining the confidence regions for $F$ and $G$. The curve $g_L$ is obtained similarly. Heuristic arguments as well as empirical studies indicate $P(\text{region covers } g \mid g) < 1$ may be minimized for $g(x) \equiv x$ so that the values of $h$ and $k$ for any confidence are obtainable from the distribution of the Smirnov $D_{mn}$ statistic. (Received 13 March 1969.)

**18. Convergence of experiments on a finite sample space.** ERIK NIKOLAI TORGERSEN, University of California, Berkeley.

This paper treats the problem of convergence of experiments which all have the same finite sample space. The pseudo distance between experiments which was introduced by LeGam in 1964 is—as has been shown by LeGam—no longer compact when the parameter set is infinite. The uniformity generated by the pseudo distance between restrictions to finite sub parameter sets is, however, still compact. We consider convergence for this uniformity as well as convergence for $k$-decision problems and for the pseudo distance mentioned above. Various convergence criterions are given. It turns out—as is the case when the parameter set is finite—that convergence may be decided by testing problems alone, and that the problem of convergence for $k$-sample experiments may be reduced to the problem of convergence for 1-sample experiments. It is shown that convergence to a minimal sufficient experiment is—up to permutations of the sample space—essentially equivalent with convergence of the individual probability measures. In general the problem of convergence for the pseudo distance may be decided by first establishing asymptotic sufficiency and then applying the criterion for convergence to a minimal sufficient experiment. (Received 24 February 1969.)

**19. On the minimum distance method** (preliminary report). SIDNEY J. YAKOWITZ, University of Arizona.

A technique developed by the author for a specific problem (A Consistent Estimator for the Identification of Finite Mixtures [to appear in *Ann. Math. Statist.*]), can be generalized to apply to a model similar but not coincident with the class of problems inspiring the minimum distance method (J. Wolfowitz [*Ann. Math. Statist.* **28** (1957) 75–88]). It is convenient to use the definitions of Wolfowitz's paper in describing our results. We drop the bound $k'$ of the dimension of the "unobservable chance variables." This necessitates some minor modifications of the parameter set $A$ and the metric $\delta$. Letting $X$, $Y$, etc. be as in the Wolfowitz paper, our method gives a consistent estimate of $(\theta, G_0)$ if the following conditions obtain: (1) $A$ is the limit of an increasing sequence of compact sets; (2) $J$ is one-to-one on $A$; (3) $J$ is continuous on $A$. ($\delta$ is the metric for $A$ and the $n$-dimensional Lévy distance is the metric on $J(A)$.) The domain of application of our method includes statistical inference in compound Poisson and branching processes. (Received 17 March 1969.)

*(Abstracts of papers to be presented at the Annual meeting, New York, New York, August 19–22, 1969. Additional abstracts have appeared in earlier issues and will appear in future issues.)*

**2. Convergence in distributions of stochastic integrals.** MARK BROWN, Cornell University. (By title)

Convergence in distribution of sequences of quadratic mean stochastic integrals is studied by developing and extending an approach introduced by J. Sethuraman (1965 Stanford technical report). A type of convergence of stochastic processes, linear law convergence, is introduced. It entails convergence of finite dimensional distributions and a condition on the product moment kernels of the processes. The condition involves the uniform boundedness of a sequence of operators defined on the reproducing kernel Hilbert space corresponding to the limiting process. The $n$th operator of the sequence depends on the kernel of the $n$th process. Various sets of conditions are derived for convergence in distribution of sequences of stochastic integrals, where the $n$th integral depends on a random function $X_n$ and a non-random function $g_n$, and the sequence $X_n$, $n = 1, 2 \cdots$, converges in linear law to a process $X$. The limiting distribution is that of a stochastic integral involving $X$ and $g$ where $g$ is a limit is some sense of the $g_n$ sequence. The convergence is obtained without a sample path analysis, in fact the stochastic integrals need not exist as pathwise integrals. Results are obtained which relate linear law convergence to weak convergence. (Received 10 April 1969.)

**3. A nonparametric selection procedure's efficiency: largest location parameter case.** EDWARD J. DUDEWICZ, University of Rochester.

For the problem of selecting that one of $k$ populations which has the highest probability of producing the largest observation, Bechhofer and Sobel [*Ann. Math. Statist.* **29** (1958) 325] have suggested a nonparametric procedure (the cdf $F_i(\cdot)$ of observations from $\pi_i$ may be different for each $i$ ($i = 1, \cdots, k$) and unknown). If $F_i(x) = F(x - \nu_i)$ ($i = 1,$

$\cdots, k$), the problem reduces to that of selecting that one of $k$ populations which is associated with the largest $\nu_i$. It is of interest to know how the procedure performs under various parametric alternatives $F(\cdot)$. In this paper we determine (under specific parametric alternatives) how much one pays (in terms of increased sample sizes) for the nonparametric procedure's certainty of guaranteeing a reasonable requirement on the performance characteristic function, in the largest location parameter case. It turns out that the nonparametric procedure has a "low" efficiency relative to the specific parametric alternatives considered and will therefore be useful against these alternatives (and, presumably, others also) only when real doubt exists as to the form of the actual distribution. (Received 17 March 1969.)

**4. On asymptotic efficiencies (ARE's) of a class of rank tests.** J. L. GASTWIRTH, Johns Hopkins University.

A partial ordering is introduced in a class of rank tests according to the weight or emphasis the tests place on the middle ranks versus the extreme ones. This notion of order is shown to be related to van Zwet's partial ordering of symmetric densities according to the probability in their tails in the sense that if $F$ has lighter tails than $G$ (in van Zwet's sense) and if $T$ weights the middle ranks *less* than $S$, then the ARE of $S$ to $T$ on samples from $F(x)$ is *less* than its value on samples from $G$. This generalizes van Zwet's theorem which states that the ARE of the Wilcoxon test to the normal scores test increases as the tails of the underlying density increase. Hájek (*A Course in Nonparametric Statistics*, Holden-Day, 1969) also introduced a weaker notion of order or rank tests and the connection between our notion and his is studied briefly. Bounds on the ARE of several tests to the Wilcoxon test are established. Also, lower bounds on the ARE of the Psi test to the Wilcoxon and normal scores test are obtained. (Received 14 April 1969.)

**5. On estimation by shrinkage.** J. S. MEHTA and R. SRINIVASAN, Temple University.

Procedures are developed for shrinking the minimum variance unbiased linear estimates of the parameters of various populations towards a point and towards an interval. The resulting estimators are shown to be much better than the ones proposed by J. T. Thompson [Some Shrinkage Techniques for Estimating the Mean, *J. Amer. Statist. Assoc.* **63** (1968) 113–122; Accuracy Borrowing in the Estimation of the Mean by Shrinkage to an Interval, *J. Amer. Statist. Assoc.* **63** (1968) 953–963] in the sense of buying decreased mean square error (MSE) in the 'natural' interval of the parameter at the expense of increased MSE for other values of the parameter. (Received 10 March 1969.)

**6. Transmission of information in a $T$-terminal discrete memoryless channel.** EDWARD C. VAN DER MEULEN, University of Rochester.

In a basic paper, Shannon [*Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** (1961) 611–644] introduced the two-way channel and analyzed the problem of how to communicate over this channel in two opposite directions as effectively as possible. The present paper investigates how to send information from one specified terminal $i$ to another specified terminal $j$ over a channel which has more than two terminals. A discrete memoryless channel with $T \geqq 3$ terminals is defined by a set of transition probabilities $P(y_1, \cdots, y_T \mid x_1, \cdots, x_T)$ where $x_1, \cdots, x_T$; $y_1, \cdots, y_T$; all range over finite alphabets. The channel capacity in the $i$-$j$ direction, denoted by $C(i, j)$, is defined by a limiting expression involving mutual informations for long sequences of inputs and outputs. It is shown, by proving a coding theorem and its weak converse, that $C(i, j)$ is in fact the transmission capacity in the $i$-$j$ direction. A necessary and sufficient condition under which $C(i, j)$ is unequal to zero is derived and stated in terms of simple relationships between the rows of the channel matrix. Several interesting examples of channels with more than two terminals are given. Lower bounds for the capacity $C(1, 3)$ are obtained for the special case $T = 3$. These bounds are more easily evaluated and turn out to be quite satisfactory for the specific examples on hand. (Received 3 March 1969.)

### 7. Some results of the problem of enumeration of the symmetrical confounded factorial designs. Pierre Robillard, Université de Montreal.

Two confounded symmetrical factorial designs of class $(s^n, s^k)$ are considered as equivalent if one can equate the confounding scheme of the other by simply renaming the factors of one design. This equivalence relation splits the set of designs of class $(s^n, s^k)$ into equivalence classes called types of designs. The problem of enumeration of the confounded designs is to obtain all the types of a given class. In the binary case, we derive an exact formula for the number of types of designs of class $(2^n, 2^k)$ which equals $\sum_{s=k}^{n} (|S(s, k)| - |S(s, k-1)|)$ where $|S(j, k)|$ is the coefficient of $W^j$ in $P_p((1 - W)^{-1}, (1 - W^2)^{-1}, \cdots)$; $P_p(X_1, X_2, \cdots)$ is the cycle index of the projective group on $PG(k - 1, 2)$. The general case is more difficult to handle, however, in the case of the designs of class $(s^n, s^2)$ the number of types equals $\sum_{i=2}^{n} p(i, s1) - (n1)$ where $p(i, j)$ is the number of partition of $i$ in at most $j$ parts. In a last part we stress the connection between the designs of class $(s^n, s^k)$ and the designs of the dual class $(s^n, s^{n-k})$ and we show that the number of types is the same for both classes. (Received 12 February 1969.)

### 8. The accuracy of approximations for the null distribution of the chi-square goodness of fit statistic. James K. Yarnold, URS System Corporation.

The accuracy of approximations for the null distribution of the simplest chi-squre goodness of fit statistic, $X^2$, is examined numerically for many different multinomial distributions. Some of these approximations are terms of an asymptotic expansion which provides much greater accuracy than the usual $\chi^2$ approximation, when all expectations are large. Outside of this region of the parameter space, however, all of these approximations fail and the $C(\mathbf{m})$ distribution is, in some cases, required. The latter is the limiting distribution of $X^2$ when some expectations remain finite while the rest increase without limit. Three additional approximations studied have been recommended by previous investigators, i.e., the $\chi^2$ approximation with continuity correction, the two-parameter $\Gamma$ distribution having the same mean and variance as $X^2$, and the multivariate Edgeworth approximation (for a continuous distribution). The accuracy of each of these approximations is compared with that of the usual $\chi^2$ approximation. Recommendations and conclusions on the use and accuracy of these approximations are given. (Received 15 April 1969.)

*(Abstracts of papers not connected with any meeting of the Institute.)*

### 1. A characterization of the geometric distribution. V. P. Bhapkar, University of Poona.

Govindarajulu [*Skand. Aktuari.* 132–36, (1966)] has characterized the exponential distribution using order statistics in a random sample. Even though the characterization is claimed in the class of all nontrivial distributions, it is here pointed out that it holds only in the class of continuous distributions. It can be shown that the properties characterizing the exponential distribution in the class of continuous distributions actually characterize the geometric distribution in the class of discrete distributions. Specifically, suppose $X_1 \leqq X_2 \leqq \cdots \leqq X_N$ denote the order statistics in a random sample of size $N$ from a distribution $F$ (of r.v. $X$) with probability mass $f(t) > 0$ at points $t > 0$ belonging to a countable set

$T$, and let $V = X_1$ and $U = X_i - X_1$, $i = 2, 3, \cdots, N$. Then $V$ and the random vector $U = (U_2, U_3, \cdots, U_N)$ are independent if, and only if $F$ is a geometric distribution with some $c > 0$ as a scale parameter and some $a$ as a location parameter (such that $a + c > 0$), i.e. $Y = (X - a)/c$ has probability mass $(1 - p)p^{k-1}$ at points $k = 1, 2, \cdots$, for some $p$, $0 \leqq p < 1$. (Received 26 March 1969.)

### 2. On the best linear unbiased estimates of the location and scale parameters of the Type I asymptotic distribution of smallest values (preliminary report). LAI K. CHAN, University of Western Ontario.

Let $X_{(r_1+1)} < X_{(r_1+2)} < \cdots < X_{(n-r_2)}$, $r_1, r_2 \geqq 0$ be a censored sample corresponding to a complete sample ($r_1 = 0, r_2 = 0$) of size $n$ from the Type I asymptotic distribution of smallest values with location and scale parameters $\mu$ and $\sigma$. Also let $X_{(n_1)} < X_{(n_2)} < \cdots < X_{(n_k)}$ be $k$ order statistics in the above censored sample. The following tables have been computed. (1) The best linear unbiased estimates (BLUE'S) of $\mu$($\sigma$ known) and of $\sigma$($\mu$ known) based on all the order statistics in censored samples with $n - r_1 - r_2 \geqq 1$ for $n = 2(1)10(5)25$: the variances, efficiencies (relative to the BLUE'S based on the corresponding complete samples), biases and coefficients. (2a) The BLUE of $\mu$($\sigma$ known) based on $k = 1, 2, 3, 4$ order statistics selected from a censored sample such that the BLUE yielded by them has the minimum variance among the $\binom{n-r_1-r_2}{k}$ BLUE'S based on the same number of $k$ order statistics taken from the same censored sample; for all possible censored samples $n - r_1 - r_2 \geqq k$ and $n = k(1)10(5)25$: the coefficients, variances, efficiencies (relative to the BLUE'S based on the corresponding censored samples), biases and coefficients. (2b) The BLUE of $\sigma$($\mu$ known) based on $k = 1, 2, 3, 4$ selected order statistics as in (2a). (2c) The BLUE, say $(\mu^*, \sigma^*)$, of $(\mu, \sigma)$ based on $k = 2, 3, 4$ order statistics selected from a censored sample which yields the minimum generalized variance $V(\mu^*)V(\sigma^*) - \mathrm{Cov}^2(\mu^*, \sigma^*)$ for $n - r_1 - r_2 \geqq k$ and $n = k(1)10(5)25$: variances and covariances, respective and generalized efficiencies in the sense of (2a) and coefficients. (Received 28 March 1969.)

### 3. The asymptotically most powerful rank test (amprt) for data for from the $t_2$ density (preliminary report). JOSEPH L. GASTWIRTH, The Johns Hopkins University.

The $t_2$ density is of interest to statisticians since it is a heavy tailed density with a finite mean. Using the notation of Chernoff and Savage [*Ann. Math. Statist.* (1958) 972–994] the a.m.p.r.t. for the two sample shift problem is shown to be specified by the function $J(u) = (u - \frac{1}{2})[(30)(1 - 4(u - \frac{1}{2})^2]^{\frac{1}{2}}$. Some properties of the A.R.E. of the test and the role of the $t_2$ density in the authors approach to developing efficiency robust rank tests are discussed. One interesting numerical result is that the test which is based on the average of $J(u)$ and the normal scores test has a minimum relative efficiency of 91% when it is compared to the best tests for each $T$ density with $k(\geqq 2)$ degrees of freedom. (Received 14 April 1969.)

### 4. Algebraic structural classification of PBIB designs (preliminary report). PANKAJ GHOSH, The Catholic University of America.

Bose and Shimamoto (*J. Amer. Statist. Assoc.* **47** (1952)) classified the PBIB designs in the light of association schemes. But this classification is not mutually exclusive. In this paper the author has developed another criterion of classification which yields mutually exclusiveness. The author has shown that the association matrices of PBIB designs introduced by Thompson (*Amer. Math. Soc.* **29** (1958)) can be split up into permutation matrices by applying Konig's theorem (Theorie der Endlichen und unendlichen Graphen) which follow algebraic structures like monoid, abelian group and nonabelian group. So one can classify the PBIB designs as monoid generating, abelian group generating and nonabelian group generating and they are mutually exclusive. (Received 3 March 1969.)

**5. The mixed effects model and simultaneous diagonalization of symmetric matrices.** ROBERT HULTQUIST, Pennsylvania State University. (Introduced by Charles E. Antle.)

This paper presents results obtained in efforts to generalize ideas reported during the past 15 years on the subject of variance components. The paper applies the theorems of Mitra and Rao (*Sankhyā* (1968)) to the mixed effects model $Y = \sum_{j=1}^{g} X_j \tau_j + \sum_{k=1}^{h} Z_k b_k$ were the $\tau$ are fixed vectors of parameters and the $b_k$ are random vectors distributed normally with Covariance Matrix $\sigma_k^2 I$. The restrictions made on the model are far less severe than those imposed by other authors. Through a nonsingular transformation of the data vector $Y$, minimal sufficient statistics are obtained. Theorems are presented which give conditions under which minimum variance unbiased estimates exist and these estimates are displayed. Properties of the model and the estimates are discussed in both the complete and noncomplete density cases. Perhaps the most important contribution relates to the simplicity with which the theoretical methods treat the general variance components situation. (Received 13 February 1969.)

**6. Asymptotic coverage and concentration probabilities for Poisson spheres.** ROGER E. MILES, Australian National University.

Suppose a sphere of radius $r$ is centred at each particle of a Poisson process of intensity $\rho$ in $E^3$. Write $H(y)$ for the number of these spheres containing $y \varepsilon E^3$, $\underline{H}(X) = \inf_{y \varepsilon X} H(y)$ and $\bar{H}(X) = \sup_{y \varepsilon X} H(y)$ $(X \subset E^3)$. Then, if $p\{i, \theta\} = 1 - q\{i, \theta\} = e^{-\theta}\{1 + \theta + \cdots + (\theta^i/i!)\}$, as $|X| \to \infty$

$$\sup_{r>0} |P\{\underline{H}(X) \geqq j\} - \exp[-(3\pi^2/32)j(j+1)\rho|X|p\{j+1, 4\pi\rho r^3/3\}]| \to 0 \qquad (j = 0, 1, 2, \cdots), \quad \text{and}$$

$$\sup_{r>0} |P\{\bar{H}(X) \leqq k\} - \exp[-\{4 + (\tfrac{3}{8})(\pi^2 + 16)(k-1) + (3\pi^2/32)(k-1)(k-2)\}\rho|X|q\{k-1, 4\pi\rho r^3/3\}]| \to 0 \qquad (k = 1, 2, 3, \cdots).$$

Of course $P\{\bar{H}(X) = 0\} = \exp[-\rho|X(r)|]$, where $X(r) = \{y + z\colon y \varepsilon X, |z| \leqq r\}$. (For related planar results, see Miles [*Ann. Math. Statist.* **39** (1968) 1371 and 1794].) (Received 1 April 1969.)

**7. Optimal designs of individual regression coefficients with a Tchebycheffian spline regression function** (preliminary report). V. N. MURTY, Purdue University.

When the regression function is a Tchebycheffian Spline Function (TSF) of class $(n, k)$ (See "Total Positivity" Volume one, by Samuel Karlin pp. 516) with $k$ fixed knots and $n \geqq 2$ which is of the form $E(y \mid x) = \sum_{i=0}^{n} \theta_i u_i(x) + \sum_{j=1}^{k} \theta_{n+j} \Phi_n(x; \xi_j)$ where $x \varepsilon [-1, 1]$, and $\{u_i(x)\}_{i=0}^{n}$ is an Extended Complete Tchebycheff System (ECT) with $u_0(x) \equiv 1$, and $\xi_j$'s are the knots, it is shown that optimal design for estimating any of the regression coefficients $\theta_j$ $(1 \leqq j \leqq n + k)$ is unique and is supported on a full set of $(n + k + 1)$ points and the unique optimal design for estimating $\theta_0$ is supported on the single point zero. (Received 6 March 1969.)

### 8. Confidence bands for the smallest and the largest of $k$ distribution functions (preliminary report). K. M. Lal Saxena, University of Nebraska.

Let the population $\pi_i$ have the continuous distribution function $F_i(x)$, $i = 1, \cdots, k$. Suppose that $F_i(x)$ are ordered, i.e., $F_{[1]}(x) \leqq \cdots \leqq F_{[k]}(x)$. There is no *a priori* knowledge as to which $\pi_i$ corresponds to $F_{[j]}(x)$. Let random samples of size $n$ be taken from each population and let the sample distribution function for the $i$th population be denoted by $H_{in}(x)$. Define $H_n^*(x) = \sup_{1 \leqq i \leqq k} H_{in}(x)$ and $H_{*n}(x) = \inf_{1 \leqq i \leqq k} H_{in}(x)$. Then the constants $a_n$ and $b_n$ are determined such that, for large $n$ and preassigned $\gamma$ $(0 < \gamma < 1)$, (i) $P[F_{[k]}(x) > H_n^*(x) - a_n$ for all $x \mid \mathbf{F}] \geqq \gamma$ and (ii) $P[F_{[1]}(x) < H_{*n}(x) + b_n$ for all $x \mid \mathbf{F}] \geqq \gamma$, for all $\mathbf{F} = (F_1, F_2, \cdots, F_k)$. (Received 25 March 1969.)

### 9. Double sampling in difference method of estimation with unequal probabilities and without replacement. Surendra K. Srivastava, Lucknow University.

The double sampling estimator $\hat{Y}_d = \hat{Y} + k(\hat{X}' - \hat{X})$ of the population total $Y$ of a finite population when sampling is done with unequal probabilities and without replacement is considered where $k$ is a constant, $\hat{X}'$ denotes an unbiased estimator of the population total $X$ of the auxiliary variable $x$ based on the first-phase sample and $\hat{Y}$ and $\hat{X}$ denote respectively unbiased estimators of $Y$ and $X$ based on the second-phase sample. The variance of $\hat{Y}_d$ when Horvitz-Thompson estimators are taken for $\hat{Y}$, $\hat{X}$ and $\hat{X}'$ is obtained. An asymptotic expression of the variance is then obtained for randomized systematic sampling procedure [Hartley and Rao *Ann. Math. Statist.* **33** (1962) 350–374] and is compared with the corresponding estimator when sampling is done with unequal probabilities and with replacement [Raj, Des *Ann. Math. Statist.* **36** (1965) 327–330]. When sampling is done by the procedure of Rao, Hartley and Cochran (1962) [*J. Roy. Statist. Soc. Ser. B* **24** 482–491], the variance of the estimator $\hat{Y}_d$ and an unbiased variance estimator is obtained. In this case also the variance of $\hat{Y}_d$ is compared to that in the case of sampling with replacement. (Received 24 February 1969.)

### 10. On the robustness of optimal designs in polynomial regression. Joseph Skwish, The Johns Hopkins University. (Introduced by Leon J. Gleser.)

For the classical regression model, designs which minimize the generalized variance of the regression coefficients have been found by P. G. Hoel (*Amer. Math. Soc.* **29** (1958) 1134–1146) to be supported on points which are zeros of the integral of Legendre polynomials. The robustness of these designs examined for the case when the usual least-squares estimator of the regression coefficients is used, but when the assumptions concerning homogeneity of the error variances and uncorrelatedness of the errors do not hold. It is found that the Legendre designs remain optimum for an error covariance matrix which is a multiple of either (a) the interclass correlation matrix or (b) the compound interclass correlation matrix, (D. E. Votaw, Jr. *Amer. Math. Soc.* **19** (1948) 447–473) when the dimension of its component blocks is a multiple of one plus the order of the polynomial. A randomized version of the Legendre design is shown to be minimax under the understanding that the generalized variance of the randomized design is the determinant of the mean covariance matrix. Other departures from the classical regression model are considered and discussed. (Received 7 March 1969.)