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THE STRONG RATIO LIMIT PROPERTY FOR SOME GENERAL MARKOV PROCESSES[1]

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1. Introduction. We shall consider a time homogeneous Markov process $X_0, X_1, \cdots$ with state space an abstract measurable space $(X, \mathcal{B})$, where the σ-field $\mathcal{B}$ is assumed generated by a countable collection of subsets of X and $\{x\}$ is assumed to be in $\mathcal{B}$ for every $x \varepsilon X$. For $n \geqq 1$ the n-step transition probability function is denoted as $P^n(x, E)$. For $n = 0$ we define $P^0(x, E) = I_E(x)$, the indicator function of the set E. The following recurrence condition will be assumed throughout.

CONDITION (C). There exists a σ-finite measure φ, $\varphi(X) > 0$, such that for every $E \varepsilon \mathcal{B}$, $\varphi(E) > 0$ implies Prob $\{X_n \varepsilon E \text{ i.o.}/X_0 = x\} = 1$ for all $x \varepsilon X$, where "i.o." means "infinitely often". This recurrence condition was first imposed by Harris [2] and the following result was established:

(i) There exists a σ-finite measure π such that φ is absolutely continuous with respect to π and for every $E \varepsilon \mathcal{B}$, $n \geqq 1$,

$$\pi(E) = \int \pi(dx)P^n(x, E). \tag{1.1}$$

(The domain of integration will be understood to be X when none is mentioned.) Such measure π is unique modulo a constant multiplier.

(ii) $\pi(E) > 0$ implies Prob $\{X_n \varepsilon E \text{ i.o.}/X_0 = x\} = 1$ for all x.

A measure satisfying (1.1) is usually called an invariant measure.

Orey [5] showed that there exists a unique integer d, which is the number of "cyclic sets" of the process. If $d = 1$ we shall call the process aperiodic, otherwise periodic. Suppose $X = \{0, 1, 2, \cdots\}$ and φ is the counting measure, then Condition (C) gives us an "irreducible recurrent Markov chain." If for any $i, j, k, h \varepsilon X$, and any integer m

$$\lim_n P^{n+m}(i, j)/P^n(k, h) = \pi(j)/\pi(h) \tag{1.2}$$

holds, then we say the "strong ratio limit property" holds for the chain. Obviously if the chain is periodic (1.2) cannot be true. Even if $d = 1$, a counterexample in [1] shows that (1.2) need not hold for an irreducible recurrent Markov chain. Orey showed [6] that a necessary and sufficient condition for (1.2) to hold in this special situation is that $P^{n+1}(0, 0)/P^n(0, 0) \to 1$ as $n \to \infty$. Our aim is to prove results similar to (1.2) when X is quite general and Condition (C) holds. The conditions that we impose and the methods that we use are natural analogues of those used by Orey [6] and Pruitt [7] in the discrete situation. Orey's result mentioned above for the discrete situation follows directly as a special case of our Theorem 2 given below.

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Before stating our results it is necessary to introduce some notation. $f^n(x, y)$ will denote the Radon-Nikodym derivative of the absolutely continuous part of $P^n(x, E)$ with respect to π (the invariant measure).

DEFINITION 1.1. Let $\mathcal{S}$ denote the class of sets $A \varepsilon \mathcal{B}$ of positive π-measure such that for some n (depending on A) $\inf_{x\varepsilon A, y\varepsilon A} f^n(x, y) > 0$.

Orey [5] showed that $\mathcal{S}$ is nonempty. It is also known [4] that X can be written as a countable union of sets in $\mathcal{S}$ plus a π-null set. If $d = 1$ (aperiodic case) then the union of two sets in $\mathcal{S}$ is again in $\mathcal{S}$. This is also given in [4]. Since $d = 1$ is necessary for the strong ratio limit property, we shall from now on assume, in addition to Condition (C), that $d = 1$.

REMARK. It is a simple matter to check that a set in $\mathcal{S}$ must have finite π-measure and that a subset of a set in $\mathcal{S}$ is again in $\mathcal{S}$ if it has positive π-measure.

CONDITION (F). There exists a set $F \varepsilon \mathcal{S}$ and a point $\theta \varepsilon F$ such that for any $x \varepsilon F$, any measurable subset E of F

$$\lim_n (P^{n+1}(x, E)/P^n(\theta, F)) = \pi(E)/\pi(F).$$

REMARK. Notice that Condition (F) is equivalent to requiring that for any integer m, any $x \varepsilon F$, and any measurable subset E of F

$$\lim_n (P^{n+m}(x, E)/P^n(\theta, F)) = \pi(E)/\pi(F).$$

The following main results will be established in the next section.

THEOREM 1. *If Condition* (F) *holds and*

$$\limsup_n (P^{n+1}(x, F)/P^n(\theta, F)) \leqq 1$$

for $x \varepsilon S$, then for all x, $y \varepsilon S$ and every integer m

$$\lim_n (P^{n+m}(x, G)/P^n(y, H)) = \pi(G)/\pi(H)$$

for any sets G, H in $\mathcal{S}$.

REMARK. It will be clear from its proof that Theorem 1 remains valid if n is replaced by n_k everywhere in its statement, $\{n_k\}$ being an increasing sequence of integers tending to $+\infty$.

THEOREM 2. *If Condition* (F) *holds, then for any sets G, H in $\mathcal{S}$ and every integer m*

$$P^{n+m}(x, G)/P^n(y, H) \to \pi(G)/\pi(H)$$

in $\pi \times \pi$-measure on sets of finite $\pi \times \pi$-measure, where $\pi \times \pi$ denotes the product of π on $(X \times X, \mathcal{B} \times \mathcal{B})$. Precisely this means, given $\epsilon > 0$ and $R \varepsilon \mathcal{B} \times \mathcal{B}$ with $\pi \times \pi(R) < \infty$,

$$\lim_n \pi \times \pi(\{(x, y) \varepsilon R: |P^{n+m}(x, G)/P^n(y, H) - \pi(G)/\pi(H)| > \epsilon\} = 0.$$

NOTATION. For any probability measure μ write $\int \mu(dy)P^n(y, E) = P^n(\mu, E)$.

THEOREM 3. *A necessary and sufficient condition in order that for any probability measures μ and ν, any integer m and any sets G and H in $\mathcal{S}$,*

$$\lim_n (P^{n+m}(\mu, G)/P^n(\nu, H)) = \pi(G)/\pi(H),$$

is that Condition (F) *be satisfied and that there exist positive integers M_1 and M_2 such* *that for all* $n \geqq M_1$ *and all* $x \varepsilon X$

$$P^{n+1}(x, F)/P^n(\theta, F) \leqq M_2. \tag{1.3}$$

REMARK. Strong ratio limit property under Condition (C) has been considered by Levitan [4]. Our results and proofs seem to be quite different. We make no attempt here to compare them with the results in [4].

We introduce the following notation which will be needed in the proofs.

$$\begin{aligned} {}_FP^n(x, A) &= \text{Prob}\{X_1 \not\varepsilon F, \cdots, X_{n-1} \not\varepsilon F, X_n \varepsilon A \mid X_0 = x\} \qquad \text{for} \quad n \geqq 2. \\ &= P(x, A) \qquad \text{for} \quad n = 1. \end{aligned}$$

2. Proofs. We start with some lemmas. Lemma 2.1 (in a weaker form) can be found in [4]. The following proof is much shorter.

LEMMA 2.1. *For any sets* F, G *in* $\mathcal{S}$ *there exist positive integers* j *and* M *such that for all* $x \varepsilon X$ *and* $m \geqq 1$,

(a) ${}_FP^m(x, G) \leqq M \sum_{k=1}^{j} {}_FP^{m+k}(x, F)$,

(b) $P^m(x, G) \leqq M \cdot P^{m+j}(x, F)$.

PROOF. We mentioned in Section 1 that if $d = 1$ then the union of two sets in $\mathcal{S}$ is again in $\mathcal{S}$. Hence $F \cup G \varepsilon \mathcal{S}$. Hence there is an integer j such that $\inf_{x \varepsilon F \cup G, y \varepsilon F \cup G} f^j(x, y) > 0$. Pick $\epsilon > 0$ so that $\epsilon \cdot \pi(F) \leqq 1$ and

$$\inf_{x \varepsilon F \cup G, y \varepsilon F \cup G} f^j(x, y) \geqq \epsilon.$$

Then for any $x \varepsilon X$,

$$\sum_{m=1}^{j} {}_FP^m(x, F) \geqq P^j(x, F) \geqq \int_F \pi(dy) f^j(x, y) \geqq \epsilon \cdot \pi(F).$$

Thus for $x \varepsilon X$,

$$\begin{aligned} \sum_{k=1}^{j} {}_FP^{m+k}(x, F) &\geqq \sum_{k=1}^{j} \int_{G \setminus F} {}_FP^m(x, dy)\, {}_FP^k(y, F) \\ &\geqq \epsilon \cdot \pi(F)\, {}_FP^m(x, G \setminus F). \end{aligned}$$

Hence $\epsilon \cdot \pi(F)\, {}_FP^m(x, G) \leqq \sum_{k=0}^{j} {}_FP^{m+k}(x, F)$, since we picked ϵ so that $\epsilon \cdot \pi(F) \leqq 1$. We set $M = (\epsilon \cdot \pi(F))^{-1}$ to get (a).

To show (b) we see that

$$P^{m+j}(x, F) \geqq \int_G P^m(x, dy) P^j(y, F).$$

But for $y \varepsilon G$, $P^j(y, F) \geqq \int_F f^j(y, z) \pi(dz) \geqq \epsilon \cdot \pi(F)$. Hence

$$P^{m+j}(x, F) \geqq \epsilon \cdot \pi(F) P^m(x, G)$$

for all $x \varepsilon X$. Again set $M = (\epsilon \cdot \pi(F))^{-1}$.

LEMMA 2.2. *Condition* (F) *implies that for any bounded* $\mathcal{B}$*-measurable function* f, $x \varepsilon F$, *and any integer* m

$$\lim_n \int_F P^{n+m}(x, dy) f(y)/P^n(\theta, F) = \int_F \pi(dy) f(y)/\pi(F). \tag{2.1}$$

PROOF. If $f = I_E$, the indicator function of some $E \varepsilon \mathcal{B}$, then for $x \varepsilon F$ (2.1) follows from Condition (F). Hence (2.1) holds for linear combinations of indicator functions. It is enough to prove (2.1) for non-negative f. Let $f_j \nearrow f$, where f_j is a linear combination of indicator functions of sets in $\mathcal{B}$. Then for $j \geqq 1$ we have

$$\lim_n \left(\int_F P^{n+m}(x, dy)f_j(y)/P^n(\theta, F)\right) = \int_F \pi(dy)f_j(y)/\pi(F).$$

Since f is bounded we can pick f_j so that $f_j \nearrow f$ uniformly. Then

$$\begin{aligned}|\textstyle\int_F P^{n+m}(x, dy)f(y)/P^n(\theta, F) - \int_F \pi(dy)f(y)/\pi(F)| \\ \leqq |\textstyle\int_F P^{n+m}(x, dy)f(y)/P^n(\theta, F) - \int_F P^{n+m}(x, dy)f_j(y)/P^n(\theta, F)| \\ + |\textstyle\int_F P^{n+m}P(x, dy)f_j(y)/P^n(\theta, F) - \int_F \pi(dy)f_j(y)/\pi(F)| \\ + |\textstyle\int_F \pi(dy)f_j(y)/\pi(F) - \int_F \pi(dy)f(y)/\pi(F)|.\end{aligned}$$

Pick j sufficiently large so that for all $y \varepsilon F$ we have $|f_j(y) - f(y)| \leqq \epsilon/3$. Then n can clearly be picked large enough so that for $x \varepsilon F$ the first and the second terms on the right side of the inequality each is less than $\epsilon/3$. This finishes the proof.

LEMMA 2.3. *Condition* (F) *implies that for any* $G \varepsilon \mathcal{B}$ *and any* $x \varepsilon X$

$$\liminf_n \left(P^{n+m}(x, G)/P^n(\theta, F)\right) \geqq \pi(G)/\pi(F) \tag{2.2}$$

holds for any integer m. (If $\pi(G) = +\infty$ *then* $\pi(G)/\pi(F)$ *is interpreted as* $+\infty$ *).*

PROOF. Clearly it is enough to establish (2.2) for sets G of finite π-measure. The following decomposition formula is easily established. The arguments are essentially the same as in [1] for the discrete case.

$$P^{n+m}(x, G) = {}_FP^{n+m}(x, G) + \textstyle\sum_{\nu=1}^{n+m-1} \int_F {}_FP^\nu(x, dy) \cdot [\sum_{\mu=1}^{m+n-\nu-\mu} \int_F P^{m+n-\nu}(y, dz)\, {}_FP^\mu(z, G)].$$

Hence for n sufficiently large we have

$$\begin{aligned}P^{n+m}(x, G) &\geqq \textstyle\sum_{\nu=1}^{N} \int_F {}_FP^\nu(x, dy)[\sum_{\mu=1}^{N} \int_F P^{m+n-\nu-\mu}(y, dz)\, {}_FP^\mu(z, G)] \\ &= s_N(x, m, n; G), \quad \text{say.}\end{aligned} \tag{2.3}$$

We will show that $\lim_N \lim_n (s_N(x, m, n; G)/P^n(\theta, F))$ exists for all integers m, all $x \varepsilon X$ and equals $\pi(G)/\pi(F)$. This fact will be needed later and is certainly enough to prove the lemma. By Lemma 2.2 we have

$$\lim_n \left(\int_F P^{m+n-\nu-\mu}(y, dz)\, {}_FP^\mu(z, G)/P^n(\theta, F)\right) = \int_F \pi(dz)\, {}_FP^\mu(z, G)/\pi(F)$$

for $y \varepsilon F$. Using this fact we immediately conclude that for every N

$$\lim_n s_N(x, m, n; G)/P^n(\theta, F) = (\pi(F))^{-1}(\textstyle\sum_{\nu=1}^{N} {}_FP^\nu(x, F)) \cdot (\sum_{\mu=1}^{N} \int_F \pi(dz)\, {}_FP^\mu(z, G)).$$

Hence

$$\lim_N \lim_n s_N(x, m, n; G)/P^n(\theta, F) = (\pi(F))^{-1}(\textstyle\sum_{\nu=1}^{N} {}_FP^\nu(x, F)) \cdot (\sum_{\mu=1}^{\infty} \int_F \pi(dz)\, {}_FP^\mu(z, G)).$$

But Condition (C) implies that $\sum_{\nu=1}^{\infty} {}_FP^{\nu}(x, F) = 1$, because the expression is simply the probability of eventually hitting F starting from x. Also,

$$\sum_{\mu=1}^{\infty} \int_F \pi(dz)\, {}_FP^{\mu}(z, G) = \pi(G) \qquad \text{(see Harris [2])}.$$

This proves the assertion.

Proof of Theorem 1. It is enough to show that under the condition of Theorem 1 for any $x \varepsilon A$ and any integer m we have

$$\limsup_n P^{n+m}(x, G)/P^n(\theta, F) \leqq \pi(G)/\pi(F).$$

The result then follows from Lemma 2.3 and similar arguments for H. We write

$$P^{n+m}(x, G) = s_N(x, m, n; G) + r_N(x, m, n; G) \tag{2.4}$$

where the first term of the right is defined already, the second term is the remainder. For $x \varepsilon S$ we have $\lim_n (P^{n+m}(x, F)/P^n(\theta, F)) = 1$ because of Lemma 2.3 and the additional condition of Theorem 1. Hence if we have F in place of G in (2.4) and we divide through by $P^n(\theta, F)$ we conclude that for $x \varepsilon S$

$$\lim_N \lim_n r_N(x, m, n; F)/P^n(\theta, F) = 0, \tag{2.5}$$

because we have already shown in the proof of Lemma 2.3 that

$$\lim_N \lim_n (s_N(x, m, n; F)/P^n(\theta, F)) = 1$$

for all x. On the other hand,

$$\limsup_n P^{n+m}(x, G)/P^n(\theta, F) \leqq \limsup_n s_N(x, m, n; G)/P^n(\theta, F) + \limsup_n r_N(x, m, n; G)/P^n(\theta, F) \qquad \text{for all } N.$$

Hence we have by Lemma 2.3 as we let $N \to \infty$ on the right side,

$$\limsup_n P^{n+m}(x, G)/P^n(\theta, F) \leqq \pi(G)/\pi(F) + \limsup_N (\limsup_n r_N(x, m, n; G)/P^n(\theta, F)).$$

It is thus enough to show that the last term equals 0 for $x \varepsilon S$. By Lemma 2.1 (a) we have for $G \varepsilon \mathcal{S}$ an integer j and a positive number M such that

$${}_FP^m(x, G) \leqq M \sum_{k=1}^{j} {}_FP^{m+k}(x, F)$$

for all $x \varepsilon X$ and all $m \geqq 1$. Using this fact we conclude after some simple arithmetic that

$$r_N(x, m, n; G)/P^n(\theta, F) \leqq M \sum_{k=1}^{j} r_N(x, m + k, n; F)/P^n(\theta, F).$$

Hence

$$\limsup_N (\limsup_n r_N(x, m, n; G)/P^n(\theta, F)) \leqq M \sum_{k=1}^{j} \limsup_N (\limsup_n r_N(x, m + k, n; F)/P^n(\theta, F)) = 0.$$

This finishes the proof of Theorem 1.

PROOF OF THEOREM 2. Let $\varphi_n(x) = P^{n+1}(x, F)/P^n(\theta, F)$. Let us assume that every subsequence of $\{\varphi_n\}$ has a further subsequence coverging to 1 a.e. (π). Then we are done, because if a subsequence $\{\varphi_{n_k}\}$ contains a further subsequence $\{\varphi_{n_k'}\}$ which converges to 1 a.e. (π), where let N be the exceptional π-null set, then by the remark after Theorem 1 we have for $(x, y) \varepsilon (X - N) \times (X - N)$ and $G, H \varepsilon \mathcal{S}$

$$\lim_{n_k'} P^{n'_k+m}(x, G)/P^{n_k'}(y, H) = \pi(G)/\pi(H).$$

This clearly finishes the proof provided we prove the assumption about $\{\varphi_n(x)\}$. We have

$$(2.6) \qquad 1 = P^n(\theta, F)/P^n(\theta, F) = \int P^{n_0}(\theta, dy)P^{n-n_0}(y, F)/P^n(\theta, F).$$

By Lemma 2.3 we have $\liminf_n P^{n-n_0}(y, F)/P^n(\theta, F) \geqq 1$ for all y. It is easy to conclude from (2.6) and this fact that $P^{n-n_0}(y, F)/P^n(\theta, F)$ converges to 1 in $P^{n_0}(\theta, dy)$-measure. But as a consequence of Condition (F) we have $P^{n+1}(\theta, F)/P^n(\theta, F) \to 1$ as $n \to \infty$, and hence we conclude that $\varphi_n(x) \to 1$ in $P^{n_0}(\theta, dy)$-measure. Since n_0 was arbitrary we have that $\varphi_n(x) \to 1$ in $p^k(\theta, dy)$-measure for every k. Let $\{n_k\}$ be any sequence of integers $\nearrow + \infty$. Then there exists a subsequence $\{n_k^{(1)}\}_{k=1}^{\infty}$ such that $\varphi_{n_k^{(1)}} \to 1$ a.e. $(P^1(\theta, dy))$. Let $\{n_k^{(2)}\}_k \subset \{n_k^{(1)}\}_k$ so that $\varphi_{n_k^{(2)}} \to 1$ a.e. $(P^2(\theta, dy))$ as $k \to \infty$. We proceed in this manner and construct the diagonal subsequence $\{n_k^{(k)}\}_{k=1}^{\infty}$. Then we have

$$\varphi_{n_k^{(k)}} \to 1 \text{ a.e. } (P^r(\theta, dy)) \qquad \text{for} \quad r = 1, 2, \cdots.$$

Let $N = \{x: \varphi_{n_k^{(k)}}(x) \nrightarrow 1\}$. Then $P^r(\theta, N) = 0$ for $r = 1, 2, \cdots$. Hence Prob $\{\text{Hitting } N \text{ eventually} \mid X_0 = \theta\} = 0$. This means $\pi(N) = 0$ as a consequence of (ii) of Harris' result mentioned in Section 1. Thus we have demonstrated that every subsequence of $\{\varphi_n\}$ has a further subsequence that converges to 1 a.e. (π). This establishes our result.

PROOF OF THEOREM 3. The necessity of the conditions is almost obvious. Condition (F) is clearly necessary. If the second requirement is violated, then there exist sequences of integers $\{n_k\}$ and $\{M_k\}$ tending to infinity and a sequence $\{x_k\}$ in X such that

$$P^{n_k+1}(x_k, F)/P^{n_k}(\theta, F) \geqq M_k, \qquad k \geqq 1.$$

We can clearly pick these sequences so that $M_k \geqq 2^{k+1}$. Let μ be the probability measure which puts all the mass in the $\{x_k\}$ with $\mu(\{x_k\}) = 1/2^k$ and let $\nu(\{\theta\}) = 1$. Then

$$P^{n_k+1}(\mu, F)/P^{n_k}(\nu, F) \geqq 2^{-k} \cdot P^{n_k+1}(x_k, F)/P^{n_k}(\theta, F) \geqq 2.$$

This shows necessity.

To prove sufficiency we first assume that $\lim_n (P^{n+1}(x, F)/P^n(\theta, F)) = 1$ for all $x \varepsilon X$. Then Theorem 1 applies and we have for any $G \varepsilon \mathcal{S}$, all $x \varepsilon X$, and any integer m

$$(2.7) \qquad \lim_n (P^{n+m}(x, G)/P^n(\theta, F)) = \pi(G)/\pi(F).$$

Lemma 2.1 (b) along with the condition (1.3) now yield the fact that for every m there is an integer M such that for all $x \varepsilon X$ and all sufficiently large n we have

$$P^{n+m}(x, G)/P^n(\theta, F) \leqq M.$$

This allows us to apply the dominated convergence theorem to (2.7) and we get

$$\lim_n (P^{n+m}(\mu, G)/P^n(\theta, F)) = \pi(G)/\pi(F).$$

The rest is now obvious. It thus remains to show that the conditions of the theorem imply

$$\lim_n (P^{n+1}(x, F)/P^n(\theta, F)) = 1 \qquad \text{for all} \quad x \varepsilon X.$$

A theorem of Jamsion and Orey [3] says that under Condition (C) if $d = 1$ then $\|P^n(\mu, \cdot) - P^n(\nu, \cdot)\| \to 0$ as $n \to \infty$, where $\|\cdot\|$ denotes the total variation of a finite signed measure. Now,

$$|[P^{n+1}(x, F) - P^n(\theta, F)]/P^n(\theta, F)|$$
$$\leqq \int \|P^{n_0+1}(x, dy) - P^{n_0}(\theta, dy)\| \, P^{n-n_0}(y, F)/P^n(\theta, F).$$

We now use condition (1.3) and the fact that $P^{n+1}(\theta, F)/P^n(\theta, F) \to 1$ (consequence of Condition (F)) to conclude that for all sufficiently large n (depending on n_0) the right side of the above inequality is dominated by

$$2M_2 \|P^{n_0+1}(x, dy) - P^{n_0}(\theta, dy)\|.$$

Hence

$$\limsup_n |[P^{n+1}(x, F) - P^n(\theta, F)]/P^n(\theta, F)| \leqq 2M_2 \|P^{n_0+1}(x, dy) - P^{n_0}(\theta, dy)\|.$$

But the left side does not depend on n_0 and so by the Jamison-Orey theorem, upon letting n_0 to tend to $+\infty$, we conclude that it must be equal to 0. This finishes the proof.

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