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ON THE DISTRIBUTION OF THE MAXIMUM AND MINIMUM OF RATIOS OF ORDER STATISTICS[1]

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1. Introduction and summary. Let X_i $(i = 0, 1, \cdots, p)$ be $(p + 1)$ independent and identically distributed nonnegative random variables each representing the jth order statistic in a random sample of size n from a continuous distribution $G(x)$ of a nonnegative random variable. Let $G_{j,n}(x)$ be the cumulative distribution function of X_i $(i = 0, 1, \cdots, p)$. Consider the ratios $Y_i = X_i/X_0$ $(i = 1, 2, \cdots, p)$. The random variables Y_i $(i = 1, 2, \cdots, p)$ are correlated and the distribution of the maximum and the minimum is of interest in problems of selection and ranking for restricted families of distribution. The distribution-free subset selection rules using the percentage points of these order statistics are investigated in a companion paper by Barlow and Gupta (1969). In the present paper, we discuss the distribution of these statistics, in general, for any $G(x)$ and then derive specific results for $G(x) = 1 - e^{-x/\theta}$, $x > 0$, $\theta > 0$. Section 2 deals with the distribution of the maximum while Section 3 discusses the distribution of the minimum. Some asymptotic results are given in Section 4, while Section 5 describes the tables of the percentage points of the two statistics.

2. Distribution of $Y_{\max}$. First we derive the joint distribution of Y_i $(i = 1, 2, \cdots, p)$. The joint density function for $X_0, X_1, \cdots, X_p$ is given by

$$(2.1)\quad f(x_0, x_1, \cdots, x_p) = [j\tbinom{n}{j}]^{p+1} \textstyle\prod_{t=0}^{p} G^{j-1}(x_t)[1 - G(x_t)]^{n-j} g(x_t)$$

where $g(x) = dG(x)/dx$.

Making appropriate transformations the joint density of $Y_1, Y_2, \cdots, Y_p$ can be written as

$$(2.2)\quad f_1(y_1, y_2, \cdots, y_p) = [j\tbinom{n}{j}]^{p+1} \int_0^\infty y_0^p [G(y_0) \textstyle\prod_1^p G(y_t y_0)]^{j-1}$$
$$\cdot [(1 - G(y_0)) \textstyle\prod_1^p (1 - G(y_t y_0))]^{n-j} g(y_0) \prod_1^p g(y_t y_0)\, dy_0 .$$

For $G(X) = 1 - e^{-x/\theta}$, (2.2) reduces to

$$(2.3)\quad f_1(y_1, y_2, \cdots, y_p) = [j\tbinom{n}{j}]^{p+1} \int_0^1 (-\log u)^p$$
$$[(1 - u)(1 - u^{y_1}) \cdots (1 - u^{y_p})]^{j-1} u^{(n-j+1)(1+y_1+\cdots+y_p)-1}\, du.$$

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For $j = 1$, we obtain

$$(2.4)\quad f_1(y_1, y_2, \cdots, y_p) = \Gamma(p+1)(1 + y_1 + \cdots + y_p)^{-(p+1)}, \quad 0 \leqq y_1, \cdots, y_p < \infty.$$

It should be noted that the distribution (2.4) is independent of n, as was also pointed out in the companion paper by Barlow and Gupta (1969). Again, (2.4) gives the joint density functions of several correlated F random variables each with (2, 2) degrees of freedom. In this case, $Y_{\max}$ and $Y_{\min}$ are the largest and the smallest of several correlated F statistics with degrees of freedom (2, 2). The distribution of the largest and the smallest of several correlated F statistics with different degrees of freedom have been discussed by Gupta (1963a) and Gupta and Sobel (1962) respectively.

The cumulative distribution function of $Y_{\max}$ can be obtained directly (without using (2.2)) as follows.

$$(2.5)\quad P\{Y_{\max} \leqq y\} \equiv H_1(y) = \int_0^\infty G_{j,n}^p(yx) g_{j,n}(x)\, dx$$

where

$$(2.6)\quad g_{j,n}(x) = j\binom{n}{j} G^{j-1}(x)[1 - G(x)]^{n-j} g(x),$$

and

$$(2.7)\quad G_{j,n}(x) \equiv \int_0^x g_{j,n}(t)\, dt = I_{G(x)}(j, n-j+1) = \sum_{t=j}^n \binom{n}{t} G^t(x)(1 - G(x))^{n-t}$$

The density of $Y_{\max}$ is

$$(2.8)\quad h_1(y) = p \int_0^\infty x G_{j,n}^{p-1}(yx) g_{j,n}(yx) g_{j,n}(x)\, dx.$$

By expanding $G_{j,n}^p(yx)$ in powers of $1 - G(yx)$, we can express (2.5) as

$$(2.9)\quad H_1(y) = j\binom{n}{j} \sum_{r=0}^{np} \int_0^\infty b(r, p; n, j)[1 - G(xy)]^r G^{j-1}(x) \cdot [1 - G(x)]^{n-j} g(x)\, dx$$

where $b(r, p; n, j)$ is the coefficient of y^r in $[\sum_{t=j}^n \binom{n}{t}(1-y)^t y^{n-t}]^p$ and is given by the following recursion relations.

$$(2.10)\quad \begin{aligned}
b(r, 1; n, j) &= 1, && r = 0,\\
&= 0, && 1 \leqq r \leqq n-j,\\
&= \binom{n}{r} \sum_{t=0}^{n-j} (-1)^{r-t}\binom{r}{t}, && n-j+1 \leqq r \leqq n,\\
&= 0, && n < r < \infty;\\
b(r, p; n, j) &= 1, && r = 0,\\
&= 0, && 1 \leqq r \leqq n-j,\\
&= b(r, p-1; n, j)
\end{aligned}$$

$$+ \sum_{t=n-j+1}^{r} b(t, 1; n, j) b(r - t, p - 1; n, j), \quad n - j + 1 \leqq r \leqq n,$$

$$(2.11) \quad = b(r, p - 1; n, j) + \sum_{t=n-j+1}^{n} b(t, 1; n, j) b(r - t, p - 1; n, j), \quad n + 1 \leqq r \leqq np - n,$$

$$= \sum_{t=\max(n-j+1, r-np+n)}^{n} b(t, 1; n, j) b(r - t, p - 1; n, j), \quad np - n + 1 \leqq r \leqq np,$$

$$= 0, \quad np < r < \infty.$$

The density $h(y)$ can be written in a similar way. For the special case $G(x) = 1 - e^{-x/\theta}$, we obtain

$$(2.12) \quad H_1(y) = 1 + \sum_{r=n-j+1}^{np} b(r, p; n, j) \cdot [(1 + ry/n)(1 + ry/(n - 1)) \cdots (1 + ry/(n - j + 1))]^{-1}.$$

For $j = 1$, the coefficients $b(r, p; n, 1) = (-1)^l \binom{p}{l}$ if $r = nl$, $l = 0, 1, 2, \cdots, p$ and zero otherwise. It follows that

$$(2.13) \quad H_1(y) = \sum_{l=0}^{p} (-1)^l \binom{p}{l} (1 + ly),^{-1}$$

which is independent of n as it should be, and

$$(2.14) \quad h_1(y) = \sum_{l=0}^{p} (-1)^{l+1} \binom{p}{l}\, l\, (1 + ly)^{-2}.$$

Incidentally, one can obtain inequalities on the right hand sides of (2.12) and (2.13) by using the fact that $H_1(y)$ and $h_1(y)$ are the cdf and the density function.

3. Distribution of $Y_{\min}$. The cdf $H_2(y)$ of $Y_{\min}$ is given by

$$(3.1) \quad H_2(y) \equiv P\{Y_{\min} \leqq y\} = 1 - \int_0^\infty [1 - G_{j,n}(yx)]^p g_{j,n}(x)\, dx,$$

where $g_{j,n}(x)$ and $G_{j,n}(x)$ are given by (2.6) and (2.7).

The density of $Y_{\min}$ is

$$(3.2) \quad h_2(y) = p \int_0^\infty x[1 - G_{j,n}(yx)]^{p-1} g_{j,n}(yx) g_{j,n}(x)\, dx.$$

Let $1 - H_2(y) = F(y)$. By expanding $[1 - G_{j,n}(yx)]^p$ in powers of $1 - G(yx)$, we can write

$$(3.3) \quad F(y) = j\binom{n}{j} \sum_{r=0}^{np} \int_0^\infty b'(r, p; n, j)[1 - G(xy)]^r G^{j-1}(x) \cdot [1 - G(x)]^{n-j} g(x)\, dx$$

where $b'(r, p; n, j)$ is the coefficient of y^r in $[\sum_{t=0}^{j-1} \binom{n}{t} (1 - y)^t y^{n-t}]^p$ and is given by the following recursion relations:

$$
\begin{aligned}
(3.4)\quad b'(r, 1; n, j) &= 0, && 0 \leqq r \leqq n - j, \\
&= \textstyle\sum_{k=0}^{j-1-n+r} (-1)^k \binom{n}{n-r+k}\binom{n-r+k}{k}, && n - j + 1 \leqq r \leqq n, \\
&= 0, && n < r < \infty;
\end{aligned}
$$

$$
\begin{aligned}
(3.5)\quad b'(r, p; n, j) &= 0, && 0 \leqq r \leqq (n - j + 1)p - 1, \\
&= \textstyle\sum_{m=\max(n-j+1, r-n(p-1))}^{\min(n, r-(p-1)(n-j+1))} b'(m, 1; n, j) b'(r - m, p - 1; n, j), && (n - j + 1)p \leqq r \leqq np \\
&= 0, && np < r < \infty.
\end{aligned}
$$

The density $h_2(y)$ can be written similarly. For the special case $G(x) = 1 - e^{-x/\theta}$, we obtain

$$
\begin{aligned}
(3.6)\quad F(y) &= \textstyle\sum_{r=(n-j+1)p}^{np} b'(r, p; n, j) \\
&\quad \cdot [(1 + ry/n)(1 + ry/(n - 1)) \cdots (1 + ry/(n - j + 1))]^{-1}.
\end{aligned}
$$

For $j = 1$, the coefficients $b(r, p; n, 1) = 1$ if $r = np$ and zero otherwise. So (3.6) reduces to

$$(3.7)\qquad F(y) = (1 + yp)^{-1},$$

which is independent of n. In this case

$$(3.8)\qquad h_2(y) = (1 + yp)^{-2}.$$

4. Asymtotic results. Let ξ_α denote the quantile of order α of the distribution $G(x)$, i.e. the root (assumed unique) of the equation $G(\xi) = \alpha$, where $0 < \alpha < 1$. We assume that, in some neighbourhood of $x = \xi_\alpha$, the density function $g(x)$ is continuous and has a continuous derivative $g'(x)$. In a sample of size n from the distribution $G(x)$, we take the jth smallest observation such that $j \leqq (n + 1)\alpha < j + 1$. Then $X_{j,n}$ is asymptotically normal with mean ξ and standard deviation $(g(\xi_\alpha))^{-1}(\alpha\bar{\alpha}/n)^{1/2}$, where $\bar{\alpha} = 1 - \alpha$.

Thus we have, as $n \to \infty$ and $j/n \to \alpha$

$$(4.1)\qquad H_1(y) \approx \int_{-\infty}^{\infty} \Phi^p(xy + (y - 1)\xi_\alpha g(\xi_\alpha)n^{\frac{1}{2}}(\alpha\bar{\alpha})^{-\frac{1}{2}})\, d\Phi(x)$$

and

$$(4.2)\qquad H_2(y) \approx 1 - \int_{-\infty}^{\infty} [1 - \Phi(xy + (y - 1)\xi_\alpha g(\xi_\alpha)n^{\frac{1}{2}}(\alpha\bar{\alpha})^{-\frac{1}{2}})]^p\, d\Phi(x)$$

where

$$\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-t^2/2}\, dt.$$

(Note: $a_n \approx b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$.) For $p = 1$, we get

$$(4.3)\qquad H_1(y) = H_2(y) \approx \int_{-\infty}^{\infty} \Phi(xy + (y - 1)\xi_\alpha g(\xi_\alpha)n^{\frac{1}{2}}(\alpha\bar{\alpha})^{-\frac{1}{2}})\, d\Phi(x).$$

Using the result

$$\int_{-\infty}^{\infty} \Phi(\alpha x + \beta)\, d\Phi(x) = \Phi(\beta)(1 + \alpha^2)^{-\frac{1}{2}},$$

this reduces to

(4.4) $H_1(y) = H_2(y) \approx \Phi((y - 1)\xi_\alpha g(\xi_\alpha)n^{\frac{1}{2}}(1 + y^2)^{-\frac{1}{2}}(\alpha\bar{\alpha})^{-\frac{1}{2}})$.

So if $H_1(y) = P^*$

$$(y - 1)\xi_\alpha g(\xi_\alpha)n^{\frac{1}{2}}(1 + y^2)^{-\frac{1}{2}}(\alpha\bar{\alpha})^{-\frac{1}{2}} = \Phi^{-1}(P^*), \tag{4.5}$$

which can be written as

$$y^2(1 - B^2) - 2y + (1 - B^2) = 0, \tag{4.6}$$

where

$$B = \Phi^{-1}(P^*)(\alpha\bar{\alpha})^{\frac{1}{2}}(\xi_\alpha g(\xi_\alpha)n^{\frac{1}{2}})^{-1}.$$

Obviously, this quadratic equation in y, has two positive roots which are reciprocals of each other. The appropriate roots can be determined using the fact that $H_1(y)$ is increasing in y and $H_1(1) = \frac{1}{2}$. So for $P^* > \frac{1}{2}$ (which will be the case for the selection procedures discussed in the companion paper), $y > 1$.

For the special case $G(x) = 1 - e^{-x/\theta}$, $e^{-\xi_\alpha/\theta} = 1 - \alpha$ and $g(\xi_\alpha) = \theta^{-1}e^{-\xi_\alpha/\theta} = (1 - \alpha)\theta^{-1}$. So (4.1) and (4.2) reduce to

$$H_1(y) \approx \int_{-\infty}^{\infty} \Phi^p(xy - (n\bar{\alpha}/\alpha)^{\frac{1}{2}}(y - 1)\log\bar{\alpha})\, d\Phi(x) \tag{4.7}$$

and

$$H_2(y) \approx 1 - \int_{-\infty}^{\infty} [1 - \Phi(xy - (n\bar{\alpha}/\alpha)^{\frac{1}{2}}(y - 1)\log\bar{\alpha})]^p\, d\Phi(x). \tag{4.8}$$

For a general p, to solve for y from $H_1(y) = P^*$ or $H_2(y) = P^*$ using (4.1) and (4.2), the Table II of Gupta (1963b) can be used with interpolations if necessary.

5. Description of the tables. Table 1 provides for the case $j = 1$ the reciprocals of the percentage points of the distribution of $Y_{\max}$ corresponding to the probability levels $\alpha = P^* = .75, .90$ and $.95$ and the percentage points of the distribution of $Y_{\min}$ corresponding to the probability levels $\alpha = 1 - P^* = .05, .10$ and $.25$ for $p = 1(1)10$. We note that when $j = 1$, the statistics $Y_{\max}$ and $Y_{\min}$ are the maximum and the minimum of several correlated F statistics with degrees of freedom (2, 2) and hence the entries in Table 1 are the same as those for $\nu = 2$ in Tables 1 A, B, C of Gupta (1963a) in the case of $Y_{\max}$ and same as those for $\nu = 2$ in Tables 3 A, B, C of Gupta and Sobel (1962) in the case of $Y_{\min}$, but are given for more places of decimals.

Tables 2 A through 2 E give the reciprocals of the percentage points of the distribution of $Y_{\max}$ corresponding to the probability levels $\alpha = P^* = .75, .90, .95$ for $p = 1$ through 5 respectively. The ranges of n are: 5(1)15 in Tables 2 A, B, C, 5(1)12 in Table 2D and 5(1)10 in Table 2E.

Tables 3A through 3E present the percentage points of the distribution of $Y_{\min}$ corresponding to the probability levels $\alpha = 1 - P^* = .25, .10, .05$ for $p = 1$ through 5 respectively. The ranges of n: 5(1)15 in Tables 3A-D & 5(1)13 in Table 3E.

TABLE I

A. *Reciprocals of 100α percentage points of* $Y_{\max} = \max_{1 \leqq i \leqq p} X_i/X_0$ *for j = 1 and all n*

p	$\alpha = P^*$		
	0.75	0.90	0.95
1	.33333	.11111	.05263
2	.20783	.07233	.03469
3	.16631	.05871	.02827
4	.14472	.05145	.02483
5	.13115	.04683	.02263
6	.12166	.04357	.02107
7	.11456	.04112	.01990
8	.10901	.03919	.01897
9	.10451	.03762	.01822
10	.10078	.03631	.01759

B. *100α percentage points of* $Y_{\min} = \min_{1 \leqq i \leqq p} X_i/X_0$ *for j = 1 and all n*

p	$1 - \alpha = P^*$		
	0.75	0.90	0.95
1	.33333	.11111	.05263
2	.16667	.05556	.02632
3	.11111	.03704	.01754
4	.08333	.02778	.01316
5	.06667	.02222	.01053
6	.05556	.01852	.00877
7	.04762	.01587	.00752
8	.04167	.01389	.00658
9	.03704	.01235	.00585
10	.03333	.01111	.00526

For given p and P^*, the entries in Tables A and B are respectively the values of c and d (c and d are independent of n) for which

$$\int_0^\infty G_{1,n}^p\,(x/c)\,dG_{1,n}(x) = P^* \quad \text{and} \quad \int_0^\infty [1 - G_{1,n}(xd)]^p\,dG_{1,n}(x) = P^*$$

where $G_{1,n}(\cdot)$ is the cdf of the smallest order statistic in a sample of size n from the exponential distribution.

TABLE 2A

Reciprocals of the percentage points of $Y_{\max} = \max_{1 \leq i \leq p} X_i/X_0$ *for* $p = 1$

n	j = 2	3	4	5	6	7	8	9	10	11	12	13	14	15
	.48307	.55596	.59583	.60462										
5	.24225	.32197	.37004	.38251										
	.15560	.22871	.27582	.28958										
	.48353	.55788	.60192	.62654	.62516									
6	.24266	.32397	.37699	.40851	.40827									
	.15591	.23045	.28228	.31441	.31548									
	.48379	.55889	.60473	.63401	.64944	.64125								
7	.24289	.32503	.38021	.41755	.43822	.42890								
	.15609	.23138	.28527	.32314	.34485	.33649								
	.48396	.55949	.60626	.63757	.65802	.66737	.65431							
8	.24303	.32566	.38198	.42189	.44902	.46206	.44593							
	.15620	.23193	.28692	.32735	.35558	.36965	.35399							
	.48407	.55988	.60720	.63958	.66222	.67685	.68189	.66520						
9	.24313	.32607	.38306	.42434	.45436	.47436	.48175	.46032						
	.15627	.23229	.28793	.32973	.36089	.38211	.39036	.36890						
	.48414	.56014	.60782	.64082	.66463	.68159	.69211	.69396	.67447					
10	.24320	.32635	.38377	.42587	.45743	.48057	.49534	.49837	.47271					
	.15633	.23253	.28860	.33121	.36396	.38843	.40435	.40801	.38182					
	.48420	.56033	.60824	.64166	.66616	.68436	.69731	.70481	.70421	.68250				
11	.24325	.32654	.38427	.42689	.45939	.48422	.50232	.51307	.51266	.48353				
	.15637	.23270	.28906	.33220	.36592	.39215	.41157	.42334	.42330	.39316				
	.48424	.56047	.60855	.64224	.66720	.68614	.70040	.71041	.71558	.71305	.68954			
12	.24329	.32669	.38462	.42761	.46071	.48657	.50648	.52074	.52833	.52512	.49311			
	.15639	.23283	.28939	.33290	.36724	.39455	.41588	.43136	.43979	.43671	.40323			
	.48427	.56057	.60878	.64267	.66793	.68736	.70240	.71377	.72153	.72488	.72078	.69578		
13	.24331	.32680	.38489	.42813	.46165	.48817	.50919	.52535	.53659	.54163	.53612	.50166		
	.15642	.23293	.28964	.33341	.36818	.39619	.41869	.43622	.44855	.45425	.44861	.41227		
	.48430	.56066	.60896	.64299	.66847	.68823	.70379	.71597	.72513	.73112	.73300	.72762	.70138	
14	.24334	.32689	.38509	.42852	.46234	.48933	.51106	.52839	.54163	.55044	.55338	.54593	.50936	
	.15643	.23300	.28982	.33380	.36888	.39737	.42064	.43941	.45389	.46365	.46707	.45926	.42044	
	.48432	.56072	.60909	.64324	.66888	.68888	.70479	.71751	.72751	.73494	.73951	.74018	.73373	.70643
15	.24336	.32696	.38525	.42883	.46287	.49019	.51242	.53052	.54497	.55585	.56266	.56384	.55474	.51636
	.15645	.23306	.28997	.33409	.36940	.39825	.42205	.44165	.45744	.46944	.47707	.47856	.46887	.42787

For given p, n, j and $P^* = .75$ (top), .90 (middle), .95 (bottom), the entries in this table are the values of c for which $\int_0^\infty G_{j,n}^p(x/c)\, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the cdf of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 2B

Reciprocals of the percentage points of $Y_{max} = \max_{1 \leq i \leq p} X_i/X_0$ *for* $p = 2$

n	j													
	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	.34892	.42508	.46809	.47527										
5	.18181	.25464	.29996	.31044										
	.11853	.18353	.22682	.23872										
	.34945	.42738	.47564	.50261	.49813									
6	.18220	.25665	.30712	.33734	.33528									
	.11882	.18521	.23315	.26322	.26304									
	.34975	.42860	.47913	.51207	.52891	.51631								
7	.18242	.25772	.31045	.34682	.36670	.35541								
	.11898	.18610	.23611	.27194	.29238	.28297								
	.34994	.42934	.48104	.51659	.53993	.54982	.53125							
8	.18256	.25835	.31228	.35139	.37819	.39056	.37217							
	.11908	.18663	.23774	.27616	.30324	.31642	.29971							
	.35006	.42980	.48221	.51915	.54535	.56214	.56697	.54382						
9	.18265	.25876	.31340	.35398	.38388	.40378	.41046	.38644						
	.11915	.18697	.23873	.27855	.30865	.32917	.33668	.31405						
	.35015	.43012	.48298	.52074	.54847	.56833	.58038	.58137	.55461					
10	.18272	.25904	.31414	.35559	.38718	.41049	.42519	.42740	.39880					
	.11920	.18720	.23939	.28005	.31178	.33567	.35111	.35408	.32655					
	.35022	.43035	.48352	.52180	.55045	.57195	.58724	.59572	.59371	.56402				
11	.18276	.25924	.31465	.35667	.38927	.41443	.43279	.44345	.44206	.40966				
	.11923	.18737	.23984	.28105	.31377	.33949	.35858	.36998	.36923	.33757				
	.35026	.43051	.48390	.52254	.55179	.57428	.59131	.60316	.60885	.60442	.57232			
12	.18280	.25939	.31502	.35743	.39069	.41697	.43732	.45184	.45926	.45492	.41930			
	.11926	.18749	.24017	.28175	.31512	.34196	.36305	.37834	.38644	.38259	.34740			
	.35030	.43064	.48419	.52309	.55274	.57588	.59396	.60762	.61679	.62026	.61386	.57973		
13	.18283	.25950	.31529	.35798	.39169	.41871	.44028	.45692	.46837	.47314	.46633	.42796		
	.11928	.18758	.24042	.28227	.31608	.34365	.36597	.38341	.39562	.40098	.39450	.35625		
	.35033	.43074	.48441	.52350	.55344	.57702	.59579	.61056	.62162	.62865	.63030	.62224	.58639	
14	.18285	.25958	.31550	.35840	.39244	.41996	.44233	.46026	.47393	.48289	.48544	.47654	.43578	
	.11930	.18766	.24060	.28265	.31679	.34487	.36800	.38675	.40123	.41088	.41394	.40520	.36427	
	.35035	.43082	.48458	.52382	.55397	.57787	.59711	.61260	.62481	.63379	.63909	.63922	.62977	.59243
15	.18287	.25965	.31567	.35872	.39300	.42089	.44381	.46259	.47763	.48889	.49578	.49647	.48575	.44291
	.11931	.18771	.24075	.28295	.31734	.34578	.36947	.38909	.40496	.41700	.42451	.42559	.41489	.37160

For given p, n, j and $P^* = .75$ (top), .90 (middle), .95 (bottom), the entries in this table are the values of c for which $\int_0^\infty G^p_{j,n}(x/c)\, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the cdf of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 2C

Reciprocals of the percentage points of $Y_{\max} = \max_{1 \leq i \leq p} X_i/X_0$ *for* $p = 3$

n	j = 2	3	4	5	6	7	8	9	10	11	12	13	14	15
	.29806	.37258	.41523	.42096										
5	.15749	.22607	.26937	.27858										
	.10324	.16388	.20487	.21565										
	.29860	.37502	.42329	.45018	.44395									
6	.15786	.22808	.27658	.30566	.30259									
	.10352	.16551	.21113	,23985	.23894									
	.29891	.37631	.42703	46038	.47708	.46236								
7	.15808	.22914	.27994	.31528	.33440	.32214								
	.10367	.16639	.21406	.24854	.26810	.25813								
	.29911	.37708	.42909	.46528	.48904	.49862	.47757							
8	.15821	.22978	.28179	.31994	.34614	.35791	.33850							
	.10377	.16691	.21568	.25276	.27899	.29153	.27432							
	.29924	.37758	.43035	.46804	.49495	.51207	.51640	.49043						
9	.15830	.23018	.28292	.32257	.35198	.37149	.37762	.35248						
	.10384	.16724	.21667	.25515	.28442	.30437	.31137	.28825						
	.29933	.37792	.43117	.46976	.49835	.51885	.53111	.53141	.50151					
10	.15837	.23047	.28366	.32422	.35535	.37840	.39281	.39447	.36463					
	.10388	.16747	.21732	.25665	.28757	.31093	.32595	.32846	.30040					
	.29939	.37816	.43174	.47091	.50051	.52283	.53865	.54720	.54431	.51121				
11	.15841	.23066	.28418	.32532	.35750	.38246	.40067	.41107	.40911	.37753				
	.10392	.16763	.21777	.25765	.28958	.31479	.33353	.34459	.34341	.31115				
	.29944	.37834	.43216	.47172	.50197	.52538	.54314	.55541	.56103	.55556	.51979			
12	.15845	.23081	.28455	.32609	.35896	.38508	.40536	.41978	.42694	.42198	.38486			
	.10394	.16775	.21809	.25835	.29094	.31729	.33806	.35310	.36091	.35662	.32075			
	.29948	.37847	.43246	.47231	.50301	.52713	.54605	.56034	.56983	.57310	.56550	.52746		
13	.15848	.23092	.28483	.32666	.35999	.38688	.40842	.42504	.43642	.44091	.43342	.39342		
	.10396	.16784	.21834	.25887	.29191	.31901	.34103	.35825	.37026	.37536	.36842	.32942		
	.29951	.37858	.43270	.47276	.50378	.52839	.54807	.56359	.57517	.58241	.58374	.57435	.53438	
14	.15850	.23101	.28505	.32709	.36076	.38817	.41054	.42851	.44221	.45108	.45334	.44369	.40119	
	.10398	.16791	.21852	.25925	.29262	.32024	.34309	.36166	.37599	.38547	.38827	.37904	.33729	
	.29954	.37866	.43289	.47310	.50436	.52932	.54953	.56585	.57871	.58812	.59352	.59323	.58232	.54068
15	.15851	.23108	.28521	.32742	.36134	.38913	.41208	.43094	.44606	.45735	.46413	.46449	.45298	.40827
	.10399	.16797	.21867	.25956	.29317	.32116	.34458	.36404	.37980	.39173	.39908	.39990	.38869	.34449

For given p, n, j and $P^* = .75$ (top), .90 (middle), .95 (bottom), the entries in this table are the values of c for which $\int_0^\infty G_{j,n}^p(x/c)\, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the cdf of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 2D

Reciprocals of the percentage points of $Y_{\max} = \max_{1 \leq i \leq p} X_i/X_j$ *for* $p = 4$

n	j = 2	3	4	5	6	7	8	9	10	11	12
	.26969	.34243	.38435	.38901							
5	.14355	.20924	.25105	.25937							
	.09440	.15215	.19156	.20155							
	.27023	.34493	.39269	.41919	.41181						
6	.14392	.21123	.25827	.28647	.28272						
	.09466	.15377	.19778	.22552	.22411						
	.27055	.34626	.39657	.42980	.44617	.43014					
7	.14414	.21229	.26164	.29617	.31468	.30181					
	.09481	.15463	.20068	.23418	.25309	.24275					
	.27075	.34705	.39870	.43490	.45867	.46788	.44534				
8	.14427	.21293	.26351	.30086	.32656	.33786	.31783				
	.09491	.15514	.20229	.23839	.26399	.27605	.25853				
	.27088	.34757	.40004	.43779	.46485	.48198	.48587	.45823			
9	.14436	.21334	.26464	.30353	.33247	.35164	.35735	.33155			
	.09497	.15547	.20328	.24079	.26943	.28893	.29555	.27213			
	.27097	.34791	.40086	.43958	.46841	.48910	.50133	.50110	.46937		
10	.14442	.21361	.26539	.30519	.33590	.35865	.37280	.37405	.34350		
	.09502	.15570	.20392	.24228	.27259	.29552	.31021	.31240	.28402		
	.27104	.34816	.40146	.44079	.47068	.49328	.50927	.51774	.51423	.47913	
11	.14447	.21381	.26591	.30631	.33808	.36279	.38080	.39097	.38860	.35404	
	.09505	.15586	.20437	.24328	.27461	.29942	.31785	.32864	.32715	.29456	
	.27109	.34835	.40188	.44163	.47220	.49596	.51399	.52640	.53188	.52572	.48779
12	.14450	.21396	.26629	.30709	.33956	.36545	.38559	.39987	.40681	.40142	.36344
	.09508	.15598	.20470	.24399	.27597	.30193	.32243	.33723	.34481	.34022	.30399

For given p, n, j and $P^* = .75$(top), .90(middle), .95(bottom), the entries in this table are the values of c for which $\int_0^\infty G_{j,n}^p(x/c)\, dG_{j,n}(x) = P^*$, where $G_{j,n}(\cdot)$ is the cdf of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 2E

Reciprocals of the percentage points of $Y_{\max} = \max_{1 \leq i \leq p} X_i/X_0$ *for* $p = 5$

n	j: 2	3	4	5	6	7	8	9	10
	.25101	.32219	.36340	.36723					
5	.13424	.19776	.23842	.24607					
	.08844	.14410	.18232	.19171					
	.25156	.32473	.37191	.39800	.38978				
6	.13461	.19975	.24565	.27314	.26889				
	.08870	.14570	.18850	.21549	.21370				
	.25188	.32609	.37587	.40887	.42490	.40796			
7	.13482	.20081	.24903	.28288	.30091	.28760			
	.08885	.14655	.19139	.22413	.24253	.23193			
	.25208	.32690	.37806	.41411	.43776	.44664	.42308		
8	.13495	.20144	.25099	.28761	.31287	.32379	.30334		
	.08895	.14706	.19299	.22834	.25342	.26511	.24738		
	.25221	.32742	.37940	.41707	.44412	.46117	.46469	.43592	
9	.13504	.20185	.25204	.29029	.31884	.33770	.34308	.31684	
	.08901	.14739	.19397	.23073	.25888	.27802	.28433	.26072	
	.25231	.32777	.38027	.41894	.44779	.46852	.48065	.48001	.44704
10	.13510	.20213	.25279	.29197	.32230	.34480	.35870	.35965	.32861
	.08905	.14762	.19462	.23223	.26205	.28463	.29905	.30097	.27240

For given p, n, j and $P^* = .75$(top), .90(middle), .95(bottom), the entries in this table are the values of c for which $\int_0^\infty G_{j,n}^p(x/c)\,dG_{j,n}(x) = P^*$, where $G_{j,n}(\cdot)$ is the cdf of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 3A

Percentage points of the distribution of $Y_{\min} = \min_{1 \leq i \leq p} X_i/X_0$ *for* $p = 1$

n	j: 2	3	4	5	6	7	8	9	10	11	12	13	14	15
	.48307	.55596	.59583	.60462										
5	.24225	.32197	.37004	.38251										
	.15560	.22871	.27582	.28958										
	.48353	.55788	.60192	.62654	.62516									
6	.24266	.32397	.37699	.40851	.40827									
	.15591	.23045	.28228	.31441	.31548									
	.48379	.55889	.60473	.63401	.64944	.64125								
7	.24289	.32503	.38021	.41755	.43822	.42890								
	.15609	.23138	.28527	.32314	.34485	.33648								
	.48396	.55949	.60626	.63757	.65802	.66737	.65431							
8	.24303	.32566	.38198	.42189	.44902	.46206	.44593							
	.15620	.23193	.28692	.32735	.35557	.36964	.35399							
	.48407	.55988	.60720	.63958	.66222	.67685	.68189	.66520						
9	.24313	.32607	.38306	.42434	.45436	.47436	.48175	.46032						
	.15627	.23229	.28793	.32972	.36089	.38211	.39036	.36890						
	.48414	.56014	.60782	.64082	.66463	.68159	.69211	.69396	.67447					
10	.24320	.32635	.38377	.42587	.45743	.48057	.49534	.49837	.47271					
	.15633	.23253	.28859	.33121	.36396	.38843	.40435	.40801	.38182					
	.48420	.56033	.60824	.64166	.66616	.68436	.69731	.70481	.70421	.68250				
11	.24325	.32654	.38427	.42689	.45939	.48422	.50232	.51307	.51266	.48353				
	.15637	.23270	.28905	.33220	.36592	.39215	.41157	.42333	.42330	.39316				
	.48424	.56047	.60855	.64224	.66720	.68614	.70040	.71041	.71558	.71305	.68954			
12	.24329	.32669	.38462	.42761	.46071	.48657	.50648	.52074	.52833	.52512	.49310			
	.15639	.23283	.28939	.33290	.36724	.39455	.41588	.43136	.43980	.43671	.40323			
	.48427	.56057	.60878	.64267	.66793	.68736	.70240	.71377	.72153	.72488	.72078	.69578		
13	.24331	.32680	.38489	.42813	.46165	.48817	.50919	.52535	.53659	.54163	.53612	.50166		
	.45642	.23293	.28964	.33341	.36818	.39619	.41869	.43621	.44855	.45425	.44860	.41226		
	.48430	.56066	.60896	.64299	.66847	.68823	.70379	.71597	.72513	.73112	.73300	.72762	.70138	
14	.24334	.32689	.38509	.42852	.46234	.48933	.51106	.52839	.54163	.55044	.55338	.54593	.50936	
	.15643	.23300	.28983	.33379	.36888	.39737	.42064	.43941	.45389	.46366	.46707	.45926	.42044	
	.48432	.56072	.60909	.64324	.66888	.68888	.70479	.71751	.72751	.73494	.73951	.74018	.73373	.70643
15	.24336	.32696	.38525	.42883	.46287	.49019	.51242	.53052	.54497	.55585	.56266	.56384	.55474	.51636
	.15645	.23306	.28997	.33409	.36940	.39825	.42205	.44165	.45745	.46945	.47707	.47856	.46887	.42787

For given p, n, j and $P^* = .75$ (top), .90 (middle), .95 (bottom), the entries in this table are the values of d for which $\int_0^\infty [1 - G_{j,n}(x\,d)]^p \, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the cdf of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 3B

Percentage points of the distribution of $Y_{\min} = \min_{1 \leqq i \leqq p} X_i/X_0$ *for* $p = 2$

n	j = 2	3	4	5	6	7	8	9	10	11	12	13	14	15
	.31553	.39895	.44798	.46243										
5	.16203	.23711	.28562	.30102										
	.10551	.17100	.21617	.23160										
	.31594	.40084	.45438	.48616	.48824									
6	.16234	.23881	.29187	.32510	.32778									
	.10574	.17244	.22180	.25390	.25725									
	.31617	.40184	.45733	.49430	.51519	.50871								
7	.16251	.23971	.29477	.33353	.35619	.34946								
	.10587	.17320	.22441	.26178	.28424	.27829								
	.31631	.40243	.45894	.49820	.52479	.53822	.52548							
8	.16262	.24024	.29636	.33758	.36652	.38147	.36752							
	.10595	.17365	.22585	.26559	.29416	.30929	.29598							
	.31641	.40282	.45993	.50040	.52952	.54905	.55708	.53956						
9	.16230	.24059	.29734	.33988	.37163	.39346	.40257	.38289						
	.10601	.17395	.22673	.26775	.29910	.32104	.33042	.31115						
	.31648	.40308	.46058	.50177	.53225	.55450	.56894	.57289	.55161					
10	.16275	.24082	.29798	.34131	.37459	.39954	.41600	.42053	.39620					
	.10604	.17415	.22731	.26909	.30196	.32701	.34379	.34857	.32436					
	.31653	.40326	.46103	.50269	.53398	.55770	.57502	.58563	.58639	.56209				
11	.16279	.24099	.29842	.34227	.37647	.40311	.42293	.43522	.43606	.40787				
	.10607	.17429	.22772	.27000	.30378	.33054	.35071	.36337	.36438	.33601				
	.31656	.40340	.46135	.50333	.53515	.55976	.57864	.59226	.59990	.59811	.57132			
12	.1628[illegible]	.24111	.29875	.34294	.37775	.40542	.42708	.44292	.45187	.44968	.41824			
	.10609	.17439	.22800	.27063	.30501	.33281	.35486	.37116	.38046	.37833	.34639			
	.31659	.40350	.46160	.50380	.53598	.56117	.58099	.59625	.60701	.61227	.60840	.57953		
13	.16283	.24121	.29899	.34343	.37865	.40700	.42977	.44758	.46026	.46647	.46175	.42753		
	.10611	.17447	.22822	.27109	.30588	.33437	.35756	.37588	.38903	.39555	.39075	.35574		
	.31662	.40358	.46178	.50415	.53659	.56218	.58262	.59888	.61134	.61981	.62314	.61754	.58691	
14	.16285	.24128	.29917	.34380	.37932	.40813	.43165	.45065	.46538	.47548	.47941	.47255	.43592	
	.10612	.17453	.22839	.27144	.30653	.33549	.35944	.37899	.39428	.40483	.40900	.40191	.36421	
	.31663	.40365	.46193	.50442	.53706	.56293	.58381	.60071	.61421	.62444	.63105	.63278	.62572	.59359
15	.16287	.24134	.29931	.34409	.37983	.40898	.43300	.45279	.46879	.48103	.48898	.49100	.48229	.44356
	.10613	.17459	.22852	.27171	.30702	.33633	.36080	.38117	.39778	.41057	.41893	.42110	.41202	.37194

For given p, n, j and $P^* = .75$ (top), .90 (middle), .95 (bottom), the entries in this table are the values of d for which $\int_0^\infty [1 - G_{j,n}(x\,d)]^p \, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the cdf of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 3C

Percentage points of the distribution of $Y_{\min} = \min_{1 \leqq i \leqq p} X_i/X_0$ *for* $p = 3$

n	j = 2	3	4	5	6	7	8	9	10	11	12	13	14	15
	.24830	.33214	.38352	.40058										
5	.12895	.19983	.24751	.26412										
	.08451	.14516	.18868	.20473										
	.24865	.33390	.38968	.42388	.42810									
6	.12920	.20134	.25329	.28677	.29094									
	.08470	.14643	.19383	.22551	.22998									
	.24885	.33483	.39253	.43191	.45493	.45008								
7	.12935	.20214	.25597	.29472	.31803	.31284								
	.08481	.14710	.19623	.23287	.25546	.25084								
	.24898	.33538	.39409	.43576	.46455	.47978	.46818							
8	.12944	.20262	.25744	.29855	.32792	.34367	.33118							
	.08488	.14750	.19755	.23644	.26486	.28038	.26847							
	.24906	.33574	.39505	.43794	.46930	.49075	.50024	.48344						
9	.12950	.20293	.25834	.30072	.33283	.35527	.36519	.34685						
	.08492	.14776	.19835	.23845	.26955	.29163	.30154	.28365						
	.24912	.33598	.39568	.43929	.47204	.49628	.51236	.51747	.49655					
10	.12955	.20314	.25894	.30208	.33567	.36117	.37830	.38359	.36046					
	.08495	.14794	.19888	.23971	.27226	.29736	.31443	.31979	.29690					
	.24916	.33616	.39611	.44020	.47378	.49954	.51859	.53059	.53225	.50798				
11	.12958	.20329	.25935	.30298	.33748	.36464	.38508	.39802	.39957	.37244				
	.08498	.14806	.19925	.24056	.27399	.30075	.32113	.33415	.33574	.50798				
	.24920	.33629	.39642	.44084	.47496	.50163	.52231	.53743	.54622	.54511	.51807			
12	.12960	.20340	.25965	.30362	.33871	.36688	.38913	.40561	.41517	.41362	.38309			
	.08499	.14816	.19952	.24115	.27516	.30293	.32514	.34173	.35143	.34985	.31910			
	.24922	.33638	.39666	.44130	.47580	.50307	.52473	.54156	.55361	.55984	.55644	.52707		
13	.12962	.20348	.25987	.30409	.33958	.36841	.39178	.41020	.42348	.43027	.42610	.39266		
	.08501	.14823	.19972	.24159	.27600	.30443	.32766	.34633	.35981	.36672	.36245	.32853		
	.24924	.33646	.39684	.44165	.47642	.50410	.52641	.54428	.55812	.56771	.57182	.56652	.53516	
14	.12963	.20355	.26004	.30444	.34022	.36952	.39362	.41323	.42856	.43922	.44368	.43730	.40132	
	.08502	.14828	.19987	.24191	.27661	.30551	.32958	.34936	.36495	.37584	.38040	.37380	.33711	
	.24926	.33652	.39698	.44193	.47689	.50487	.52762	.54619	.56112	.57255	.58013	.58249	.57556	.54250
15	.12964	.20360	.26017	.30471	.34071	.37034	.39495	.41535	.43195	.44476	.45323	.45572	.44741	.40921
	.08503	.14833	.19999	.24217	.27708	.30631	.33090	.35149	.36838	.38149	.39019	.39272	.38409	.34494

For given p, n, j and $P^* = .75$ (top), .90 (middle), .95 (bottom), the entries in this table are the values of d for which $\int_0^\infty [1 - G_{j,n}(x\,d)]^p \, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the cdf of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 3D

Percentage points of the distribution of $Y_{\min} = \min_{1 \leq i \leq p} X_i/X_0$ for $p = 4$

n	j = 2	3	4	5	6	7	8	9	10	11	12	13	14	15
	.21024	.29285	.34497	.36357										
5	.10994	.17752	.22430	.24157										
	.07234	.12953	.17176	.18812										
	.21055	.29650	.35081	.38624	.39188									
6	.11017	.17891	.22974	.26316	.26830									
	.07251	.13060	.17658	.20783	.21301									
	.21073	.29537	.35364	.39407	.41825	.41462								
7	.11029	.17965	.23226	.27076	.29436	.29023								
	.07260	.13131	.17883	.21483	.23741	.23368								
	.21084	.29589	.35514	.39784	.42774	.44401	.43341							
8	.11037	.18009	.23365	.27442	.30391	.32010	.30866							
	.07266	.13168	.18007	.21822	.24644	.26216	.25121							
	.21092	.29622	.35606	.39996	.43243	.45491	.46532	.44930						
9	.11043	.18037	.23450	.27650	.30865	.33138	.34179	.32446						
	.07270	.13192	.18082	.22014	.25094	.27303	.28325	.26635						
	.21097	.29645	.35666	.40129	.43514	.46043	.47744	.48332	.46299					
10	.11046	.18056	.23506	.27779	.31140	.33711	.35460	.36040	.33821					
	.07273	.13208	.18132	.22134	.25355	.27858	.29579	.30151	.27960					
	.21101	.29661	.35708	.40218	.43686	.46367	.48368	.49650	.49881	.47494				
11	.11049	.18070	.23545	.27866	.31315	.34050	.36124	.37457	.37660	.35034				
	.07275	.13219	.18167	.22214	.25521	.28186	.30231	.31554	.31751	.29134				
	.21104	.29673	.35738	.40280	.43802	.46576	.48741	.50339	.51290	.51231	.48551			
12	.11051	.18080	.23573	.27927	.31433	.34268	.36522	.38203	.39198	.39087	.36114			
	.07276	.13228	.18192	.22271	.25634	.28398	.30622	.32295	.33288	.33169	.30184			
	.21106	.29682	.35761	.40326	.43885	.46719	.48984	.50756	.52037	.52721	.52422	.49495		
13	.11053	.18088	.23594	.27972	.31518	.34418	.36782	.38655	.40018	.40733	.40358	.37085		
	.07277	.13234	.18210	.22312	.25714	.28543	.30877	.32745	.34111	.34827	.34438	.31132		
	.21108	.29689	.35778	.40360	.43947	.46822	.49153	.51031	.52494	.53520	.53984	.53484	.50345	
14	.11054	.18094	.23610	.28005	.31580	.34526	.36962	.38954	.40521	.41621	.42101	.41499	.37966	
	.07278	.13239	.18224	.22343	.25773	.28648	.31055	.33042	.34616	.35725	.36206	.35582	.31994	
	.21109	.29695	.35792	.40387	.43993	.46899	.49275	.51223	.52798	.54012	.54829	.55109	.54438	.51116
15	.11055	.18099	.23623	.28031	.31627	.34606	.37092	.39163	.40856	.42170	.43050	.43329	.42532	.38769
	.07279	.13243	.18236	.22368	.25818	.28726	.31184	.33250	.34953	.36281	.37172	.37449	.36621	.32783

For given p, n, j and $P^* = .75$ (top), .90 (middle), .95 (bottom), the entries in this table are the values of d for which $\int_0^\infty [1 - G_{j,n}(x\,d)]^p \, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the cdf of the jth order statistic in a sample of size n from the exponential distribution.

TABLE 3E

Percentage points of the distribution of $Y_{\min} = \min_{1 \leq i \leq p} X_i/X_0$ *for* $p = 5$

n	j										
	2	3	4	5	6	7	8	9	10	11	12
5	.18512	.26616	.31847	.33808							
	.09728	.16219	.20805	.22583							
	.06419	.11872	.15989	.17643							
	.18541	.26772	.32418	.36014	.36682						
6	.09748	.16350	.21333	.24659	.25241						
	.06433	.11981	.16447	.19533	.20101						
	.18557	.26855	.32682	.36779	.39268	.39000					
7	.09760	.16419	.21573	.25391	.27766	.27431					
	.06442	.12038	.16660	.20205	.22457	.22150					
	.18567	.26904	.32827	.37146	.40201	.41899	.40921				
8	.09767	.16460	.21706	.25744	.28693	.30340	.29278				
	.06447	.12073	.16778	.20531	.23331	.24915	.23894				
	.18574	.26935	.32916	.37354	.40663	.42977	.44082	.42550			
9	.09772	.16486	.21787	.25944	.29154	.31441	.32517	.30864			
	.06451	.12095	.16850	.20716	.23767	.25974	.27017	.25402			
	.18579	.26957	.32974	.37484	.40930	.43524	.45286	.45931	.43955		
10	.09775	.16504	.21840	.26070	.29421	.32002	.33773	.34389	.32246		
	.06453	.12110	.16897	.20831	.24020	.26514	.28242	.28840	.26725		
	.18582	.26972	.33015	.37570	.41100	.43845	.45908	.47245	.47525	.45184	
11	.09777	.16517	.21877	.26153	.29591	.32333	.34425	.35783	.36022	.33468	
	.06455	.12120	.16930	.20908	.24181	.26833	.28880	.30215	.30442	.27899	
	.18585	.26984	.33044	.37631	.41215	.44053	.46279	.47934	.48935	.48917	.46272
12	.09779	.16527	.21904	.26212	.29706	.32546	.34815	.36518	.37539	.37463	.34557
	.06456	.12128	.16954	.20963	.24290	.27040	.29262	.30943	.31952	.31863	.28951

For given p, n, j and $P^* = .75$(top), .90(middle), .95(bottom), the entries in this table are the values of d for which $\int_0^\infty [1 - G_{j,n}(x\,d)]^p \, dG_{j,n}(x) = P^*$ where $G_{j,n}(\cdot)$ is the cdf of the jth order statistic in a sample of size n from the exponential distribution.

In all these tables the probability levels are chosen such that P^*, the infimun of the probability of correct selection in the companion paper by Barlow and Gupta is .75, .90 and .95 and the entries are either the percentage points or the reciprocals of the percentage points so that they will be the values of the constants d or c $(0 < c, d < 1)$ to be used in the selection procedures discussed in the companion paper.

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